KÄHLER GEOMETRY FROM A RIEMANNIAN PERSPECTIVE

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1. Preliminaries

Let (M,g) be a Riemannian manifold of dimension n. We always assume that M is connected and all functions, vector and tensor fields under considerations are C^∞ differentiable. The symbols $\nabla, R, \mathrm{r}, \mathrm{s}$ denote the Levi-Civita connection, curvature tensor, Ricci tensor and scalar curvature of g. Thus,

$$R(u,v)w = \nabla_v\nabla_u w - \nabla_u\nabla_v w + \nabla_{[u,v]}w \qquad \text{for vector fields} \quad u,v,w \tag{1.1}$$

and $\mathrm{r}(u,w) = \mathrm{tr}\,[v \mapsto R(u,v)w]$ for vectors $u,v,w \in T_xM$ at any point $x \in M$. Given vector fields u,v, we denote by $R(u,v)$ the vector-bundle morphism

$$R(u,v) : TM \to TM\,, \qquad \text{acting on vector fields by} \quad w \mapsto R(u,v)w\,. \tag{1.2}$$

Remark 1.1. The metric g will often be used to identify twice-covariant tensors a on M with bundle morphisms $A : TM \to TM$ by requiring that $g(Av,w) = a(v,w)$ for all vector fields v,w. Symmetry/skew-symmetry of a amounts to self-adjointness/skew-adjointness of A. We denote by $\langle\,,\rangle$ the inner product of twice-covariant tensors, so that $\langle a,b\rangle = \langle A,B\rangle$ for A,B related to a,b as above, with $\langle A,B\rangle = \mathrm{tr}\,AB^*$, where A^* is the (pointwise) adjoint of A. The symbols $|\;|$ and tr_g will stand for the corresponding norm and the g-trace. Thus, $\mathrm{tr}_g a = \langle g,a\rangle$ and $\mathrm{s} = \langle g,\mathrm{r}\rangle = \mathrm{tr}_g\mathrm{r}$.

Remark 1.2. The curvature tensor of (M,g) gives rise to the bundle morphism $\hat R : [T^*M]^{\wedge 2} \to [T^*M]^{\wedge 2}$, known as the *curvature operator* acting on exterior 2-forms ω, and uniquely characterized by $[\hat R(\xi\wedge\eta)](w,w') = g(R(u,v)w,w')$ for $x \in M$, $u,v,w,w' \in T_xM$ and $\xi = \imath_u g$, $\eta = \imath_v g$. In local coordinates, $2(\hat R\zeta)_{jk} = \zeta^{lm}R_{jklm}$. Our convention about $\xi\wedge\eta$ is

$$(\xi\wedge\eta)(w,w') \,=\, \xi(w)\eta(w') \,-\, \eta(w)\xi(w')\,. \tag{1.3}$$

We let $\pounds_w$ stand for the Lie derivative in the direction of a vector field w on M. Thus, $\pounds_w f$ for a function f coincides with the directional derivative $d_w f$. Given a twice-covariant symmetric tensor a, the usual expression $(\pounds_w a)(u,v) = d_w[a(u,v)] - a([w,u],v) - a(u,[w,v])$ for vector fields u,v can be rewritten as $(\pounds_w a)(u,v) = (\nabla_w a)(u,v) + a(\nabla_u w,v) + a(u,\nabla_v w)$, that is,

$$\pounds_w a \;=\; \nabla_w a \;+\; a\nabla w \;+\; (\nabla w)^* a\,, \tag{1.4}$$

the two multiplications by a on the right-hand side being the compositions with A that corresponds to a as in Remark 1.1. Also, with ∇f denoting the g-gradient of a function f,

$$\text{a)} \quad \pounds_w g = \nabla w + (\nabla w)^*, \qquad \text{b)} \quad \pounds_w g = 2\nabla df \;\text{ if }\; w = \nabla f. \tag{1.5}$$

(In fact, (a) follows from (1.4), and implies (b).) Here ∇w is treated as a vector-bundle morphism $TM \to TM$ sending any vector (or vector field) v to $\nabla_v w$, while

$(\nabla w)^* : TM \to TM$ stands for its (pointwise) adjoint, and $a = \pounds_w g$ is identified with $A = \nabla w + (\nabla w)^*$ as in Remark 1.1. For a vector field w and a twice-covariant symmetric tensor a, we have

$$\text{(1.6)}\quad \text{i)}\ \delta w = \operatorname{tr}\nabla w, \quad \text{ii)}\ 2\delta\imath_w a \,=\, 2\imath_w\delta a \,+\, \langle a, \pounds_w g\rangle, \quad \text{iii)}\ \langle g, \pounds_w g\rangle \,=\, 2\delta w.$$

Here (i) defines the divergence operator δ, (iii) is obvious from (ii) (or (1.5.a)), and (ii) follows from (1.5.a) via the local-coordinate calculation $2(w^j a_{jk})^{,k} = 2w^j a_{jk}{}^{,k} + (w^{j,k} + w^{k,j})a_{jk}$. Next, for a vector field w and a function f,

$$\text{(1.7)}\qquad \text{a)}\ \ d_w f \,=\, \delta(fw) \,-\, f\delta w, \quad \text{where} \quad \text{b)}\ \ d_w f = \imath_w df = g(w, \nabla f).$$

We can also apply δ to vector-bundle morphisms $A : TM \to TM$, such as ∇w, resulting in the 1-form δA that sends any vector field v to the function

$$\text{(1.8)}\qquad (\delta A)v \,=\, \delta(Av) \,-\, \operatorname{tr}(A\nabla v)\,,$$

the "product" of A and ∇v being the composite. We then further extend δ to twice-covariant symmetric tensors a by setting $\delta a = \delta A$, where A corresponding to a as in Remark 1.1. Given such a (an example of which is the Ricci tensor r), and a vector field v, we define the 1-form $\imath_v a$ by the usual formula $\imath_v a = a(v, \cdot\,)$. Thus, $v \mapsto \imath_v g$ is the "index-lowering" isomorphism $TM \to T^*M$. The relations

$$\text{(1.9)}\quad \text{i)}\ \imath_v g \,=\, df \ \text{ if }\ v = \nabla f, \quad \text{ii)}\ 2\imath_v a \,=\, dQ \ \text{ if }\ v = \nabla f,\ \ Q = |v|^2 \ \text{and}\ \ a = \nabla df,$$

valid for any function $f : M \to \mathbf{R}$, follow since $d_w f = g(w, v)$ for all vectors w, while $2f^{,j} f_{,jk} = [f^{,j} f_{,j}]_{,k}$ in local coordinates. The divergence $\delta\xi$ of a 1-form ξ is given by $\delta\xi = \delta v$ for the vector field v with $\xi = \imath_v g$. Now δ may be applied twice in a row to a bundle morphism $A : TM \to TM$ such as ∇w or $(\nabla w)^*$. In addition, $\delta\xi$ has an obvious generalization to once-contravariant tensor fields on (M, g), with any number of covariant arguments, and

$$\text{(1.10)}\qquad \begin{array}{ll} \text{a)}\ \ d\nabla w \,=\, -R(\,\cdot\,,\,\cdot\,)w\,, & \text{b)}\ \ \imath_w\mathrm{r} \,=\, \delta\nabla w \,-\, d\delta w\,,\,, \\ \text{c)}\ \ 2\delta\mathrm{r} \,=\, d\mathrm{s}\,, & \text{d)}\ \ \delta R \,=\, -d\mathrm{r}\,, \\ \text{e)}\ \ \langle \mathrm{r}, \pounds_w g\rangle \,=\, 2\delta\imath_w\mathrm{r} \,-\, d_w\mathrm{s}\,, & \text{f)}\ \ \delta\delta\nabla w \,=\, \delta\delta(\nabla w)^* \end{array}$$

for any vector field w. Equalities (1.10.a) – (1.10.d) have the local-coordinate forms

$$\text{(1.11)}\qquad \begin{array}{ll} \text{a)}\ \ w^j{}_{,kl} \,-\, w^j{}_{,lk} \,=\, R_{kls}{}^j w^s, & \text{b)}\ \ R_{kl}w^k \,=\, w^k{}_{,lk} \,-\, w^k{}_{,kl}\,, \\ \text{c)}\ \ 2R_j{}^k{}_{,k} \,=\, \mathrm{s}_j & \text{d)}\ \ R_{jkl}{}^s{}_{,s} \,=\, R_{jl,k} - R_{kl,j}\,. \end{array}$$

The first three of them are known as the *Ricci identity*, the *Bochner* (or or *Weitzenböck*) *formula*, and the *Bianchi identity for the Ricci tensor.* To justify (1.10), note that (1.10.a) is, essentially, the definition of the curvature tensor R, (1.10.b), (1.10.d) and (1.10.c) are immediate if one applies a contraction to (1.10.a), the second Bianchi identity for R and, respectively, (1.10.d), while (1.10.e) follows from (1.6.ii) and (1.10.c). Finally, (1.10.f) is obvious since $\delta^2 = 0$ for the divergence operator δ acting on differential forms; namely, being skew-adjoint, $\nabla w - (\nabla w)^*$ corresponds, as in Remark 1.1, to a 2-form. Here is a direct local-coordinate verification of (1.10.f): $\delta\delta\nabla w - \delta\delta(\nabla w)^* = w^j{}_{,kj}{}^k - w^j{}_{,k}{}^k{}_j = 0$, immediate from (1.11.a) and symmetry of the Ricci tensor.

For functions $f : M \to \mathbf{R}$, (1.10.b) gives

$$\text{(1.12)}\qquad \imath_v\mathrm{r} \,=\, \delta a \,-\, dY \ \ \text{if}\ \ v = \nabla f,\ \ a = \nabla df \ \ \text{and}\ \ Y = \Delta f.$$

The symbol Δ will also stand for the 'rough Laplacian' acting on arbitrary tensors A, so that ΔA is obtained from the second covariant derivative of A by

g-contraction applied to the differentiation arguments. Thus, for a function f we have $\Delta f = \delta\xi$, with the 1-form $\xi = df$, while, for any vector field w,

$$\text{(1.13)} \qquad \text{i)}\ \ \Delta f \,=\, \delta\nabla f \,=\, \mathrm{tr}_g \nabla df \,=\, \langle g, \nabla df\rangle, \qquad \text{ii)}\ \ \Delta \imath_w g \,=\, \delta(\nabla w)^*.$$

Relation (1.13.ii) is easily verified in local coordinates, using (1.4) and the Ricci identity (1.11.a).

We denote by $\mathrm{d}g$ and $\mathrm{V} = \int_M \mathrm{d}g \in (0,\infty]$ the volume element of g and the total volume of M relative to g. If M is compact, f_{max} and f_{min} stand for the extrema of a function $f : M \to \mathbf{R}$, while $f_{\mathrm{avg}} = \mathrm{V}^{-1}\int_M f\,\mathrm{d}g$ is its average value. We will repeatedly use the *divergence theorem*:

$$\text{(1.14)} \qquad \int_M \delta w\,\mathrm{d}g \,=\, 0 \quad \text{for any compactly supported vector field } w.$$

Given a function $f : M \to \mathbf{R}$ on a compact Riemannian manifold (M, g),

$$\text{(1.15)} \qquad f_{\mathrm{avg}} = 0 \ \text{ if and only if } \ f = \Delta\phi \ \text{ for some } \ \phi : M \to \mathbf{R}.$$

Recall that a function is, by definition, C^∞-differentiable.

