MATH 7721, SPRING 2018

Homework #4, January 17

PROBLEMS

1. Recall that, given a manifold M such that the Betti numbers $b_r(M) = \dim H^r(M,\mathbf{R})$ are all finite, one defines the *Poincaré polynomial* $\mathrm{I\!P}[M]$ in the variable t by $\mathrm{I\!P}[M] = \sum_{r=0}^n b_r t^r$, where $b_r = b_r(M)$ and $n = \dim M$. We know that $\mathrm{I\!P}[\mathbf{R}^n] = 1$ and $\mathrm{I\!P}[S^n] = 1 + t^n$, including the case of the two-point space S^0. The *Künneth formula*

$$\mathrm{I\!P}[M \times N] \;=\; \mathrm{I\!P}[M] \cdot \mathrm{I\!P}[N],$$

which was not proved in Math 6701, allows one to determine the Betti numbers of a product manifold from those of the factors. (For instance, when both M and N have finite Betti numbers, so does $M \times N$.) Using the Mayer-Vietoris sequence, prove the following special case of the Künneth formula (Hint below):

$$\mathrm{I\!P}[S^1 \times M] \;=\; (1+t)\,\mathrm{I\!P}[M].$$

2. Verify that, for the torus T^n,

$$\mathrm{I\!P}[T^n] \;=\; (1+t)^n.$$

3. Establish *naturality* of the first Chern class: $c_1(F^*\mathcal{E}) = F^*[c_1(\mathcal{E})]$, whenever $\mathcal{E}$ is a complex vector bundle over a manifold M and $F^*\mathcal{E}$ denotes its pull-back under a C^∞ mapping $F : N \to M$ (which makes $F^*\mathcal{E}$ a complex vector bundle over the other manifold N), with F^* on the right-hand side standing for the action of F is cohomology.

4. Verify that $c_1(\mathcal{E} \otimes \mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{F})$ and $c_1(\mathcal{E}^*) = -c_1(\mathcal{E})$ for complex line bundles over any manifold M.

Hint. In Problem 1, the general form of the Mayer-Vietoris sequence

$$\ldots \underset{\text{conn.}}{\overset{\delta^*}{\to}} H^sN \underset{\text{rstr.}}{\to} H^sU \times H^sU' \underset{\text{sbtr.}}{\to} H^s(U \cap U') \underset{\text{conn.}}{\overset{\delta^*}{\to}} H^{s+1}N \underset{\text{rstr.}}{\to} \ldots\,,$$

exact whenever U, U' of open subsets of a manifold N such that $U \cup U' = N$, becomes

$$\underset{\text{conn.}}{\overset{\delta^*}{\to}} H^sN \underset{\text{rstr.}}{\to} H^sM \times H^sM \underset{\text{sbtr.}}{\to} H^sM \times H^sM \underset{\text{conn.}}{\overset{\delta^*}{\to}} H^{s+1}N \underset{\text{rstr.}}{\to} H^{s+1}M \times H^{s+1}M\,,$$

where we use $U = (S^1 \smallsetminus \{p\}) \times M$ and $U' = (S^1 \smallsetminus \{q\}) \times M$ for two different points $p, q \in S^1$, and the identifications $H^sU = H^sM$ etc. are induced by the projections $U' \to M$, etc. (which, obviously, are homotopy equivalences). Thus, the subtraction operator acts on cohomology classes by $(\alpha, \beta) \mapsto (\alpha - \beta, \alpha - \beta)$. Its image is therefore the diagonal subspace of $H^sM \times H^sM$, of dimension $b_s(M)$, and this is also the kernel of δ^*, so that the image of δ^* is of dimension $2b_s(M) - b_s(M) = b_s(M)$, which, in turn, is the dimension of the kernel of the rightmost restriction operator. The latter has an image of dimension $b_{s+1}(M)$ (from what we already know about the subtraction operator, combined with exactness), and so $b_{s+1}(N) = b_{s+1}(M) + b_s(M)$, as required.