MATH 7721, SPRING 2018

Homework #11, February 2

PROBLEMS

1. Let $\langle\,,\rangle$ be a (positive-definite) Euclidean/Hermitian inner product in a real/complex vector space V of any finite dimension. One calls an endomorphism $A : V \to V$ *self-adjoint* if $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in V$, and *positive* if $\langle Av, v\rangle > 0$ for all nonzero vectors $v \in V$. Prove that positive self-adjoint endomorphism of V form an open subset U of the real vector space W of all self-adjoint endomorphism of V, and the mapping $F : U \to U$ given by $F(A) = A^2$ is a diffeomorphism. (Hint below)

2. Suppose that g and h are (positive-definite) Euclidean/Hermitian inner products in a finite-dimensional real/complex vector space V, and B is the g-self-adjoint g-positive endomorphism $V \to V$, uniquely characterized by the condition $g(Bv, w) = h(v, w)$ for all $v, w \in V$. According to Problem 1, B has a unique g-self-adjoint g-positive square root $A : V \to V$, which depends smoothly on the pair g, h. Verify that A is a linear isometry of (V, g) onto (V, h).

3. Given two (smooth) Riemannian/Hermitian fibre metrics g and h in a real/complex vector bundle $\mathcal{E}$ over a manifold M, show that there exists a unique smooth vector-bundle isomorphism $A : \mathcal{E} \to \mathcal{E}$ sending g to h and g-self-adjoint as well as g-positive at every point.

Hint. In Problem 1, F is surjective since every $A \in U$ has, in some orthonormal basis, a diagonal matrix with positive (real) diagonal entries; it is injective since, using such a basis, we see that A^2 determines A to be the operator acting in each eigenspace of A^2 as a specific multiple of Id; and it is locally diffeomorphic by the inverse mapping theorem: if two endomorphisms A, B anticommute, B must send each eigenspace $\mathrm{Ker}\,(A - \lambda)$ into $\mathrm{Ker}\,(A + \lambda)$. To compute the differential of the mapping $U \to U$ given by $A \mapsto A^2$ you may pretend that A depends smoothly on a real parameter t and then differentiate A^2 with respect to t.