MATH 7721, SPRING 2018

Homework #13, February 7

PROBLEMS

1. For two twice-covariant tensor fields a, b on a manifold M, we write $a \leq b$, or $b \geq a$, if $b - a$ is positive-semidefinite at every point of M. Prove that $a = b$ whenever a, b are Hermitian 2-tensor fields on a compact Kähler manifold, $a \leq b$, the 2-forms aJ, bJ are closed, and $[aJ] = [bJ]$ in $H^2(M, \mathbf{R})$. (Hint below)

2. Let two Kähler metrics $g, \hat{g}$ with the Ricci tensors $\mathrm{r}, \hat{\mathrm{r}}$ on a compact almost-complex manifold satisfy the condition $\mathrm{r} \leq \hat{\mathrm{r}}$. Show that $\mathrm{r} = \hat{\mathrm{r}}$.

3. Prove the Gauss-Bonnet Theorem: for a compact oriented surface Σ, the following oriented integral (with K denoting the Gaussian curvature of g):

$$\chi(\Sigma) = \frac{1}{2\pi} \int_{\Sigma} K \, \mathrm{d}g,$$

does not depend on the choice of the Riemannian metric g, or the orientation.

Hint. In Problem 1, the $\partial\overline{\partial}$ Lemma gives $aJ - bJ = -i\partial\overline{\partial}f$ for some $f : M \to \mathbf{R}$, so that $0 \leq b - a = (aJ - bJ)J = -[\partial\overline{\partial}f]J$. Applying tr_g, we get $\Delta f \geq 0$, and hence f is constant (see formulae (11.3) and (11.2) in the day-by-day list of topics).

Hint. In Problem 3, independence of the orientation is clear, as reversing the orientation changes the signs of both $\mathrm{d}g$ and the oriented integral. Once the orientation is fixed, the claim follows since $\chi(\Sigma) = \int_{\Sigma} c_1(\Sigma)$ (Problem 3 in Homework #12), and we may combine Problem 1 in Homework #12 with Problem 4 in Homework #5.