The 'if' part of (1.15) is obvious from (1.13.i) and (1.14). The 'only if' claim in (1.15) is one of the very few facts from analysis that are used in this exposition.

From (1.14) and (1.7.a) it follows that, for a function f and a vector field w,

$$\text{(1.16)} \qquad \int_M f\delta w\,\mathrm{d}g \,=\, -\int_M d_w f\,\mathrm{d}g \ \text{ if } M \text{ is compact.}$$

For instance, given a function f on a compact Riemannian manifold (M, g),

$$\text{(1.17)} \qquad \int_M d_u f\,\mathrm{d}g \,=\, 0 \quad \text{if } u \text{ is a Killing field,}$$

since $\delta u = 0$. If $w = \nabla\phi$ is the gradient of a function ϕ, (1.16) becomes

$$\text{(1.18)} \qquad \int_M f\Delta\phi\,\mathrm{d}g \,=\, -\int_M g(\nabla f, \nabla\phi)\,\mathrm{d}g = \int_M \phi\Delta f\,\mathrm{d}g \ \text{ if } M \text{ is compact,}$$

which, applied to $\phi = f$, shows that

$$\text{(1.19)} \qquad \begin{array}{ll} \text{a)} & \int_M f\Delta f\,\mathrm{d}g \,=\, -\int_M |\nabla f|^2\mathrm{d}g \ \text{ if } M \text{ is compact, and so} \\ \text{b)} & \text{a function } f : M \to \mathbf{R} \text{ is constant if } M \text{ is compact and } \Delta f \geq 0. \end{array}$$

(Namely, as $\int_M \Delta f\,\mathrm{d}g \,=\, 0$ by (1.14), the inequality $\Delta f \,\geq\, 0$ yields $\Delta f \,=\, 0$.) Another consequence of (1.14) is *Bochner's integral formula*

$$\text{(1.20)} \qquad \int_M \mathrm{r}(w, w)\,\mathrm{d}g \;=\; \int_M (\delta w)^2\mathrm{d}g \;-\; \int_M \mathrm{tr}\,(\nabla w)^2\,\mathrm{d}g,$$

valid for all compactly supported vector fields w on a Riemannian manifold (M, g) (and easily derived from (1.11.b)). An important special case of (1.20) arises when $w = \nabla\varphi$ is the gradient of a function:

$$\text{(1.21)} \qquad \int_M \mathrm{r}(\nabla f, \nabla f)\,\mathrm{d}g \;=\; \int_M (\Delta f)^2\mathrm{d}g \;-\; \int_M |\nabla df|^2\,\mathrm{d}g.$$

In the case of oriented manifolds, (1.14) may be restated as the *Stokes formula* (which we need only in Appendix H): on an oriented n-dimensional manifold M,

$$\text{(1.22)} \qquad \int_M d\eta \,=\, 0 \quad \text{for any compactly supported } (n-1)\text{-form } \eta.$$

In fact, as M is oriented, we may treat the volume element $\mathrm{d}g$ of any fixed metric g as a positive differential n-form, and then $d\eta = (\delta w)\,\mathrm{d}g$ for the unique vector field w corresponding to η under the Hodge-star isomorphism $TM \to [T^*M]^{\wedge(n-1)}$

(in the sense that $\eta = \imath_w \, \mathrm{d}g$). The exterior derivative of a 1-form ξ or 2-form ζ acts on vector fields u, v, w by

$$\begin{array}{ll} & \text{a)} \quad (d\xi)(u,v) \,=\, d_u[\xi(v)] \,-\, d_v[\xi(u)] \,-\, \xi([u,v])\,, \\ (1.23) & \text{b)} \quad (d\xi)(u,v) \,=\, [\nabla_u\xi](v) \,-\, [\nabla_v\xi](u)\,, \\ & \text{c)} \quad (d\zeta)(u,v,w) \,=\, [\nabla_u\zeta](v,w) \,+\, [\nabla_v\zeta](w,u) \,+\, [\nabla_w\zeta](u,v)\,. \end{array}$$

Here (a) expresses our convention about $d\xi$, while (b) and (c) easily follow from the Leibniz rule, for any torsionfree connection ∇, such as the Levi-Civita connection of a Riemannian metric.

Remark 1.3. Only one result from global analysis is used in this text. It is the assertion that, if f is a C^∞ function on a compact Riemannian manifold and $\int_M \phi \, \mathrm{d}g = 0$, then $\phi = \Delta f$ for some C^∞ function f.

2. The first Chern class

Given a manifold M and an integer r, let F^rM be the vector space of all differential r-forms on M (that is, C^∞ sections of $[T^*M]^{\wedge r}$). Thus, F^rM is infinite-dimensional if $\dim M = n \geq 1$ and $0 \leq r \leq n$, while, by definition, $F^rM = \{0\}$ if $r < 0$ or $r > \dim M$. The spaces Z^rM and B^rM of *closed* or, respectively, *exact* r-forms are defined to be, respectively, the kernel of the exterior derivative $d : F^rM \to \Omega^{r+1}M$ and the image of $d : \Omega^{r-1}M \to F^rM$. Consequently, $B^rM \subset Z^rM \subset F^rM$, as $dd = 0$. The quotient space $H^r(M, \mathbf{R}) = Z^rM/B^rM$ is known as the rth *de Rham cohomology space* of M. We denote by $[\zeta] \in H^r(M, \mathbf{R})$ the cohomology class of $\zeta \in Z^rM$ (that is, its equivalence class in Z^rM/B^rM).

As an example, the (real) *first Chern class* $c_1(\mathcal{L}) \in H^2(M, \mathbf{R})$ of a complex line bundle $\mathcal{L}$ over a manifold M is given by $2\pi c_1(\mathcal{L}) = [\mathrm{Im}\,\zeta]$, where $\mathrm{Im}\,\zeta$ is the imaginary part of the curvature form ζ of any given connection ∇ in $\mathcal{L}$. More precisely, the curvature tensor of ∇ is defined as in (1.1), except that the vector field w has to be replaced by a section ψ of $\mathcal{L}$. Since the fibre dimension is 1, for vector fields u, v on M and sections ψ of $\mathcal{L}$, the section $R(u,v)\psi$ equals the product of ψ and a function $\zeta(u,v) : M \to \mathbf{C}$, which gives rise to the (complex-valued) curvature form ζ. A fixed section ψ of $\mathcal{L}$ without zeros, defined on an open set $U \subset M$, leads to the complex-valued *connection form* Γ of ∇ (relative to ψ), with $\nabla_v\psi = \Gamma(v)\psi$ for all vector fields v on U. Now, by (1.1) and (1.23.a), $\zeta = -d\Gamma$, and so ζ is closed (although not necessarily exact, as Γ is defined only locally). Thus, $\mathrm{Im}\,\zeta$ is closed as well. Finally, $c_1(\mathcal{L})$ does not depend on the choice of the connection ∇. In fact, for another connection ∇', with the corresponding ζ' and Γ', we clearly have $\zeta' - \zeta = d\Gamma - d\Gamma' = d\xi$, for the complex-valued 1-form ξ on M such that $\nabla' = \nabla - \xi$.

One also defines the first Chern class $c_1(\mathcal{E})$ of a complex vector bundle $\mathcal{E}$ of any fibre dimension $m \geq 1$ over a manifold M by setting $c_1(\mathcal{E}) = c_1(\mathcal{L})$ for the line bundle $\mathcal{L} = \mathcal{E}^{\wedge m}$, that is, the highest complex exterior power of $\mathcal{E}$.

The exterior multiplication $\wedge$ of differential forms preserves closedness, and descends to a multiplication $\cup$ of cohomology classes, known as the *cup product*; explicitly, $[\zeta] \cup [\eta] = [\zeta \wedge \eta]$. This is clear from the Leibniz rule for $\wedge$ and d.

3. Almost complex manifolds

An *almost complex manifold* is a manifold M carrying a fixed *almost complex structure* (a C^∞ vector-bundle morphism $J : TM \to TM$ with $J^2 = -\mathrm{Id}$). In other words, TM then is the underlying real bundle of a complex vector bundle, in which J is the multiplication by i. This allows us to define the first Chern class $c_1(M) \in H^2(M, \mathbf{R})$ by $c_1(M) = c_1(TM)$. (See Section 2.)

We always use the symbol J for the almost complex structure under consideration, while the almost complex manifold in question is simply denoted by M (rather than, for instance, (M, J)). The *complex dimension* of M is then defined to be $\dim_{\mathbf{C}} M = n/2$, where n stands for the ordinary (real) dimension of M.

The automorphism group $\mathrm{GL}(V) \approx \mathrm{GL}(m, \mathbf{C})$ of any complex vector space V with $1 \leq \dim V = m < \infty$ is connected, since every automorphism of V is represented in some basis by a triangular matrix, and that matrix can be joined to Id by an obvious curve of nonsingular triangular matrices. The underlying real space of V thus becomes naturally oriented, as it has a distinguished connected set of real bases, namely, $e_1, ie_1, \ldots, e_m, ie_m$, where $e_1, \ldots, e_m$ runs through the set of all complex bases of V (and the latter set is connected, being an orbit of the connected group $\mathrm{GL}(V)$). This has the following obvious consequence:

(3.1) Every almost complex manifold is canonically oriented.

Given an almost complex manifold M, we denote by $i\partial\overline{\partial}$ the operator sending every C^∞ function $f : M \to \mathbf{R}$ to the exact 2-form $i\partial\overline{\partial}f$ such that

$$2i\partial\overline{\partial}f \;=\; -d[(df)J]. \tag{3.2}$$

Here $(df)J$ is the 1-form equal, at any point $x \in M$, to the composite in which $J_x : T_xM \to T_xM$ is followed by $df_x : T_xM \to \mathbf{R}$. For our purposes, $i\partial\overline{\partial}$ may be treated as a single symbol, even though the notation reflects an actual factorization.

Remark 3.1. A twice-covariant tensor field a on an almost complex manifold M gives rise to two more such tensor fields, $b = aJ$ (or, $b = Ja$), characterized by $b(u, v) = a(Ju, v)$ (or, respectively, $b(u, v) = -a(u, Jv)$) for any vector fields u, v on M. The tensor field a is said to be *Hermitian* (or, *skew-Hermitian*) if it is symmetric (or, skew-symmetric) at every point and $aJ = Ja$, that is, if $a(Ju, Jv) = a(u, v)$ for all vector fields u, v on M. Clearly, a is Hermitian if and only if aJ is skew-Hermitian, while $(aJ)J = J(Ja) = -a$.

Note that a twice-covariant skew-symmetric tensor field is nothing else than a differential 2-form.

Remark 3.2. By a *Hermitian metric* on a given almost complex manifold M we mean a Riemannian metric g on M which is a Hermitian tensor, that is, $gJ = Jg$. This amounts to g-skew-adjointness of J at every point; equivalently, J is required to act in every tangent space as a linear isometry.

