MATH 7721, SPRING 2018

Homework #1, January 8

PROBLEMS

1. Let g be a Riemannian metric on an almost complex manifold M with the structure tensor J. Prove that g is a Hermitian metric if and only if $g = \mathrm{Re}\, h$ for some (complex-valued) Hermitian fibre metric h in the complex vector bundle TM. Verify that such h is uniquely determined by g and, explicitly, $h = g - ig(J\,\cdot\,,\,\cdot\,)$.

2. For g, M, J as above, verify that g is a Hermitian metric if and only if J is g-skew-adjoint at each point or, equivalently, J_x constitutes, at each point x, a linear isometry of the tangent space.

3. For any finite-dimensional complex vector space V, we introduced a natural orientation in the underlying real space of V by declaring the real basis $e_1, ie_1, \ldots, e_m, ie_m$ to be positive oriented whenever $e_1, \ldots, e_m$ is a complex basis of V. There is an obvious direct-sum operation both for complex vector spaces and for oriented real vector spaces. Verify that our assignment (complex) $\mapsto$ (real oriented) is "additive" relative to these direct-sum operations. Would it still be the case if, rather than $e_1, ie_1, \ldots, e_m, ie_m$, we used $e_1, \ldots, e_m, ie_1, \ldots, ie_m$ instead?

4. Prove the claims made in the first sentence of the second paragraph of Remark 3.2, and in the last sentence of the first paragraph of Remark 3.1.

5. For A and a related as in Remark 2.1, on a Riemannian manifold (M, g), verify the local coordinate relation $A_j^k = a_j{}^k$, where $a_j{}^k = a_{jp} g^{pk}$. Show that the adjoint $B = A^*$ then has the components $B_j^k = a^k{}_j$, with $a^k{}_j = g^{kp} a_{pj}$.

6. Given a (complex) basis of a finite-dimensional complex vector space V and a Hermitian inner product $\langle\,,\rangle$ in V, verify that $e_1, \ldots, e_m$ is $\langle\,,\rangle$-orthonormal if and only if $e_1, ie_1, \ldots, e_m, ie_m$ is $(\mathrm{Re}\,\langle\,,\rangle)$-orthonormal (as a basis of the underlying real space of V, in which $\mathrm{Re}\,\langle\,,\rangle$ obviously constitutes a Euclidean inner product).