MATH 7721, SPRING 2018

Homework #18, February 19

PROBLEMS

1. Given a connection ∇ with the curvature tensor R in a real/complex vector bundle $\mathcal{E}$ over a manifold M, and a torsion-free connection in TM (also denoted by ∇), prove the Ricci identity

$$\varPhi^b_{a,jk} - \varPhi^b_{a,kj} = R_{jkc}{}^b\varPhi^c_a - R_{jka}{}^c\varPhi^b_c$$

for any smooth section $\varPhi$ of $\mathrm{Hom}(\mathcal{E},\mathcal{E})$ or, equivalently,

$$[\nabla_w(\nabla\varPhi)]v - [\nabla_v(\nabla\varPhi)]w = [R(v,w),\varPhi],$$

where v, w are any smooth vector fields and $[\ ,\ ]$ on the right-hand side is the commutator of bundle morphisms $\mathcal{E} \to \mathcal{E}$, while $[\ldots]v$ and $[\ldots]w$ on the left-hand side are the images of v and w under bundle morphisms $TM \to \mathrm{Hom}(\mathcal{E},\mathcal{E})$. (Hint below)

2. Use Problem 1 to conclude that $w_{j,k}{}^{jk} = w_{j,k}{}^{kj}$ for any smooth vector field w on a Riemannian manifold.

3. Generalize Problem 1 to the case of two real/complex vector bundles $\mathcal{E}, \mathcal{E}'$ over a manifold M, both endowed with connections, and any smooth section $\varPhi$ of $\mathrm{Hom}(\mathcal{E},\mathcal{E}')$.

Hint. In Problem 1, note that, with $u = \nabla_w v$,

$$[\nabla_w(\nabla\varPhi)]v = \nabla_w\nabla_v\varPhi - \nabla_u\varPhi$$

and so, for any smooth section ψ of $\mathcal{E}$,

$$\{[\nabla_w(\nabla\varPhi)]v\}\psi = \nabla_w[(\nabla_v\varPhi)\psi] - (\nabla_v\varPhi)\nabla_w\psi - \nabla_u(\varPhi\psi) + \varPhi\nabla_u\psi,$$

while

$$\nabla_w[(\nabla_v\varPhi)\psi] - (\nabla_v\varPhi)\nabla_w\psi = \nabla_w[\nabla_v(\varPhi\psi) - \varPhi\nabla_v\psi] - \nabla_v(\varPhi\nabla_w\psi) + \varPhi\nabla_v\nabla_w\psi.$$

Since $R(v,w)\psi = \nabla_w\nabla_v\psi - \nabla_v\nabla_w\psi + \nabla_{[v,w]}\psi$, the difference $\{[\nabla_w(\nabla\varPhi)]v\}\psi - \{[\nabla_v(\nabla\varPhi)]w\}\psi$ thus equals $R(v,w)(\varPhi\psi) - \varPhi[R(v,w)\psi]$. .