If g is Hermitian, the operation $a \mapsto b = Ja$ (or, $a \mapsto b = aJ$), defined in Remark 3.1 for twice-covariant tensor fields a, coincides with the ordinary composition $B = JA$ (or, $B = AJ$) of bundle morphisms $TM \to TM$, provided that one identifies a, b with A, B as in Remark 1.1. In the case where a is also symmetric (or, skew-symmetric) at every point, its being Hermitian (or, skew-Hermitian) is obviously equivalent to complex-linearity of the corresponding bundle morphism $A : TM \to TM$, which in turn means that A commutes with J.

Let M be an almost complex manifold. If a Riemannian metric g on M is Hermitian, the formula $\Omega = gJ$ clearly defines a skew-symmetric twice-covariant tensor field (that is, a differential 2-form), which is also skew-Hermitian. Moreover,

$$\Omega^{\wedge m} \;=\; m!\,\mathrm{d}g, \qquad \text{where} \;\; m \;=\; \dim_{\mathbf{C}} M. \tag{3.3}$$

(Since M is oriented according to (3.1), the volume element $\mathrm{d}g$ may be treated as a positive differential $2m$-form.) In fact, let $x\in M$ and let a complex basis $e_1,\dots,e_m$ of T_xM be orthonormal relative to the Hermitian inner product with real part g_x. Now $\varOmega_x=\xi_1\wedge\xi_2+\ldots+\xi_{2m-1}\wedge\xi_{2m}$ for the real basis $\xi_1,\dots,\xi_{2m}$ of T_x^*M, dual to the g_x-orthonormal real basis $e_1,Je_1,\dots,e_m,Je_m$ of T_xM, which one easily sees using (1.3) to evaluate both sides on any pair of vectors from the basis $e_1,Je_1,\dots,e_m,Je_m$. Thus, $\varOmega_x^{\wedge m}=m!\,\xi_1\wedge\ldots\wedge\xi_{2m}$, as required.

4. Kähler metrics

By a *Kähler manifold* we mean a Riemannian manifold (M,g) which is simultaneously an almost complex manifold, such that g is Hermitian (Remark 3.2) and $\nabla J=0$, where ∇ is the Levi-Civita connection of g.

The simplest example of a Kähler manifold (M,g) arises when a finite-dimensional complex vector space V with a Hermitian inner product $\langle\,,\rangle$ is given: we then set $M=V$, let J operate in each tangent space T_xM via the ordinary multiplication by i (with the standard identification $T_xV=V$), and choose g to be the constant (translation-invariant) metric $\mathrm{Re}\,\langle\,,\rangle$. Another example is provided by any oriented 2-dimensional Riemannian manifold (M,g), with J that acts in each tangent plane T_xM as the positive rotation by the angle $\pi/2$. Further examples are provided by locally symmetric Kähler manifolds, described below in Section 7.

Speaking of a Kähler manifold (M,g), we usually skip the word 'almost' and call J the (underlying) *complex structure* of (M,g), while g is referred to as *a Kähler metric on the complex manifold* M. See also the end of Section 7.

By the *Ricci form* of a Kähler manifold (M,g) one means the twice-covariant tensor field $\rho=\mathrm{r}J$ (cf. Remark 3.1), where r the Ricci tensor of g. We have

$$\text{(4.1)}\quad \text{a) } \mathrm{tr}_{\mathbf{R}}\,J[R(v,w)]=-2\rho(v,w),\ \text{ b) } \delta[J(\nabla w)^*]=\imath_w\rho,\ \text{ c) } R(Jv,Jw)=R(v,w),$$

for δ as in (1.8) and any vector fields v,w on M. In coordinates, (a) – (c) read $R_{klp}{}^qJ_q^p=-2\rho_{kl}$, $J_q^pw_{k,}{}^q{}_p=\rho_{lk}w^l$ and $J_k^rJ_l^sR_{rsp}{}^q=R_{klp}{}^q$.

In fact, as $\nabla J=0$, the Levi-Civita connection ∇ is a connection in the *complex* vector bundle TM, and so, for any vector fields u,v on M, the vector-bundle morphism $R(u,v):TM\to TM$ in (1.2) is complex-linear (commutes with J). At every point, the commuting morphisms $R(u,v)$ and J are skew-adjoint, and so their composite is self-adjoint. Hence $R_{qlsp}J_k^p=R_{qlkp}J_s^p$, which, contracted against J_r^k or g^{qs}, gives (4.1.c) or, respectively, $\rho_{kl}=R_{pkl}{}^qJ_q^p$. However, due to the well-known symmetries of R and skew-adjointness of J, the expression $R_{pkl}{}^qJ_q^p$ is skew-symmetric in k,l, so that, from the first Bianchi identity, $0=(R_{kpl}{}^q+R_{kl}{}^q{}_p+R_k{}^q{}_{pl})J_q^p=-2R_{pkl}{}^qJ_q^p-R_{klp}{}^qJ_q^p$, and (4.1.a) follows. Finally, since J is skew-adjoint, $2J_q^pw_{k,}{}^q{}_p=J_q^p(w_{k,}{}^q{}_p-w_{k,p}{}^q)=J_q^pR^q{}_{plk}w^l=J_q^pR_{klp}{}^qw^l$, by the Ricci identity (1.11.a). Now (4.1.a) yields (4.1.b).

For any vector field v on a Kähler manifold (M,g), we have, with δ as in (1.6.i),

$$\text{(4.2)}\qquad \begin{array}{ll}\text{i)} & \mathrm{tr}\,JAJA\,=\,(\mathrm{tr}\,JA)^2\,-\,\mathrm{r}(v,v)\,+\,\delta[JAJv-(\mathrm{tr}\,JA)Jv]\quad\text{and}\\ \text{ii)} & \mathrm{tr}\,JAJA^*\,=\,\delta(JA^*Jv)\,-\,\mathrm{r}(v,v)\,,\qquad\text{where }\ A=\nabla v:TM\to TM\,,\end{array}$$

A^* being the (pointwise) adjoint of A. Namely, in local coordinates, $\mathrm{tr}\,JAJA=J_q^pv^q{}_{,k}J_l^kv^l{}_{,p}=(J_q^pv^q{}_{,k}J_l^kv^l)_{,p}-J_q^pv^q{}_{,kp}J_l^kv^l$. Next, $(J_q^pv^q{}_{,k}J_l^kv^l)_{,p}=\delta(JAJv)$ and, by the Ricci identity (1.11.a), $-J_q^pv^q{}_{,kp}J_l^kv^l=-J_q^pv^q{}_{,pk}J_l^kv^l+J_q^pJ_l^kR_{pks}{}^qv^sv^l$, while $-J_q^pv^q{}_{,pk}J_l^kv^l=-(J_q^pv^q{}_{,p}J_l^kv^l)_{,k}+J_q^pv^q{}_{,p}J_l^kv^l{}_{,k}=-\delta[(\mathrm{tr}\,JA)Jv]+(\mathrm{tr}\,JA)^2$

(as $J_q^p v^q{}_{,p} = \operatorname{tr} JA$). Also, by (4.1.c), $J_q^p J_l^k R_{pks}{}^q v^s v^l$ equals $R_{qls}{}^q v^s v^l$, that is, $-\operatorname{r}(v,v)$. This proves (4.2.i). Finally, $\operatorname{tr} JAJA^* = J_q^p v^q{}_{,k} J_l^k v_{p,}{}^l = (J_q^p v^q J_l^k v_{p,}{}^l)_{,k} - J_q^p v^q J_l^k v_{p,}{}^l{}_k$, while $(J_q^p v^q J_l^k v_{p,}{}^l)_{,k} = \delta(JA^*Jv)$ and, by (4.1.b), $-J_q^p v^q J_l^k v_{p,}{}^l{}_k = -J_q^p v^q \rho_{kp} v^k = \rho(Jv,v) = -\operatorname{r}(v,v)$, which gives (4.2.ii).

Remark 4.1. If (M,g) is a Kähler manifold,

(i) the Ricci tensor r of (M,g) is Hermitian;
(ii) its Ricci form $\rho = \operatorname{r} J$ is a *closed* differential 2-form;
(iii) as g is Hermitian, $\varOmega = gJ$ is a skew-Hermitian 2-form on M, called the *Kähler form* of (M,g). Being parallel, $\varOmega$ is closed as well.

In fact, (i) amounts to skew-symmetry of ρ (obvious from (4.1.a)), while the relation $d\rho = 0$, that is, $\rho_{sk,l} + \rho_{kl,s} + \rho_{ls,k} = 0$ (cf. (1.23.c)), is immediate from the coordinate version of (4.1.a) and the second Bianchi identity (since $\nabla J = 0$).

For any function $f : M \to \mathbf{R}$ on a Kähler manifold (M,g), we have

$$\text{(4.3)} \qquad \text{i)}\ \ 2i\partial\overline{\partial} f = (\nabla df)J + J(\nabla df), \quad \text{ii)}\ \ \operatorname{tr}_g[(i\partial\overline{\partial} f)J] = -\varDelta f$$

(notation of (1.13.i), (3.2) and Remark 3.1 for $a = \nabla df$). Namely, (3.2) and (1.23.b) give (i), which in turn implies (ii). Thus, by (1.19.b),

$$\text{(4.4)} \qquad \text{a function } f : M \to \mathbf{R} \text{ is constant if } M \text{ is compact and } i\partial\overline{\partial} f = 0.$$

Lemma 4.2. *Let an exact differential* 2*-form* ζ *on a compact Kähler manifold* (M,g) *be skew-Hermitian in the sense that* $J\zeta = \zeta J$*, cf.* Remark 3.1.

(a) *There exists a* C^∞ *function* $\theta : M \to \mathbf{R}$ *with* $\zeta = i\partial\overline{\partial}\theta$.
(b) *The function* θ *in* (a) *is unique up to an additive constant.*
(c) *Denoting by* $\|\ \|$ *the* L^2 *norm, both for functions and tensor fields on* M*, we have* $\sqrt{2}\,\|\zeta\| = \|\operatorname{tr}_g \zeta J\|$.
(d) *If* $\operatorname{tr}_g \zeta J = 0$*, then* $\zeta = 0$.

Proof. We first prove (c). Let v be a vector field with $\zeta = d\xi$ for the 1-form $\xi = \imath_v g$, and let $A = \nabla v$, so that, by (1.23.b), $A - A^*$ is the vector-bundle morphism $TM \to TM$ corresponding to ζ as in Remark 1.1. We clearly have $\operatorname{tr}(A - A^*)A^* = -\operatorname{tr}(A - A^*)A$. Thus, $\|\zeta\|^2 = -\int_M \operatorname{tr}(A - A^*)^2 \,\mathrm{d}g = -2\int_M \operatorname{tr}(A - A^*)A \,\mathrm{d}g$. Since ζ is skew-Hermitian, $[J, A - A^*] = 0$, that is, $A - A^* = JA^*J - JAJ$. Thus, $\|\zeta\|^2 = 2\int_M \operatorname{tr} JAJ(A - A^*)\,\mathrm{d}g$, and so (c) follows from (4.2) and (1.14), as $2\operatorname{tr} JA = \operatorname{tr} J(A - A^*) = \operatorname{tr}_g \zeta J$ due to skew-adjointness of J.

