MATH 7721, SPRING 2018

Homework #7, January 24

PROBLEMS

1. Prove that $\operatorname{tr}_{\mathbf{R}} A = 2 \operatorname{Re} \operatorname{tr}_{\mathbf{C}} A$ for any linear endomorphism A of a finite-dimensional complex vector space V (also viewed as an endomorphisms of its underlying real vector space). (Hint below)

2. Show that if a real-valued function of three variables ranging over any set is skew-symmetric in the first two variables and symmetric in the last two, then it must be identically zero.

3. Verify the local-coordinate relation $b_{kl} = J_k^q a_{ql}$ for any twice-covariant tensor field a and $b = aJ$ on an almost-complex manifold. Note that, in particular, $\rho_{kl} = J_k^p R_{pl}$ for the Ricci tensor r and Ricci form ρ of any Hermitian metric.

4. A Riemannian manifold is said to have *harmonic curvature* if $R_{kl,p} = R_{kp,l}$, that is, if $[\nabla_u \mathrm{r}](v, w) = [\nabla_v \mathrm{r}](u, w)$ for its Ricci tensor r and any vector fields u, v, w. (This is always the case when $\nabla \mathrm{r} = 0$.) Prove that a Kähler manifold with harmonic curvature necessarily has $\nabla \mathrm{r} = 0$. (Hint below)

Hint. In Problem 1, determine how the matrix of A in a fixed complex basis $e_1, \ldots, e_m$ of V is related to its real matrix in the real basis $e_1, ie_1, \ldots, e_m, ie_m$.

Hint. In Problem 4, note that Problem 3 implies symmetry of $\rho_{kl,q} = J_k^p R_{pl,q}$ in l, q, and use Problem 2.