MATH 7721, SPRING 2018

Homework #2, January 10

PROBLEMS

1. Let M be an almost complext manifold, with the structure tensor J. Its *Nijenhuis* tensor field (or, *torsion* tensor field) is the section $\varPhi$ of $\mathrm{Hom}([TM]^{\wedge 2}, TM)$ (where $[\ ]^{\wedge 2}$ denotes the *real* exterior power), characterized by

$$\varPhi(v,w) \,=\, [Jv,Jw] - [v,w] - J[v,Jw] - J[Jv,w]$$

for any two C^∞ vector fields v, w on M.

(a) Prove that $\varPhi$ is really a tensor field, that is, for any point $x \in M$, the value $[\varPhi(v,w)]_x$ depends only on v_x and w_x.
(b) Verify that $\varPhi = 0$ if J is integrable.
(c) Show that $\varPhi = 0$ if $\nabla J = 0$ for the Levi-Civita connection of some Riemannian metric g on M (in other words, if (M,g) is a Kähler manifold).

Vanishing of the Nijenhuis tensor is known to be not only necessary, but also sufficient for integrability of J (the Newlander-Nirenberg theorem).

2. Show that a C^∞ section ψ of a vector bundle $\mathcal{E}$ over a manifold M is parallel relative to a connection ∇ in $\mathcal{E}$ if and only if $\nabla_{\dot x}\psi = 0$ for every C^∞ curve $t \mapsto x(t)$ in M.

3. Given a connection ∇ ion the tangent bundle TM of a manifold M, a Riemannian metric g on M, and an almost complex structure J on M, prove the following two statements. (Hint below.)

(i) $\nabla g = 0$ if and only if all ∇-parallel transports are linear isometries.
(ii) $\nabla J = 0$ if and only if all ∇-parallel transports are complex-linear.

Hint. In Problem 3, use parallel vector fields, or parallel orthonormal vector fields, along the curve.