Next, (d) is obvious from (c). To prove (a), let us choose $\theta : M \to \mathbf{R}$ with $\varDelta\theta = -\operatorname{tr}_g \zeta J$. (Such θ exists by (1.15), since, as we just saw, $\operatorname{tr}_g \zeta J = 2\operatorname{tr} JA$, so that $\operatorname{tr}_g \zeta J = 2J_q^p v^q{}_{,p} = 2\delta(Jv)$, and $\int_M \operatorname{tr}_g \zeta J \,\mathrm{d}g = 0$.) Applying (d) to $\zeta - i\partial\overline{\partial}\theta$ rather than ζ, and noting that the premise of (d) is then satisfied in view of (4.3.ii), we now see that $\zeta = i\partial\overline{\partial}\theta$. Finally, (b) is immediate from (4.4). $\square$

5. Almost-Kähler manifolds

An *almost-Kähler metric* on an almost complex manifold M is any Hermitian metric g on M (cf. Remark 3.2) for which the skew-Hermitian 2-form $\varOmega = gJ$ is closed. Such pairs (M,g) are referred to as *almost-Kähler manifolds*; obvious examples are provided by Kähler manifolds (cf. Remark 4.1(iii)).

Remark 5.1. If g is just a Hermitian metric, the differential 2-form $\varOmega = gJ$ is skew-Hermitian, but need not, in general, be parallel relative to the Levi-Civita connection ∇, or even closed. The condition $\nabla\varOmega = 0$ is necessary and sufficient for a given Hermitian metric g to be a Kähler metric: it means the same as $\nabla J = 0$, since $\varOmega = gJ$ and $\nabla g = 0$.

One easily finds examples of non-Kähler, almost-Kähler metrics, also on compact manifolds. On the other hand, as we will see below (Theorem 5.3), for an almost complex manifold M on which a Kähler metric exists, all almost-Kähler metrics on M are Kähler metrics. By Lemma 5.2, the same conclusion holds even if one replaces the existence of a Kähler metric with the weaker requirement that J be parallel relative to some torsionfree connection on M. Goldberg's conjecture[1] (stating that a compact almost-Kähler Einstein manifold is necessarily a Kähler manifold) is still open[2].

Lemma 5.2. *Let torsionfree connections $\nabla, \hat\nabla$ and a Hermitian tensor field h on an almost complex manifold M satisfy the conditions $\hat\nabla J = 0$ and $\nabla h = 0$. Then, for the skew-Hermitian 2-form $\zeta = hJ$ and any vector field w on M, we have $2\nabla_w\zeta = \imath_w d\zeta + J(\imath_w d\zeta)J$, in the notation of* Remark 3.1.

Proof. Let v, w always stand for arbitrary vector fields on M. Denoting by B the section of $\mathrm{Hom}([TM]^{\odot 2}, TM)$ with $\hat\nabla = \nabla - B$, we have $\hat\nabla_w = \nabla_w - B_w$, and B sends v, w to a vector field $B_v w = B_w v$, its symmetry being due to the fact that $\hat\nabla, \nabla$ are both torsionfree. As J is $\hat\nabla$-parallel, $\nabla_w J = [B_w, J]$, where $[\,,]$ denotes the commutator of bundle morphisms $TM \to TM$. In coordinates, $B_w, \nabla_w J, , \hat\nabla_w h \nabla_w\zeta$ and $\hat\nabla_w\zeta$ have the components $(B_w)_k^l = w^s B_{sk}^l, (\nabla_w J)_k^l = w^s J_{sk}^l, (\hat\nabla_w h)_{kl} = w^s H_{skl}, (\nabla_w\zeta)_{kl} = w^s Z_{skl}$, and $(\hat\nabla_w\zeta)_{kl} = w^s\hat Z_{skl}$, for some functions $B_{pk}^l, J_{pk}^l, H_{pkl}, Z_{pkl}, \hat Z_{pkl}$ such that

$$
\begin{aligned}
&\text{(a)}\ J_{pk}^l = J_k^s B_{ps}^l - J_s^l B_{pk}^s,\ \text{(b)}\ B_{kl}^r = B_{lk}^r,\ \text{(c)}\ Z_{pkl} = J_{pk}^s h_{sl},\ \text{(d)}\ \hat Z_{pkl} = J_k^s H_{psl},\\
&\text{(e)}\ H_{pkl} = B_{pk}^s h_{sl} + B_{pl}^s h_{ks},\ \text{(f)}\ Z_{lpk} = -Z_{lkp},\ \text{(g)}\ J_k^s\hat Z_{pls} = -J_k^s\hat Z_{psl} = H_{pkl},\\
&\text{(h)}\ (d\zeta)_{pkl} = Z_{pkl} + Z_{klp} + Z_{lpk},\ \text{(i)}\ (d\zeta)_{pkl} = \hat Z_{pkl} + \hat Z_{klp} + \hat Z_{lpk}.
\end{aligned}
$$

In fact, (a) is the coordinate version of $\nabla_w J = [B_w, J]$, (b) expresses symmetry of B, the relation $\zeta = hJ$ along with $\nabla h = 0$ (or, $\hat\nabla J = 0$) yields (c) (or, respectively, (d)), while (e) follows since $\nabla h = 0$ and $\hat\nabla = \nabla + B$, (f) is due to skew-symmetry of ζ and $\nabla_w\zeta$, and (d) implies (g) as $J^2 = -\mathrm{Id}$. Finally, (h) (or, (i)) amounts to (1.23.c) for $\zeta = \zeta$ and the torsionfree connection ∇ (or, $\hat\nabla$).

We need to prove the equality $2\nabla_w\zeta - \imath_w d\zeta = J(\imath_w d\zeta)J$, equivalent, in view of (h) and (i), to $Z_{pkl} - Z_{klp} - Z_{lpk} = -J_k^r J_l^s(\hat Z_{prs} + \hat Z_{rsp} + \hat Z_{spr})$. (Note that $2Z_{pkl} - (Z_{pkl} + Z_{klp} + Z_{lpk}) = Z_{pkl} - Z_{klp} - Z_{lpk}$.) First, (f) and (c) give $Z_{pkl} - Z_{klp} - Z_{lpk} = Z_{lkp} - Z_{klp} + Z_{pkl} = (J_{lk}^s - J_{kl}^s)h_{sp} + J_{pk}^s h_{sl}$. In view of (a), this equals $J_k^r B_{lr}^s h_{sp} - J_l^r B_{kr}^s h_{sp} + J_k^r B_{pr}^s h_{sl} - J_r^s B_{pk}^r h_{sl}$ (two other terms cancel each other by (b)). Using (e) and (b) we can rewrite the last expression as $J_k^r H_{rlp} - J_l^r B_{kr}^s h_{sp} - J_r^s B_{pk}^r h_{sl}$, which equals $J_k^r H_{rlp} - J_l^r B_{kr}^s h_{sp} + J_l^r B_{pk}^s h_{rs}$ (where $J_r^s h_{sl} = -J_l^s h_{sr}$ as $J_r^s h_{sl} = \zeta_{rl}$, and the indices r, s have been switched). Applying (e) and (b) again, we see that this coincides with $J_k^r H_{rlp} + J_l^r(H_{pkr} - H_{rkp})$.

[1]S. I. Goldberg, *Integrability of almost Kaehler manifolds*, Proc. A. M. S. **21** (1969), 96–100

[2]T. Oguro and K. Sekigawa, *Notes on the Goldberg conjecture in dimension four*, Complex, contact and symmetric manifolds, 221–233, Progr. Math., **234**, Birkhäuser, Boston, MA, 2005

On the other hand, by (d) and (g), $-J_k^r J_l^s(\hat{Z}_{prs} + \hat{Z}_{rsp} + \hat{Z}_{spr})$ is equal to $J_k^r H_{rlp} + J_l^r(H_{pkr} - H_{rkp})$ as well, which completes the proof. □

Suppose that (M, g) is an almost-Kähler manifold. As in the Kähler case, we call $\Omega = gJ$ the *Kähler form* of (M, g). Being closed, Ω gives rise to a cohomology class $[\Omega] \in H^2(M, \mathbf{R})$ (see Section 2) known as the *Kähler cohomology class* of (M, g), or, briefly, its *Kähler class.*

Theorem 5.3. *Let $\mathcal{A}$ be the set of all almost-Kähler metrics on a given almost complex manifold M.*

(i) *$\mathcal{A}$ is a convex subset of the vector space of all Hermitian twice-covariant C^∞ tensor fields a on M such that the differential 2-form aJ is closed.*
(ii) *The set of all Kähler metrics on M is either empty, or coincides with $\mathcal{A}$.*

Proof. Assertion (i) is obvious since $\mathcal{A}$ is defined by imposing on a metric g the linear equations $gJ = Jg$ and $d(gJ) = 0$. To prove (ii), let us suppose that M admits a Kähler metric. For an arbitrary almost-Kähler metric g on M, denoting by ∇ and Ω the Levi-Civita connection and Kähler form of g, we have $\nabla\Omega = 0$ by Lemma 5.2, and so g is a Kähler metric (Remark 5.1), as required. □

For an almost-Kähler metric g on a compact almost complex manifold M,

(5.1) its volume $\mathrm{V} = \int_M \mathrm{d}g$ depends only on its Kähler class $[\Omega] \in H^2(M, \mathbf{R})$.

In fact, the oriented integral $\int_M \sigma$ of a differential $2m$-form σ, for $m = \dim_{\mathbf{C}} M$, depends only on the cohomology class $[\sigma]$ (as $\int_M \sigma = 0$ when σ is exact, by Stokes's formula (1.22)). That $\mathrm{V} = \int_M \mathrm{d}g$ depends on g only through $[\Omega]$ is clear from (3.3), since $[\Omega^{\wedge m}] = [\Omega]^{\cup m}$, where $\cup$ is the cup product (Section 2).

Also, $[\Omega] \neq 0$ in $H^2(M, \mathbf{R})$, for the Kähler form Ω of any compact almost-Kähler manifold (M, g). Namely, if we had $\Omega = d\xi$ for some 1-form ξ, it would follow that $\Omega^{\wedge m} = d[\xi \wedge \Omega^{\wedge(m-1)}]$, and so $\mathrm{V} = 0$ by (3.3) and (1.22).

Given a compact almost complex manifold M, one calls an element of $H^2(M, \mathbf{R})$ *positive* (or *negative*) if it equals $[\Omega]$ (or, $-[\Omega]$) for the Kähler form Ω of some almost-Kähler metric on M. A cohomology class in $H^2(M, \mathbf{R})$ cannot be simultaneously positive and zero, or zero and negative, or positive and negative: if it were, a suitable difference would be both positive and zero, giving $[\Omega] = 0$ for the Kähler form Ω of some almost-Kähler metric, contrary to the last paragraph.

6. Comparing Kähler metrics

For any C^1 curve $t \mapsto F = F(t) \in \mathrm{GL}(V)$ of linear automorphisms of a finite-dimensional real/complex vector space V, setting $(\ )^\cdot = d/dt$ we have

$$(\det F)^\cdot = (\det F)\,\mathrm{tr}\,(F^{-1}\dot{F}). \tag{6.1}$$

In fact, shifting the variable, we see that it suffices to establish (6.1) at $t = 0$. When $F(0) = \mathrm{Id}$, (6.1) at $t = 0$ means that tr the differential of the homomorphism det at $\mathrm{Id} \in \mathrm{GL}(V)$, and so (6.1) follows since $1 + (\mathrm{tr}\,A)\,t$ is the first-order part of $\det(\mathrm{Id} + tA)$ treated as a polynomial in t. The general case is reduced to the above by replacing the curve $t \mapsto F(t)$ with $t \mapsto [F(0)]^{-1}F(t)$.

