MATH 7721, SPRING 2018

Homework #30, March 26

PROBLEMS

1. Verify that a real-valued real-bilinear form on a real vector space $\mathcal{V}$ has a unique complex-valued sesquilinear extension to its complexification $\mathcal{V}_{\mathbf{C}}$.

2. Suppose that $(\,,\,)$ is a (real) inner product on a real vector space $\mathcal{F}$ and $\Lambda : \mathcal{F} \to \mathcal{X}$ is a real-linear operator into a complex vector space $\mathcal{X}$ endowed with a (complex-valued sesquilinear) inner product $(\,,\,)'$. According to Problem 1, the complexification $\mathcal{F}_{\mathbf{C}}$ also carries the complex-valued sesquilinear inner product $(\,,\,)_{\mathbf{C}}$ arising as the unique extension of $(\,,\,)$. Prove that, if $L : \mathcal{X} \to \mathcal{F}$ is the adjoint of Λ relative to $\mathrm{Re}\,(\,,\,)'$ and $(\,,\,)$, then the operator P corresponding to L as in Problem 1 of Homework #29 is the adjoint, relative to $(\,,\,)'$ and $(\,,\,)_{\mathbf{C}}$, of the operator Π corresponding to Λ as in Problem 2 of Homework #29.

3. Show that the L^2 inner product in the complex space $\mathfrak{h}(M)$ of holomorphic vector fields on a compact Kähler manifold is the real part of a unique complex-valued sesquilinear inner product.