MATH 7721, SPRING 2018

Homework #11, February 2

PROBLEMS

1. Verify formulae (11.2) and (11.8). (Hint on the reverse.)

2. Prove (11.4). (Hint on the reverse.)

3. Given two endomorphisms A, J of a real/complex vector space, such that $J^2 = -\mathrm{Id}$, show that the commutator $[J, A] = JA - AJ$ anticommutes with J, and the anticommutator $JA + AJ$ commutes with J.

4. Using the second identification in (11.1) and formula (11.7) to define the (pointwise) g-inner product $g(a, b) = \langle a, b\rangle$ of two twice-covariant tensor fields a, b on a Riemannian manifold (M, g), establish the local-coordinate equality

$$\langle a, b\rangle \,=\, a_{jk} b^{jk},$$

that is, $\langle a, b\rangle \,=\, g^{jp} g^{kq} a_{jk} b_{pq}$, and the relations $\mathrm{tr}_g\, a \,=\, \langle g, a\rangle$, $\langle g, g\rangle \,=\, \dim M$. (Hint on the reverse.)

Hint. In Problem 1, for (11.2), the Leibniz rule gives $g(Aw,\,\cdot\,) = g(\nabla_{\!w}v,\,\cdot\,) = \nabla_{\!w}[g(v,\,\cdot\,)] = \nabla_{\!w}\xi = \nabla\xi(w,\,\cdot\,)$, where $A = \nabla v$ and w is any vector field. For (11.8), note that $(d\xi)(u,v) = [\nabla_{\!u}\xi](v) - [\nabla_{\!v}\xi](u)$ for any smooth 1-form ξ, any torsion-free connection ∇, and vector fields u, v, cf. [**KG**], formula (1.23,b)].

Hint. In Problem 2, note again that $(d\xi)(u,v) = [\nabla_{\!u}\xi](v) - [\nabla_{\!v}\xi](u)$ (see above). Then apply the Leibniz rule to (11.3).

Hint. In Problem 4, use (10.2).