Suppose that g and $\hat{g}$ are Riemannian metrics on a manifold M of any (real) dimension n and $\gamma : M \to (0, \infty)$ is the ratio of their volume elements, in the sense that $\mathrm{d}\hat{g} = \gamma\,\mathrm{d}g$. Then, with tr_g denoting the g-trace, as in Remark 1.1,

$$\text{(6.2)} \qquad \text{a)}\ \ \det{}_g\hat{g} = \gamma^2, \qquad \text{b)}\ \ \mathrm{tr}_g\hat{g} \ge n\gamma^{2/n}.$$

Here $\det_g\hat{g} : M \to \mathbf{R}$ assigns to each $x \in M$ the determinant, at x, of the vector-bundle morphism $A : TM \to TM$ corresponding to $\hat{g}$ (via the fixed metric g) as in Remark 1.1. Namely, (6.2.a) follows since, in local coordinates, $\det A = (\det g)^{-1}\det\hat{g}$, while the component function of $\mathrm{d}g$ is $(\det g)^{1/2}$, and similarly for $\hat{g}$. (By $\det g$ we mean the coordinate-dependent function $\det[g_{jk}]$.) Next, for A as above, $\mathrm{tr}_g\hat{g} = \mathrm{tr}\,A$. As the eigenvalues of A at any given point $x \in M$ are positive, (6.2.b) is obvious from (6.2.a) and the inequality between the arithmetic and geometric means, that is, (**??**) with $k = n$ and $c_1 = \ldots = c_n = 1/n$.

Remark 6.1. Let ρ be the Ricci form of a Kähler manifold (M, g).

- (i) The curvature form ζ (see Section 2) of the connection ∇ which the Levi-Civita connection of g, also denoted by ∇, induces in the complex exterior power $[TM]^{\wedge m}$, for $m = \dim_{\mathbf{C}}M$, is given by $\zeta = i\rho$.
- (ii) In cohomology, $[\rho] = 2\pi c_1(M) \in H^2(M, \mathbf{R})$, cf. Section 2.
- (iii) The Ricci form $\hat{\rho}$ of any other Kähler metric $\hat{g}$ on the same underlying complex manifold M is related to ρ by $\hat{\rho} = \rho - i\partial\overline{\partial}\log\gamma$, where $\mathrm{d}\hat{g} = \gamma\,\mathrm{d}g$, that is, $\gamma : M \to (0, \infty)$ is the ratio of the volume elements.

In fact, let the vector fields e_a, $a = 1, \ldots, m$, trivialize the *complex* vector bundle TM over an open set $U \subset M$, and let Γ_a^b be the corresponding (complex-valued) *connection forms* on U, with $\nabla_v e_a = \Gamma_a^c(v)e_c$. (Here and below repeated indices are summed over, and v, w are arbitrary vector fields on U.) Thus, by (1.1), $R(v,w)e_a = R_a^c(v,w)e_c$, where $R_a^b = -d\Gamma_a^b + \Gamma_a^c \wedge \Gamma_c^b$, with d and $\wedge$ as in (1.23.a) and (1.3). On the other hand, $i\rho(v,w)$ equals the complex trace of the complex-linear bundle morphism $R(v,w) : TM \to TM$ defined as in (1.2). To see this, note that, at each point, $R(v,w)$ is skew-adjoint relative to g, as a real operator, and hence also relative to the Hermitian fibre metric $g^{\mathbf{C}}$ in TM with $\mathrm{Re}\,g^{\mathbf{C}} = g$. Consequently, $i\,\mathrm{tr}_{\mathbf{C}}[R(v,w)]$ is real and coincides with the complex trace of the self-adjoint composite morphism $J[R(v,w)] = [R(v,w)]J$, which equals $1/2$ of its real trace, and so $i\,\mathrm{tr}_{\mathbf{C}}[R(v,w)] = -\rho(v,w)$ by (4.1.a).

In other words, $\rho = i\,d\Gamma_a^a$ on U, as $i\rho = R_a^a = -d\Gamma_a^a$, with $\Gamma_a^c \wedge \Gamma_c^a = 0$ due to obvious pairwise cancellations. Now (i) and (ii) are immediate from the discussion in the second paragraph of Section 2 applied to $\mathcal{L} = [TM]^{\wedge m}$ and $\psi = e_1 \wedge \ldots \wedge e_m$ (with the connection form $\Gamma = \Gamma_a^a$).

The formulae $\mathfrak{G} = [g^{\mathbf{C}}(e_a, e_b)]$ and $\mathfrak{D} = \det_{\mathbf{C}}\mathfrak{G}$ define functions on U valued in $m \times m$ Hermitian matrices and, respectively, in positive real numbers. For any vector field w on U we have $d_w \log\mathfrak{D} = \mathrm{tr}_{\mathbf{C}}(\mathfrak{G}^{-1}d_w\mathfrak{G})$, in view of (6.1) for $F = \mathfrak{G}$ treated as a function of the parameter t of any given integral curve of w. As $\nabla g^{\mathbf{C}} = 0$, the Leibniz rule gives $d_w h_{ab} = \Gamma_a^c(w)h_{cb} + \overline{\Gamma_b^c(w)h_{ca}}$ for the entries $h_{ab} = g^{\mathbf{C}}(e_a, e_b)$ of $\mathfrak{G}$, that is, $d_w\mathfrak{G} = \mathfrak{T}\mathfrak{G} + (\mathfrak{T}\mathfrak{G})^*$, where * stands for the conjugate transpose, and $\mathfrak{T}$ is the matrix-valued function with the entries $\Gamma_a^b(w)$. (In both $\Gamma_a^b(w)$ and h_{ab}, the index a is the row number and b the column number.) This gives $d_w \log\mathfrak{D} = \mathrm{tr}_{\mathbf{C}}(\mathfrak{G}^{-1}d_w\mathfrak{G}) = \mathrm{tr}_{\mathbf{C}}[\mathfrak{G}^{-1}\mathfrak{T}\mathfrak{G} + (\mathfrak{G}^{-1}\mathfrak{T}\mathfrak{G})^*] = 2\,\mathrm{Re}\,\mathrm{tr}_{\mathbf{C}}\mathfrak{T} = 2\,\mathrm{Re}\,\Gamma_a^a(w)$. Hence $d\log\mathfrak{D} = 2\,\mathrm{Re}\,\Gamma$, where $\Gamma = \Gamma_a^a$ denotes, as in the previous paragraph, the connection form in $[TM]^{\wedge m}$ with $\rho = i\,d\Gamma$.

Let $\hat{g}^{\mathbf{C}}, \hat{h}_{ab}, \hat{\mathfrak{D}}, \hat{\nabla}, \hat{\Gamma}_a^b$ and $\hat{\Gamma}$ be the analogous objects for another Kähler metric $\hat{g}$ on M (with the same vector fields e_a on U), and let $H : TM \to TM$ be the complex-linear bundle morphism such that $\hat{g}^{\mathbf{C}}(v,w) = g^{\mathbf{C}}(Hv,w)$. For the matrix H_a^b of functions $U \to \mathbf{C}$ given by $He_a = H_a^c e_c$ we thus have $\hat{h}_{ab} = H_a^c h_{cb}$, and so $\hat{\mathfrak{D}}/\mathfrak{D} = \det_{\mathbf{C}} H$. Moreover, since H is, at each point, a self-adjoint positive operator, $\det_{\mathbf{C}} H$ is real-valued and equals $[\det_{\mathbf{R}} H]^{1/2}$. Finally, taking the real part of the equality $\hat{g}^{\mathbf{C}}(v,w) = g^{\mathbf{C}}(Hv,w)$ we obtain $\hat{g}(v,w) = g(Hv,w)$, and so, by (6.2.a), $[\det_{\mathbf{R}} H]^{1/2} = [\det_g \hat{g}]^{1/2} = \gamma$. Consequently, $\hat{\mathfrak{D}}/\mathfrak{D} = \gamma$.

The equalities $\rho = i\,d\varGamma$, $d\log\mathfrak{D} = 2\,\mathrm{Re}\,\varGamma$ and their analogues for $\hat{g}$ now give $d\log\gamma = 2\,\mathrm{Re}\,(\hat{\varGamma} - \varGamma)$ and $\rho - \hat{\rho} = i\,d(\varGamma - \hat{\varGamma}) = d[i(\varGamma - \hat{\varGamma})]$. However, $\varGamma - \hat{\varGamma}$ is, at every point x, complex-linear as a mapping $T_xM \to \mathbf{C}$. (In fact, so is $\varGamma_a^b - \hat{\varGamma}_a^b$ for each pair of indices a, b, since $\hat{\nabla}_v w - \nabla_v w$ depends on v, w symmetrically and complex-bilinearly: symmetry follows as both connections are torsionfree, while $\mathbf{C}$-linearity in v is immediate from symmetry and $\mathbf{C}$-linearity in w, the latter being due to the relations $\nabla J = \hat{\nabla} J = 0$.) Therefore, $\rho - \hat{\rho} = d[(\varGamma - \hat{\varGamma})J]$. Since ρ and $\hat{\rho}$ are real-valued, this equals $d\,\mathrm{Re}\,[(\varGamma - \hat{\varGamma})J] = -d[(d\log\gamma)J]/2 = i\partial\overline{\partial}\,\log\gamma$ (see (3.2)), which proves (iii).

The next result is due to Calabi[3]. The proof of assertion (a) given here comes from Yau[4], p. 375. See also Bérard Bergery's exposition[5].

Theorem 6.2. *Let $g, \hat{g}$ be two Kähler metrics on a compact complex manifold M, with the Ricci tensors* r *and* $\hat{\mathrm{r}}$*, and the Kähler classes $[\varOmega], [\hat{\varOmega}] \in H^2(M, \mathbf{R})$.*

(a) *If $\mathrm{r} = \hat{\mathrm{r}}$ and $[\varOmega] = [\hat{\varOmega}]$, then $g = \hat{g}$.*
(b) *If $\mathrm{r} = -g$ and $\hat{\mathrm{r}} = -\hat{g}$, then $g = \hat{g}$.*

Proof. Let $\gamma : M \to (0, \infty)$ be the ratio of the volume elements, with $\mathrm{d}\hat{g} = \gamma\,\mathrm{d}g$.

The assumption $\mathrm{r} = \hat{\mathrm{r}}$ made in (a) gives $\rho = \hat{\rho}$ for the Ricci forms. Hence γ is constant in view of Remark 6.1(iii) and (4.4). The other assumption, $[\varOmega] = [\hat{\varOmega}]$, now has two consequences. First, by (5.1), the constant γ must be equal to 1. Secondly, the 2-form $\varOmega - \hat{\varOmega}$ is exact, so that, in view of Lemma 4.2(a), $\varOmega = \hat{\varOmega} - i\partial\overline{\partial}\alpha$ for some C^∞ function $\alpha : M \to \mathbf{R}$. Taking the g-trace of both sides of the corresponding equality $\hat{g} = g - (i\partial\overline{\partial}\alpha)J$ involving the metrics $g = -\varOmega J$ and $\hat{g} = -\hat{\varOmega} J$, we see, using (4.3.ii), (6.2.b) with $\gamma = 1$ and $\mathrm{tr}_g g = n$, for $n = \dim_{\mathbf{R}} M$, that $n = n\gamma^{2/n} \le \mathrm{tr}_g\,\hat{g} = \mathrm{tr}_g[g - (i\partial\overline{\partial}\alpha)J] = n + \varDelta\alpha$. Hence $\varDelta\alpha \ge 0$. Thus, by (1.19.b), α is constant, and so $g = \hat{g}$, which proves (a).

