MATH 7721, SPRING 2018

Homework #25, March 7

PROBLEMS

1. Verify that a Euclidean space M with its (constant) flat metric g satisfies the Ricci-soliton equation $\pounds_w g + \mathrm{r} = \lambda g$ with *any* $\lambda \in \mathbf{R}$ and w chosen to be the linear vector field $\lambda/2$ times the identity (in other words, $w_x = \lambda x/2$ for $x \in M$).

2. Prove that $\Delta f - |\nabla f|^2 + 2\lambda f$ is constant for any gradient Ricci soliton (M, g) with $\nabla df + \mathrm{r} = \lambda g$, where $\lambda \in \mathbf{R}$. (Hint below)

3. Generalize Problem 2 in Homework #23 by using the $\partial\overline{\partial}$ Lemma to show that, for *any* compact Kähler manifold and any $v \in \mathfrak{h}(M)$, the 1-form $g(v, \cdot)$ has an exact-coclosed decomposition. Make sure that you do not invoke the assumptions preceding equality (23.7) in the day-by-day list of topics. (Hint below)

Hint. In Problem 2, rewrite the relation $2\nabla df(\nabla f, \cdot) + 2\mathrm{r}(\nabla f, \cdot) = 2\lambda g(\nabla f, \cdot)$ as $\nabla|\nabla f|^2 + 2\mathrm{r}\nabla f = 2\lambda\nabla f$ (see Problem 3 in Homework #10), and note that $2\mathrm{r}\nabla f = -\nabla\Delta f$ from (25.4), in the day-by-day list of topics, combined with (18.1) for $w = \nabla f/2$.

Hint. Problem 3: relation (19.6) in the day-by-day list of topics implies that the exact 1-form $d[g(Jv, \cdot)]$ skew-Hermitian (as $\pounds_w g$ is symmetric). Thus, the $\partial\overline{\partial}$ Lemma allows us to choose a smooth function θ with $d[g(Jv, \cdot)] = 2i\partial\overline{\partial}\theta$. From the definition of $i\partial\overline{\partial}\theta$ (see (9.5) in the day-by-day list of topics) we now conclude that $g(Jv, \cdot) = -(d\theta)J + \xi$, where $\xi = g(Jv, \cdot) + (d\theta)J$ is a closed 1-form. Multiplying both sides of the last equality from the right by J (that is, composing them with J), we obtain $g(v, \cdot) = d\theta + \xi J$, which is the required decomposition since, locally, $\xi = g(\nabla f, \cdot)$ for some function f, and so Problem 1 in Homework #23 gives $\delta(\xi J) = 0$.