MATH 7721, SPRING 2018

Homework #31, March 28

PROBLEMS

1. Prove (31.2) in the day-by-day list of topics. (Hint below.)

2. Verify (31.5) in the day-by-day list of topics. (Hint below.)

3. Establish a generalized version of Futaki's result (31.5) in the day-by-day list of topics: the same conclusion, while the assumption, instead of (31.3), is just the inequality

$$-2\zeta J \,\ge\, 2\lambda g, \tag{1}$$

that is, positive semidefiniteness of $-2\zeta J - 2\lambda g$ at every point.

Hint. In Problem 1, use the coordinate form

$$R_{jk}v^k \,=\, v^k{}_{,jk} \,-\, v^k{}_{,kj} \tag{2}$$

of the Bochner identity, that is, formula (16.2) in the day-by-day list of topics. If one multiplies (2) by $v^j e^{-f}$ and integrates by parts, the right-hand side yields four terms, namely, the integrals against $e^{-f}\mathrm{d}g$ of

$$-v^k{}_{,j}v^j{}_{,k}, \quad v^k{}_{,j}v^j f_{,k}, \quad v^k{}_{,k}v^j{}_{,j}, \quad -v^k{}_{,k}v^j f_{,j}, \tag{3}$$

which add up to the integral against $e^{-f}\mathrm{d}g$ of

$$-\mathrm{tr}\,(\nabla v)^2 \,+\, v^k{}_{,j}v^j f_{,k} \,+\, (\delta v)^2 \,-\, (\delta v)\, d_v f. \tag{4}$$

Integrating by parts the second term in (4), multiplied by e^{-f}, we obtain the integral against $e^{-f}\mathrm{d}g$ of

$$-v^k v^j{}_{,j} f_{,k} - v^k v^j f_{,kj} + v^k v^j f_{,k} f_{,j} = -(\delta v)\, d_v f - (\nabla df)(v,v) + (d_v f)^2. \tag{5}$$

Now (31.2) easily follows if one replaces the second term in (4) with the right-hand side of (5).

Hint. In Problem 2, formula (30.2) in the day-by-day list of topics gives $A^* = A$ in (31.5), so that $|A|^2 = \mathrm{tr}\, A^2 = \mathrm{tr}\,[J,B]^2$ equals the left-hand side.