Under the hypotheses of (b), $\rho = -\varOmega$ and $\hat{\rho} = -\hat{\varOmega}$, so that, for $\alpha = \log\gamma$, Remark 6.1(iii) yields $\varOmega = \hat{\varOmega} - i\partial\overline{\partial}\alpha$. As in the preceding paragraph, this gives $\hat{g} = g - (i\partial\overline{\partial}\alpha)J$. By (4.3.i), $-2[(i\partial\overline{\partial}\alpha)J](u,v) = (\nabla d\alpha)(u,v) + (\nabla d\alpha)(Ju,Jv)$ for any point $x \in M$ and any vectors $u, v \in T_xM$. Hence, as $\alpha = \log\gamma$, we have $\hat{g} \le g$ (or, $\hat{g} \ge g$) at points where $\gamma = \gamma_{\max}$ (or, respectively, $\gamma = \gamma_{\min}$). The inequalities between tensors have here the usual meaning: for instance, $\hat{g} \le g$ states that $\hat{g} - g$ is negative semidefinite, or, equivalently, that if $\hat{g}$ is treated,

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[3] E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic geometry and topology: A symposium in honor of S. Lefschetz, pp. 78–89. Princeton University Press, Princeton, NJ, 1957, pp. 86–87

[4] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411

[5] L. Bérard Bergery, *Exposé n° VI*, Première classe de Chern et courbure de Ricci: preuve de la conjecture de Calabi, Séminaire Palaiseau, 1978, Astérisque **58**, Soc. Math. de France, Paris, 1978, 89–102

with the aid of g, as a bundle morphism $A : TM \to TM$ (see Remark 1.1), then its eigenvalues do not exceed 1 at the point in question. Since the eigenvalues of $\hat{g}$ are all positive, we now have, from (6.2.a), $\gamma^2 = \det_g \hat{g} \le 1$ wherever $\gamma = \gamma_{\max}$ and, similarly, $\gamma^2 \ge 1$ wherever $\gamma = \gamma_{\min}$. Consequently, $\gamma_{\max} \le 1 \le \gamma_{\min}$ and so $\gamma = 1$ everywhere in M, that is, $\alpha = 0$ and $g = \hat{g}$. $\square$

7. Holomorphic vector fields

We say that a C^∞ mapping $F : M \to N$ between almost complex manifolds M and N is *holomorphic* if, at every $x \in M$, the differential $dF_x : T_xM \to T_{F(x)}N$ is complex-linear. A diffeomorphism $F : M \to N$ which is holomorphic is referred to as a *biholomorphism*, and, if such F exists, M and N are called *biholomorphic*. By a (real) *holomorphic* vector field on an almost complex manifold M we mean any C^∞ vector field w on M for which $\pounds_w J = 0$, that is, the flow of w consists of (local) biholomorphisms. For more on terminology, see the end of this section.

Remark 7.1. Let w be a vector field on a Kähler manifold (M, g). We treat the covariant derivative ∇w of M, as well as the complex structure J, as bundle morphisms $TM \to TM$, while $[\,,\,]$ denotes the commutator of such morphisms.

(a) For $u = Jw$, we have $\nabla u = J\nabla w$.
(b) The Lie derivative $\pounds_w J$ equals $[J, \nabla w]$. Thus, w is holomorphic if and only if $[J, \nabla w] = 0$.
(c) If w is holomorphic, so is Jw.
(d) The following three conditions are equivalent:
  i) w is holomorphic and is, locally, the gradient of a function;
  ii) Jw is a holomorphic Killing field;
  iii) the tensor field $\nabla\xi$, where $\xi = \imath_w g$, is symmetric and Hermitian.

In fact, as $\nabla J = 0$, we get (a) and $\pounds_w J = [J, \nabla w]$, which yields (b). (The relation $\pounds_w u = [w, u] = \nabla_w u - \nabla_u w$, for any vector field u, gives $(\pounds_w J)u = \pounds_w(Ju) - J(\pounds_w u) = [J, \nabla w]u$.) Now (c) is obvious from (a) and (b). Next, in (d), let $u = Jw$. Condition (i) states that $[J, \nabla w] = 0$ (cf. (b)) and $(\nabla w)^* = \nabla w$, and so $[J, \nabla u] = 0$ and $(\nabla u)^* = -\nabla u$ (as $\nabla u = J\nabla w$ by (a)); hence (ii) follows. Assuming (ii) we similarly get $[J, \nabla u] = 0$ and $(\nabla u)^* = -\nabla u$, while $\nabla w = -J\nabla u$, which yields $[J, \nabla w] = 0$ and $(\nabla w)^* = \nabla w$, that is, (i). Finally, as (i) amounts to $[J, \nabla w] = 0$ and $(\nabla w)^* = \nabla w$, it is equivalent to (iii) (cf. Remark 3.2), since $a = \nabla\xi$ in (iii) corresponds to $A = \nabla w$ as in Remark 1.1.

Remark 7.2. The real vector space $\mathfrak{h}(M)$ of all holomorphic vector fields on a Kähler manifold (M, g) is a *complex* Lie algebra: in addition to being closed under the Lie bracket, it has the structure of a complex space, with $v \mapsto Jv$ serving as the multiplication by i (cf. Remark 7.1(c)).

By a *locally symmetric Kähler manifold* we mean any Riemannian manifold (M, g) which is simultaneously an almost complex manifold, such that the metric g is Hermitian (Remark 3.2) and, for every $x \in M$, there exists a holomorphic g-isometry $\varPhi$ between some neighborhoods of x in M sending x to x, and having the differential at x equal to $-\mathrm{Id} : T_xM \to T_xM$. The terminology makes sense as such (M, g) is automatically a Kähler manifold (and, in addition, its curvature tensor is parallel). In fact, any k-times covariant tensor field T on M, for odd k, which is invariant under $\varPhi_x$ for every x, must vanish identically (since the differential of $\varPhi_x$ at x sends T_x to T_x and, at the same time, to $-T_x$). Applying this to $T = \nabla\Omega$ and $T = \nabla R$, for $\Omega = gJ$ and the four-times covariant curvature tensor R, we get $\nabla\Omega = 0$ and $\nabla R = 0$, as required.

In any complex dimension m, one prominent example of a locally symmetric Kähler manifold is the standard $\mathbf{C}^m$. Another is the complex projective space $\mathbf{C}\mathrm{P}^m$, formed by all complex lines through 0 in $\mathbf{C}^{m+1}$, and hence equal to the quotient S^{2m+1}/S^1 of the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$ under the action, by multiplication, of the unit circle $S^1 \subset \mathbf{C}$. Since S^1 acts on the ambient space $\mathbf{C}^{m+1}$ by holomorphic isometries, a Riemannian metric and an almost complex structure on $\mathbf{C}\mathrm{P}^m$ can be uniquely defined by projecting them, via the isomorphism $d\pi_y$, from the orthogonal complement of $\mathrm{Ker}\, d\pi_y$ in $T_y S^{2m+1}$ onto $T_x\mathbf{C}\mathrm{P}^m$, where $\pi : S^{2m+1} \to \mathbf{C}\mathrm{P}^m$ is the quotient projection, while $y \in S^{2m+1}$ and $x = \pi(y)$. The holomorphic isometry Φ_x required in the last paragraph is provided by the unitary reflection about the line $\mathbf{C}y$ in $\mathbf{C}^{m+1}$, which obviously descends to $\mathbf{C}\mathrm{P}^m$.

The *Fubini-Study* metric g on $\mathbf{C}\mathrm{P}^m$, described above, is also an Einstein metric. In fact, the unitary automorphisms of $\mathbf{C}^{m+1}$ keeping a given unit vector y fixed descend to isometries $\mathbf{C}\mathrm{P}^m \to \mathbf{C}\mathrm{P}^m$ which fix the point $x = \pi(y)$. The differentials of these isometries at x form a group acting on $T_x\mathbf{C}\mathrm{P}^m$ in a manner equivalent to how $\mathrm{U}(m)$ acts on $\mathbf{C}^m$ (as one sees identifying $y^\perp \approx \mathbf{C}^m$ with $T_x\mathbf{C}\mathrm{P}^m$ via the isomorphism $d\pi_y$). The Ricci tensor of g at x now must be a multiple of g_x, or else its eigenspaces would correspond to nontrivial proper $\mathrm{U}(m)$-invariant real subspaces of $\mathbf{C}^m$ (which do not exist, since $\mathrm{U}(m)$ acts transitively on the unit sphere $S^{2m-1} \subset \mathbf{C}^m$).

Here is the reason why we are speaking of Kähler metrics on *complex* manifolds (without the word 'almost'). One normally defines a *complex manifold* to be any almost complex manifold M whose almost complex structure J is *integrable* in the sense that every point of M has a connected neighborhood biholomorphic to an open set in $\mathbf{C}^m$, $m = \dim_{\mathbf{C}} M$. In other words, M is required to be covered by a collection of $\mathbf{C}^m$-valued charts, the transition mappings between which are all holomorphic. The term 'holomorphic' that we used for F or w at the beginning of this section is usually reserved for objects on complex manifolds; in the general almost-complex case, such F and w are called *pseudoholomorphic*. However, in a Kähler manifold, J is always integrable (which is a well-known fact, not used here). Our terminology thus agrees, in the end, with the standard usage.

8. The Futaki and Tian-Zhu invariants

By a *compact complex manifold with* $c_1(M) > 0$, or $c_1(M) < 0$, we mean any compact almost complex manifold M that admits a Kähler metric with the Kähler cohomology class $c_1(M)$ or, respectively, $-c_1(M)$. (Cf. the text preceding Theorem 5.3.) This is equivalent to the requirement that M be a compact almost complex manifold admitting a Kähler metric and, at the same time, having a positive (or, respectively, negative) first Chern class in the sense defined at the end of Section 5. Namely, an almost-Kähler metric with the Kähler form Ω such that $c_1(M) = \pm[\Omega]$ must then be a Kähler metric by Theorem 5.3(ii).

