MATH 7721, SPRING 2018

Homework #33, April 2

PROBLEMS

1. Let $f : M \to \mathbf{R}$ be a smooth function on a Kähler manifold (M, g). Show that the following three conditions are equivalent:

(i) the gradient ∇f is holomorphic,
(ii) $J\nabla f$ is a Killing field,
(iii) the Hessian ∇df is Hermitian.

(Hint below.)

2. For a smooth function $f : M \to \mathbf{R}$ on a Kähler manifold (M, g), let

$$i\partial\overline{\partial} f \,+\, \rho \,=\, \lambda\Omega$$

with a constant λ. Prove that, if $u = \nabla f$ is holomorphic (or, equivalently, Ju is a Killing field, cf. Problem 1), then

$$\nabla df \,+\, \mathrm{r} \,=\, \lambda g,$$

and so (M, g) is a gradient Kähler-Ricci soliton. (Hint below.)

3. Given finite-dimensional complex vector spaces V, W and a holomorphic mapping $F : U \to W$ defined on an open subset U of V, verify that the differential $dF : U \to \mathrm{Hom}(V, W)$ is holomorphic. Conclude that, for every $v \in V$, the directional derivative $d_v F : U \to W$ is holomorphic. (Hint below.)

Hint. In Problem 1, let $u = \nabla f$ and $B = \nabla u$, so that $B^* = B$ and $\nabla(Ju) = JB$. Condition (i) amounts to $JB = BJ$, and so it implies that $[\nabla(Ju)]^* = (JB)^* = (BJ)^* = J^*B^* = -JB = -[\nabla(Ju)]^*$, which is nothing else than (ii). Let us now assume (ii): $(JB)^* = -JB$. Thus, $JB = -(JB)^* = -B^*J^* = BJ$, that is, (i) follows. Finally, the bundle morphism $B = \nabla u : TM \to TM$ corresponds via index raising to ∇df, condition (iii), which reads $(\nabla df)J = J(\nabla df)$, is equivalent to $JB = BJ$, that is, (i).

Hint.