MATH 7721, SPRING 2018

Homework #8, January 26

PROBLEMS

1. Given a compact oriented n-dimensional manifold M, verify that integration over M is a well-defined linear functional $H^n(M,\mathbf{R}) \to \mathbf{R}$. In other words, the integral over M of a smooth differential n-form μ depends only on its cohomology class $[\mu] \in H^n(M,\mathbf{R})$.

2. A differential n-form μ on an oriented n-dimensional manifold M is said to be *positive* (or, *negative, nonpositive, nonnegative*) if so is $\mu_x(v_1,\dots,v_n)$ for any $x \in M$ and some/any positive-oriented basis $v_1,\dots,v_n$ of T_xM. Observe that, on a compact manifold M, one then has $\int_M \mu > 0$ whenever μ is also continuous. Show that, in the case where $n = 2$ and the orientation is induced by an almost-complex structure J on M, positivity of a smooth differential 2-form on M is equivalent to its being the Kähler form of some Kähler metric for J.

3. For oriented surfaces $\Sigma_1,\dots,\Sigma_m$ and the product $M = \Sigma_1 \times \ldots \times \Sigma_m$ endowed with the direct-sum orientation (cf. Problem 2 in Homework #1), prove that the differential $2m$-form $\mu = \zeta_1 \wedge \ldots \wedge \zeta_m$ on M is positive if each ζ_j, for $j = 1,\dots,m$, is the pullback of a positive smooth differential 2-form on Σ_j under the jth-factor projection mapping $M \to \Sigma_j$. (Hint below)

4. Show that $\int_M \mu < 0$ for the compact almost-complex manifold M obtained as the product $\Sigma_1 \times \ldots \times \Sigma_m$ of closed almost-complex surfaces, where $\mu = [c_1(M)]^{\cup m}$ and, for some odd integer k, the first Chern classes of k factor surfaces are negative, and those of the remaining $m-k$ are positive. Conclude that this almost-complex manifold M carries no Kähler-Einstein metric. (Hint below)

Hint. In Problem 3, use the easily-verified fact that a differential n-form μ on an oriented n-dimensional manifold M is positive if and only if at each point x one has $\mu_x = \xi^1 \wedge \ldots \wedge \xi^n$ for some basis $\xi^1,\dots,\xi^n$ of T_x^*M, dual to a positive-oriented basis $v_1,\dots,v_n$ of T_xM.

Hint. In Problem 4, note that $c_1(M) = \sum_j c_1(\mathcal{E}_j)$ (cf. Problem 2 in Homework #5), where $\mathcal{E}_j$, for $j = 1,\dots,m$, is the pullback of $T\Sigma_j$ under the jth-factor projection mapping $M \to \Sigma_j$. As $[c_1(\mathcal{E}_j)] \cup [c_1(\mathcal{E}_j)] = 0$ for dimensional reasons, this gives $[c_1(M)]^{\cup m} = m![c_1(\mathcal{E}_1)] \cup \ldots \cup [c_1(\mathcal{E}_m)]$, and one can apply Problem 3.