The *Futaki invariant*[6] of a compact Kähler manifold (M, g) is the real-linear functional $\boldsymbol{F} : \mathfrak{h}(M) \to \mathbf{R}$ on the Lie algebra $\mathfrak{h}(M)$ (see Remark 7.2), defined as follows. With Ω and ρ denoting, as usual, the Kähler and Ricci forms, and with

[6] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), 437–443

$\mathrm{s_{avg}}$ standing for the average value of the scalar curvature s, let $f: M \to \mathbf{R}$ be a function such that $\Delta f + \mathrm{s} = \mathrm{s_{avg}}$. We set

$$\boldsymbol{F}v = \mu \int_M d_v f \, \mathrm{d}g \quad \text{for } v \in \mathfrak{h}(M), \quad \text{where } \mu = (\mathrm{s_{avg}})^m \text{ and } m = \dim_{\mathbf{C}} M. \tag{8.1}$$

The Futaki invariant $\boldsymbol{F}$ is particularly interesting for compact complex manifolds M with $c_1(M) > 0$, since M then admits a Kähler metric g with $[\rho] = \lambda[\Omega]$ for some $\lambda \in (0, \infty)$ (e.g., $\lambda = 1$), and $\boldsymbol{F}$ turns out to be the same for all such metrics g. In other words, $\boldsymbol{F}$ then is an invariant of the complex structure of M. As such, it constitutes a well-known obstruction[7] to the existence of Kähler-Einstein metrics on compact complex manifolds M with $c_1(M) > 0$. All of this is summarized by the following result of Futaki[8].

Theorem 8.1. *Given a compact complex manifold (M, g) with $c_1(M) > 0$, the Futaki invariant $\boldsymbol{F} : \mathfrak{h}(M) \to \mathbf{C}$, defined with the aid of a Kähler metric g such that $[\rho] = \lambda[\Omega]$ for a constant λ, does not depends on the choice of such g. Furthermore, $\boldsymbol{F} = 0$ if M admits a Kähler-Einstein metric.*

The final clause of Theorem 8.1 is immediate from its first part: using a Kähler-Einstein metric g to evaluate $\boldsymbol{F}$, we get $\boldsymbol{F} = 0$, since f in (8.1) is constant.

Theorem 8.1 can be derived from the following result, due to Tian and Zhu [**?**, p. 305],

Theorem 8.2. *For any compact complex manifold M with $c_1(M) > 0$, the Tian-Zhu invariant $\mathcal{F} : \mathfrak{h}(M) \to \mathbf{C}$, defined with the aid of a Kähler metric g satisfying the condition $[\rho] = \lambda[\Omega] \in H^2(M, \mathbf{R})$ for some $\lambda \in \mathbf{R}$, depends only on the complex structure of M, and not on the choice of such a metric g.*

However, we will establish the two theorems separately, since a direct proof of Theorem 8.1 is much shorter than one needed for Theorem 8.2.

We begin with two lemmas, in which Ω, ρ and $\mathrm{s_{avg}}$ denote, as before, the Kähler form, Ricci form, and the average value of the scalar curvature s.

Lemma 8.3. *If (M, g) is a compact Kähler manifold, $\lambda \in \mathbf{R}$, and $[\rho] = \lambda[\Omega]$ in $H^2(M, \mathbf{R})$, then $\lambda = \mathrm{s_{avg}}/n$, where $n = \dim_{\mathbf{R}} M$, and*

$$i\partial\overline{\partial} f \; + \; \rho \; = \; \lambda \Omega \qquad \text{for } f : M \to \mathbf{R} \text{ such that } \Delta f + \mathrm{s} = \mathrm{s_{avg}}. \tag{8.2}$$

Proof. We have $i\partial\overline{\partial} f + \rho = \lambda\Omega$ for *some* function f, as $[\rho] = \lambda[\Omega]$ (see Lemma 4.2(a)). Now (4.3.ii) gives $\Delta f + \mathrm{s} = n\lambda$, and so $\lambda = \mathrm{s_{avg}}/n$. ☐

On a Kähler manifold (M, g) with a smooth function f such that $i\partial\overline{\partial} f + \rho = \lambda\Omega$, where λ is a constant, setting $A = \pounds_v J$ (that is, $A = [J, \nabla v]$) and $Lv = \delta v - d_v f$, we obtain, for any smooth vector field v,

$$v^k{}_{,kl} - (v^k f_{,k})_{,l} + J_l^q J_k^p v^k{}_{,pq} - J_l^q J_k^p (v^k f_{,p})_{,q} = -2\lambda v_l + J_l^q (A^p_{q,p} - f_{,p} A^p_q). \tag{8.3}$$

In fact, $-v^k f_{,kl} - J_l^q J_k^p v^k f_{,pq}$ is the lth component of $-\imath_v \nabla df - (\imath_u \nabla df) J = 2\imath_u (i\partial\overline{\partial} f)$ for $u = Jv$ (by (4.3.i)), which equals $2\imath_u(\lambda\Omega - \rho) = -2\lambda\imath_v g + 2\imath_v \mathrm{r}$. Next, $-J_l^q J_k^p v^k{}_{,q} f_{,p} = -J_l^q J_q^k v^p{}_{,k} f_{,p} - J_l^q A^p_q f_{,p} = v^p{}_{,l} f_{,p} - J_l^q A^p_q f_{,p}$, as $[J, \nabla v] = A$, while the Ricci identity (1.11.a) implies that $J_l^q J_k^p v^k{}_{,pq} = J_l^q J_k^p v^k{}_{,qp} + J_l^q J_k^p R_{pqs}{}^k v^s$.

[7] A. Futaki, *Kähler-Einstein metrics and integral invariants*, Lecture Notes in Math. **1314**, Springer, Berlin, 1988

[8] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), 437–443

The relation $[J,\nabla v] = A$ also gives $J_l^q J_k^p v^k{}_{,qp} = J_l^q J_q^k v^p{}_{,kp} + J_l^q A^p_{q,p} = -v^p{}_{,lp} + J_l^q A^p_{q,p}$. Moreover, by (4.1.c), $J_l^q J_k^p R_{pqs}{}^k v^s = R_{kls}{}^k v^s = -R_{ls} v^s$. Combining these equalities and using (1.11.b), we get (8.3).

In the next lemma, only parts (a) and (b) are needed for a proof of Theorem 8.1. The symbol L denotes the operator given by (**??**), with f as in (8.2), while P in (e) is defined as in (**??**).

Lemma 8.4. *Let (M,g) be a compact Kähler manifold such that $[\rho] = \lambda[\Omega]$ in $H^2(M,\mathbf{R})$ for some $\lambda \in \mathbf{R}$. Then, for any holomorphic vector field v on M,*

(a) $\nabla Lv - J\nabla LJv = -2\lambda v$, (b) $\Delta Lv = -2\lambda\delta v$,
(c) $|\nabla Lv|^2 + 2\lambda d_v Lv = |\nabla LJv|^2 + 2\lambda d_{Jv} LJv$,
(d) $g(\nabla Lv, \nabla LJv) + \lambda(d_{Jv} Lv + d_v LJv) = 0$.
(e) $\lambda\psi - d_w\psi + \Delta\psi/2 = i\,d_{Jw}\psi$, *where* $\psi = Pv$ *and* $w = \nabla f/2$.

Proof. Assertion (a) is obvious from (8.3) with $A = 0$. Now (b) follows if we apply the divergence operator δ to (a), where $\delta(J\nabla\phi) = 0$ for any function ϕ, as $\delta(J\nabla\phi) = J_l^k \phi,{}^l{}_k$, while J is skew-adjoint and $\nabla d\phi$ is symmetric. Next, $|\nabla Lv|^2 + 2\lambda d_v Lv = g(\nabla Lv, \nabla Lv + 2\lambda v) = g(\nabla Lv, J\nabla LJv)$ by (a). The same equality for Jv rather than v, cf. Remark 7.1(c), reads $|\nabla LJv|^2 + 2\lambda d_{Jv} LJv = -g(\nabla LJv, J\nabla Lv)$, and, as J is skew-adjoint, the two equalities together prove (c). The left-hand side in (d) is $1/2$ times $g(\nabla LJv, \nabla Lv + 2\lambda v) + g(\nabla Lv, \nabla LJv + 2\lambda Jv) = g(\nabla LJv, J\nabla LJv) - g(\nabla Lv, J\nabla Lv)$ (by (a)); now (d) follows due to skew-adjointness of J. Finally, (b) applied to both v and Jv (see Remark 7.1(c)) gives $\Delta\psi = -2\lambda[\delta v - i\delta(Jv)]$ for $\psi = Pv$, since $Pv = Lv - iLJv$. Now, by (**??**), $\Delta\psi = -2\lambda(Lv + d_v f) + 2i\lambda(LJv + d_{Jv} f) = -2\lambda(\psi + d_v f - i d_{Jv} f)$, and so (a) implies (e). □

Let us now suppose that (M,g) is a compact Riemannian manifold, $f : M \to \mathbf{R}$, and u, v are vector fields on M. With $Lw = \delta w - d_w f$, we get, for any vector field w and any $\phi : M \to \mathbf{R}$,

$$\text{(8.4)}\qquad \begin{array}{l} \text{a) } \delta\nabla(e^{-f}u) = \delta(e^{-f}\nabla u) - e^{-f}[\nabla_u df + (Lu)df], \;\; \text{b) } e^{-f}Lu = \delta(e^{-f}u), \\ \text{c) } -d\delta w = \imath_w \mathrm{r} - \delta\nabla w, \;\; \text{d) } \int_M \phi\,\delta v\,\mathrm{d}g = -\int_M \imath_v d\phi\,\mathrm{d}g. \end{array}$$

In fact, (b) – (d) are trivial special cases of (1.7.a), (1.10.b) and (1.16), while (a) follows since $\nabla(e^{-f}u) = e^{-f}\nabla u - e^{-f} df \otimes u$, and $\delta(e^{-f} df \otimes u) = e^{-f}\nabla_u df + e^{-f}(Lu)df$ due to the definition of L. Let us denote by $(\ ,\ )_f$ the weighted L^2 inner product with $(\phi,\phi)_f = \int_M \phi^2 e^{-f}\,\mathrm{d}g$, by $\|\ \|_f$ the corresponding norm, both for functions and vector fields, and by $(\ ,\ )$ the ordinary L^2 inner product. Using, respectively, (8.4.b), (8.4.d) (for $\phi = \delta(e^{-f}u)$), and (8.4.c) (for $w = e^{-f}u$), we see that $(Lu, \delta v)_f = (\delta(e^{-f}u), \delta v) = -\int_M \imath_v d\delta(e^{-f}u)\,\mathrm{d}g = (\mathrm{Ric}\,u, v)_f - \int_M \imath_v \delta\nabla(e^{-f}u)\,\mathrm{d}g$, where Ric is the bundle morphism $A : TM \to TM$ corresponding as in Remark 1.1 to the Ricci tensor $a = \mathrm{r}$ of (M,g). Thus, by (8.4.a), $(Lu, \delta v)_f =$

Given a compact Riemannian manifold (M,g) and a function $f : M \to \mathbf{R}$, let us denote by Ric_f the bundle morphism $A : TM \to TM$ corresponding as in Remark 1.1 to $a = \nabla df + \mathrm{r}$, where r is the Ricci tensor of (M,g), by δ_f the operator sending a vector field w to the function $\delta_f w = e^f\delta(e^{-f}w)$ (so that, when f is the zero function, δ_f becomes the ordinary divergence δ, cf. (1.6.i)), and by $(\ ,\ )_f$ the weighted L^2 inner product of tensor fields on M with $(A,B)_f = \int_M \langle A,B\rangle e^{-f}\,\mathrm{d}g$. The symbol $\langle\,,\rangle$ in the integrand represents the inner product induced by g, including the ordinary product (when A,B are functions) and g (when they are vector fields). The divergence theorem (1.14) now implies that

$$\text{(8.5)}\qquad \text{a) } (\delta_f w, 1)_f \;=\; 0,\;\; \text{b) } (\nabla\chi, w)_f = -(\chi, \delta_f w)_f \;\text{ whenever } \chi : M \to \mathbf{R}.$$

Also, for any vector fields u, v on M, and any $f : M \to \mathbf{R}$,

$$\text{(8.6)}\qquad (\mathrm{Ric}_f u, v)_f \;=\; (\delta_f u, \delta_f v)_f \;-\; (\nabla u, (\nabla v)^*)_f\,,$$

$(\nabla v)^*$ being the (pointwise) adjoint of $\nabla v : TM \to TM$. If $f = 0$, (8.6) is nothing else than Bochner's integral formula (1.20).

