MATH 7721, SPRING 2018

Homework #26, March 9

PROBLEMS

1. For a holomorphic vector field v on a Kähler manifold (M, g), verify that the following two conditions are equivalent:

(i) v is locally a gradient, or – in other words – the 1-form $g(v, \cdot)$ is closed,
(ii) $u = Jv$ is a Killing field.

(Hint below.)

2. With the same assumptions as in Problem 2 of Homework #25, denoting by c the constant $\Delta f - |\nabla f|^2 + 2\lambda f$, show that $\Delta e^{-f} = (c - 2\lambda f)e^{-f}$.

3. Let the gradient Ricci-soliton equation $\nabla df + \mathrm{r} = 0$ (with $\lambda = 0$) be satisfied on a compact oriented Riemannian manifold (M, g). Prove that f is constant, and so the metric g is Ricci-flat. (Hint below.)

Hint. In Problem 1, observe that for $A = \nabla v$ and $B = \nabla u$ one has $B = JA = AJ$, and so $B + B^* = J(A - A^*)$, while (i) amounts to $A - A^* = 0$, and (ii) to $B + B^* = 0$.

Hint. In Problem 3, use Problem 2 along with Bochner's lemma (that is, (11.2) in the day-by-day list of topics) for $\theta = \pm e^{-f}$.