MATH 7721, SPRING 2019

Homework #27, March 19

PROBLEMS

1. Verify identity (27.6) in the day-by-day list of topics, that is,

$$\Delta e^{\psi} = [\Delta\psi + g(\nabla\psi, \nabla\psi)]\,e^{\psi}$$

for smooth complex-valued functions ψ on a Riemannian manifold, with $\Delta\psi = \Delta\operatorname{Re}\psi + i\Delta\operatorname{Im}\psi$ and $g(\nabla\psi, \nabla\psi) = |\nabla\operatorname{Re}\psi|^2 - |\nabla\operatorname{Im}\psi|^2 + 2ig(\nabla\operatorname{Re}\psi, \nabla\operatorname{Im}\psi)$.

2. Assuming the gradient Ricci-soliton equation $\nabla df + \mathrm{r} = \lambda$ with a constant $\lambda < 0$ on a compact Riemannian manifold (M, g), prove that f is constant and, consequently, g is an Einstein metric. (Hint below.)

3. For a smooth function f on a compact Riemannian manifold, taking values in an interval I and satisfying the equation $\Delta f = \sigma(f)$ for some strictly increasing function $\sigma : I \to \mathbf{R}$, prove that f must be constant. (Hint below.)

Hint. In Problem 2, note that, for c chosen as in Problem 2 of Homework #26, relation (9.4) in the day-by-day list of topics gives $2\lambda\min f \le c \le 2\lambda\max f$.

Hint. Problem 3: use (9.4) in the day-by-day list of topics.