MATH 7721, SPRING 2018

Homework #10, January 31

PROBLEMS

1. Prove that $(\delta w)\mu = -d\alpha$ for any positive smooth differential n-form μ on an oriented n-dimensional manifold M, any smooth vector field w on M, and the differential $(n-1)$-form $\alpha = \mu(w, \cdot, \ldots, \cdot)$. (Hint below)

2. Derive the divergence theorem from the Stokes theorem.

3. Denoting by ∇f and ∇df the gradient and Hessian of a smooth function f on a Riemannian manifold (M, g), show that $dQ = 2[\nabla df](\nabla f, \cdot)$, where the function $Q : M \to \mathbf{R}$ is given by $Q = g(\nabla f, \nabla f)$. Conclude that, more generally, for smooth functions ψ, ϕ, f on a Riemannian manifold (M, g), one has $d[g(\nabla\psi, \nabla\phi)] = (\nabla d\psi)(\nabla\phi, \cdot) + (\nabla d\phi)(\nabla\psi, \cdot)$. (Hint below)

Hint. In Problem 1, evaluate $(d\alpha)_{1\ldots n}$ from the general formula

$$(d\alpha)_{j_0\ldots j_p} = \sum_{q=0}^{p} (-1)^q \partial_{j_q} \alpha_{j_0\ldots\widehat{j_q}\ldots j_p},$$

for any smooth differential p-form α, where $\widehat{\ }$ stands for 'delete'.

Hint. In Problem 3, use local coordinates: $(f_{,k} f^{,k})_{,j} = 2f_{,kj} f^{,k}$.