MATH 7711, AUTUMN 2019

Totally Geodesic Submanifolds

[**DG**] stands for *Differential Geometry* at

http://www.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf

For a manifold M with a fixed connection ∇ in TM, let $\exp_z : U_z \to M$ denote the exponential mapping of ∇ at a point $z \in M$. Recall that $\exp_z$ has the domain $U_z \subseteq T_zM$ formed by all $v \in T_zM$ such that there exists a ∇-geodesic $[0,1] \ni t \mapsto x(t)$ having $(x(0), \dot x(0)) = (z, v)$, and $\exp_z v$ then equals $x(1)$. The set U_z is open and star-shaped about 0 in the sense of being closed under multiplications by all scalars $c \in 0,1]$. A submanifold P of M (with or without the subset topology) is called *totally geodesic at a point* $z \in P$ if, given any $v \in T_zP$, one has $\exp_z tv \in P$ for all t sufficiently close to 0 in $\mathrm{I\!R}$. When this is the case for all $z \in P$, one calls P a *totally geodesic submanifold* of M.

We proceed to show that *a submanifold P of M is totally geodesic relative to ∇ if and only if, for every $z \in P$, some neighborhood of z in P is the $\exp_z$-diffeomorphic image of a neighborhood of 0 in T_zP, where $\exp_z$, as before, corresponds to ∇.*

The 'if' part is obvious: the $\exp_z$-images of short constant-speed line segments through 0 are ∇-geodesics. For the 'only if' part, fix $z \in P$, a neighborhood U' of 0 in T_zM mapped by $\exp_z$ diffeomorphically onto a neighborhood U of z in M, and a linear projection operator $\varPhi : T_zM \to T_zP$. With P' denoting the pre-image of $P \cap U$ under the diffeomorphism $\exp_z : U' \to U$, we see that $\mathrm{Id} : T_zP \to T_zP$ is the differential at 0 of the restriction $\varPhi : P' \to T_zP$. Using the inverse mapping theorem and replacing P' with a suitable neighborhood of 0 in P' we may thus assume that $\varPhi : P' \to \varPhi(P')$ is a diffeomorphism onto $\varPhi(P')$, a bounded star-shaped neighborhood of 0 in T_zP. It follows now that $P' = \varPhi(P')$ and $\varPhi : P' \to \varPhi(P')$ equals the identity. Namely, let $\{tv : 0 \le t < 1\}$ be the intersection of $\varPhi(P')$ with a half-line in T_zP emanating from 0. Since P is totally geodesic, $\{tv : 0 \le t < c\} \subseteq P'$ for some $c \in (0,1]$. The greatest such c must equal 1, or else the ∇-geodesic $[0,c] \ni t \mapsto \exp_z tv$ would be contained entirely in P (due to continuity of the inverse of the diffeomorphism $\varPhi : P' \to \varPhi(P')$), and hence admit an extension to another such geodesic with a domain $[0,c')$, where $c' > c$, contrary to maximality of c.

In the statement italicized above, the words starting from ... *totally geodesic* and ending with *some neighborhood* ... cannot be replaced by ... *totally geodesic at a point* $z \in P$ *if and only some neighborhood* ....

Here is a counterexample. Generally, given a smooth mapping $F : N \to \hat N$ between manifolds, the graph $Z = \{(y, F(y)) : y \in N\}$ is a submanifold of $N \times \hat N$ with the subset topology, diffeomorphic to N, a diffeomorphism being provided by $y \mapsto (y, F(y))$ (and its inverse by the factor projection $N \times \hat N \to N$). The function $\lambda : \mathrm{I\!R} \to \mathrm{I\!R}$ given by $\lambda(t) = e^{-1/t}$ for $t > 0$ and $\lambda(t) = 0$ for $t \le 0$ is of class C^∞ [**DG**, Problems 13-15 on p. 217], and hence so is $F : \mathbf{R}^2 \to \mathrm{I\!R}$ defined by $F(x,y) = y^2\lambda(x^4 - y^2)$. In $\mathbf{R}^3$ with the standard flat connection, let P to be the graph of F. The submanifold P of $\mathbf{R}^3$ is totally geodesic at $(0,0,0)$, as it contains both the x and y coordinate axes, and its intersectino with each line of the form $\{(x,y,z) : z = y - cx = 0\}$, where $c \in \mathrm{I\!R} \smallsetminus \{0\}$, constitutes the segment $\{(x, cx, 0) : |x| \le |c|\}$. However, P contains no neighborhood of $(0,0,0)$ in the plane $\mathbf{R}^2 \times \{0\}$ tangent to P at $(0,0,0)$.