MATH 7721, SPRING 2018

Homework #28, March 21

PROBLEMS

1. Verify that a real-linear operator from a real vector space $\mathcal{V}$ into a complex vector space $\mathcal{W}$ has a unique complex-linear extension $\mathcal{V}_{\mathbf{C}} \to \mathcal{W}$, where $\mathcal{V}_{\mathbf{C}}$ denotes the complexification of $\mathcal{V}$.

2. Show that, if ψ is a smooth complex-valued function on a compact Kähler manifold and its complex gradient

$$\partial\psi \,=\, \nabla\mathrm{Re}\psi \,+\, J\nabla\mathrm{Im}\psi$$

vanishes identically, then ψ must be constant. (Hint below.)

3. Prove that a holomorphic vector field w on a compact Kähler manifold is a Killing field if and only if its divergence δw vanishes identically. (Hint below.)

Hint. In Problem 2, use integration by parts and Problem 1 in Homework #23 to conclude that $(\nabla\phi, J\nabla\chi) = 0$ for smooth real-valued functions ϕ, χ, and so

$$\|\partial\psi\|^2 \,=\, \|\nabla\mathrm{Re}\psi\|^2 \,+\, \|J\nabla\mathrm{Im}\psi\|^2,$$

$(\ ,\ )$ being the L^2 inner product and the L^2 norm.

Hint. Problem 3: by (19.6) in the day-by-day list of topics, $\zeta = (\pounds_w g)J$ is an exact skew-Hermitian 2-form, while, in view of (15.3), $\delta w = 0$ if and only if, for this ζ, the right-hand side of (12.1) equals 0.