MATH 7721, SPRING 2018

Homework #32, March 30

PROBLEMS

1. For any Killing vector field on a Riemannian manifold (M, g) and any smooth function $f : M \to \mathbf{R}$, verify that

$$\nabla(d_w f) = [w, \nabla f] \qquad \text{and} \qquad d_w \Delta f = \Delta(d_w f).$$

In other words, the gradient ∇ and Laplacian Δ both commute with the Lie derivative $\pounds_w$, that is,

$$\nabla(\pounds_w f) = \pounds_w \nabla f \qquad \text{and} \qquad \pounds_w \Delta f = \Delta(\pounds_w f).$$

2. Let w be a holomorphic Killing vector field on a Kähler manifold (M, g). Show that the Lie derivative $\pounds_w$ commutes with both the complex gradient operator ∂ and with the operator Θ defined by formula (28.2) in the day-by-day list of topics (where, in the latter case, M is assumed compact).

3. Given a compact gradient Kähler-Ricci soliton (M, g), with $\nabla df + \mathrm{r} = \lambda g$, where f is a smooth function and $\lambda \in \mathbf{R}$, prove that, setting $u = \nabla f$, one has $\tau \geq -2\lambda$ for every eigenvalue τ of $\mathrm{Ad}\, u : \mathfrak{h}(M) \to \mathfrak{h}(M)$. (Hint below.)

Hint. In Problem 3, use the fact that $\mathrm{Ad}\, u : \mathfrak{h}(M) \to \mathfrak{h}(M)$ corresponds, under the complex-linear isomorphism ∂ (see (28.4) in the day-by-day list of topics) to the operator (32.2), in which $-\Delta + d_u$ is $(\ ,\ )_f$-nonnegative, as it equals $\nabla_f^* \nabla$, cf. (30.4) and (23.2).