To verify (8.6), note that $\delta_f[\nabla_v u - (\delta_f u)v] = \mathrm{tr}\,(\nabla u)(\nabla v) + (\mathrm{r} + \nabla df)(u,v) - (\delta_f u)\delta_f v$ (as one easily sees in local coordinates, using (1.11.b) and the Leibniz rule); then apply (8.5.a).

It is obvious from (8.6) and (8.5.b), for $\chi = \delta_f u$ and $w = v$, that

$$\text{(8.7)}\qquad -(\nabla\delta_f u, v)_f \;=\; (\mathrm{Ric}_f u, v)_f \;+\; (\nabla u, (\nabla v)^*)_f\,.$$

Lemma 8.5. *Suppose that u, v are vector fields on a Kähler manifold (M, g) and $f : M \to \mathbf{R}$. Then, for $A = \nabla u$ and $B = \nabla v$,*

i) $\pounds_{\nabla f} J = [J, \nabla\nabla f]$,
ii) the bundle morphism $(\pounds_{\nabla f} J)J : TM \to TM$ is self-adjoint at every point,
iii) $\langle Ju, \nabla_{Jv}\nabla f\rangle = (\nabla df)(u,v) + \langle u, (\pounds_{\nabla f} J)Jv\rangle$,
iv) $\delta_f[(d_{Ju}f)Jv) = (\nabla df)(u,v) + (d_{Ju}f)(\mathrm{tr}\,JB - d_{Jv}f)$
$\qquad + \langle\nabla f, JAJv\rangle + \langle u, (\pounds_{\nabla f} J)Jv\rangle$.

Proof. Assertion (i) is obvious from Remark 7.1(b). By (i), $\pounds_{\nabla f} J$ anticommutes with J. As $J^* = -J$ and $(\nabla\nabla f)^* = \nabla\nabla f$, (i) also implies that $\pounds_{\nabla f} J$ is self-adjoint at every point, and (ii) follows.

Next, $(\nabla df)(u,v) = \langle u, (\nabla\nabla f)v\rangle = \langle Ju, J(\nabla\nabla f)v\rangle$, which is nothing else than $\langle Ju, (\nabla\nabla f)Jv\rangle - \langle Ju, [J, \nabla\nabla f]v\rangle$, so that (i) yields (iii).

Finally, in local coordinates, $\delta_f[(d_{Ju}f)Jv] = e^f[e^{-f}(Ju)^l f_{,l}(Jv)^k]_{,k}$ equals

$$-f_{,k}(Ju)^l f_{,l}(Jv)^k + J_s^l u^s{}_{,k} f_{,l}(Jv)^k + (Ju)_s f_{,}{}^s{}_k (Jv)^k + (Ju)^l f_{,l} J_s^k v^s{}_{,k}.$$

These four terms are, respectively, $-(d_{Ju}f)d_{Jv}f$, $\langle\nabla f, JAJv\rangle$, $\langle Ju, \nabla_{Jv}\nabla f\rangle$ and $(d_{Ju}f)\,\mathrm{tr}\,JB$. Now (iv) is immediate from (iii). □

The expression $(\mathrm{Ric}_f u, v)_f$ also appears in another integral identity, requiring additional hypotheses. Specifically, we have the following lemma.

Lemma 8.6. *Let $f : M \to \mathbf{R}$ be a function on a compact Kähler manifold (M, g). If vector fields u, v on M are local gradients, that is, the 1-forms $\imath_u g, \imath_v g$ are closed, then, with J denoting the complex-structure tensor of (M, g),*

$$\text{(8.8)}\quad (\pounds_u J, \pounds_v J)_f/2 = (\nabla u, \nabla v)_f - (\mathrm{Ric}_f u, v)_f + (d_{Ju}f, d_{Jv}f)_f - ((\pounds_{\nabla f} J)u, Jv)_f\,,$$

where $\pounds$ stands for the Lie derivative. Furthermore,

$$\text{(8.9)}\quad -(\nabla\delta_f u, v)_f = 2(\mathrm{Ric}_f u, v)_f - (d_{Ju}f, d_{Jv}f)_f + ((\pounds_{\nabla f} J)u, Jv)_f + (\pounds_u J, \pounds_v J)_f/2.$$

In the remainder of Section 8, all tensor fields, such as a Riemannian metric g, and operators (including connections), depend C^∞-differentiably on a *time parameter* t varying in a fixed interval, in the sense that their components in a local coordinate system are C^∞ functions of the coordinates and t. Their dependence on t will, however, be suppressed in our notation. The same will apply to the volume element $\mathrm{d}g$, divergence operator δ, and the g-inner product $\langle\,,\rangle$ of twice-covariant symmetric tensors. Rather than speaking of curves of metrics, connections, etc., we will refer to such objects as *time-dependent* (and call them *time-independent* when appropriate). Writing $(\;)^\cdot$ for d/dt, we have

$$\text{(8.10)}\quad \text{a) } \dot\delta = d\varphi \text{ and b) } \langle g,\dot g\rangle = 2\varphi \text{ for } \varphi : M\to\mathbf{R} \text{ such that: c) } (\mathrm{d}g)^\cdot = \varphi\,\mathrm{d}g,$$

(a) meaning that $(\delta w)^\cdot = d_w\varphi$ for any time-independent vector field w on M. In fact, contracting the Christoffel symbol formula $2\Gamma^l_{jk} = g^{ls}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})$ we get $2\Gamma^j_{jk} = g^{jl}\partial_k g_{jl}$, that is, by (6.1), $2\Gamma^j_{jk} = \partial_k \log\det[g_{jl}]$. Also, $\mathrm{d}g$ has the component function $(\det[g_{jl}])^{1/2}$, and hence (6.1) gives $\langle g,\dot g\rangle = g^{jl}\dot g_{jl} = 2\varphi$. Finally, applying d/dt to $\delta w = \partial_j w^j + \Gamma^j_{jk}w^k = \partial_j w^j + w^k\partial_k\log\det[g_{jl}]$ and switching d/dt with ∂_k, we obtain (8.10.a).

Lemma 8.7. *Suppose that* $\dot\Omega = 2i\partial\overline{\partial}\chi$ *for some time-dependent function* χ *and the Kähler form* Ω *of a time-dependent Kähler metric* g *on a given complex manifold* M *with a time-independent complex structure* $J : TM \to TM$*, where* $(\;)^\cdot = d/dt$*. Then, for* ρ, L *as above and* φ *given by* (8.10.c)*,*

(i) $\varphi = \Delta\chi$, (ii) $\dot\rho = -i\partial\overline{\partial}\Delta\chi$, (iii) $\dot L = -2\lambda d\chi$,

(iv) f *with* (8.2) *may be chosen so that* $\dot f = \Delta\chi + 2\lambda\chi$*,*

(iii) *meaning that* $(Lw)^\cdot = -2\lambda d_w\chi$ *for all time-independent vector fields* w*.*

Proof. As $\dot gJ = \dot\Omega = 2i\partial\overline{\partial}\chi$, we have $\dot g = -2(i\partial\overline{\partial}\chi)J$. Hence, by (4.3.ii), $\langle g,\dot g\rangle = 2\Delta\chi$, and (8.10.b) yields (i). By (i), Remark 6.1(iii) and (8.10.c), $\dot\rho = -i\partial\overline{\partial}\varphi = -i\partial\overline{\partial}\Delta\chi$, and (ii) follows. Next, choosing $f : M\to\mathbf{R}$ so that $\Delta f + \mathrm{s} = \mathrm{s}_{\mathrm{avg}}$ for *some* t and $\dot f = \Delta\chi + 2\lambda\chi$ for all t, and then applying d/dt to $i\partial\overline{\partial}f + \rho - \lambda\Omega$, we see that, by (ii), $i\partial\overline{\partial}f + \rho = \lambda\Omega$ for all t, which proves (iv). Using (**??**) and (8.10.a) with $\varphi = \Delta\chi$ we now obtain (iii). □

We now proceed to prove Theorems 8.1 and 8.2. Rescaling two given Kähler metrics with the stated property, we may assume that they have the same value of λ, which will also be the case for all intermediate metrics in a line segment of Kähler metrics joining them (Theorem 5.3). We thus have a C^∞ curve $t\mapsto g = g(t)$ of Kähler metrics on the complex manifold M, with Kähler forms Ω such that $\dot\Omega = 2i\partial\overline{\partial}\chi$ for some function $\chi : M\to\mathbf{R}$. (We use the shorthand conventions of the last paragraph.) We will from now on ignore the fact that the curve is a line segment, although we do make use of its consequence in the form of differentiability of the assignment $t\mapsto\chi$ (which is in fact constant).

Proof of Theorem 8.1. Applying d/dt to $-\mu^{-1}\boldsymbol{F}v = \int_M Lv\,\mathrm{d}g$ (cf. (8.1) and (1.14)), we obtain the integral of $(\Delta\chi)Lv - 2\lambda d_v\chi$. Integration by parts shows that this equals the L^2 inner product of χ and the function $\Delta Lv + 2\lambda\delta v$, which vanishes by Lemma 8.4(b). □

Proof of Theorem 8.2. The relation $\dot L = -2\lambda d\chi$ gives $(Pw)^\cdot = 2i\lambda d_{Jw}\chi - 2\lambda d_w\chi$, and, since $(\mathrm{d}g)^\cdot = (\Delta\chi)\,\mathrm{d}g$, we get $\mu^{-1}\dot{\mathcal{F}}(w) = \int_M(2i\lambda d_{Jw}\chi - 2\lambda d_w\chi + \Delta\chi)e^{Pw}\,\mathrm{d}g$

from (**??**). Integrating by parts we see that this is equal to the integral of χ times

$$\Delta e^{Pw} + 2\lambda(d_w e^{Pw} - i d_{Jw} e^{Pw}) + 2\lambda[\delta w - i\delta(Jw)]e^{Pw}. \tag{8.11}$$

To prove that (8.11) vanishes for every holomorphic vector field w, we use the identity $\Delta e^{\psi} = e^{\psi}[\Delta\psi + g(\nabla\psi,\nabla\psi)]$, immediate when the function ψ is real-valued, but also easily verified to complex-valued functions ψ, with g extended complex-bilinearly to complex vector fields (sections of the complexified tangent bundle). Thus, $g(\nabla\psi,\nabla\psi) = |\nabla\,\mathrm{Re}\,\psi|^2 - |\nabla\,\mathrm{Im}\,\psi|^2 + 2ig(\nabla\,\mathrm{Re}\,\psi, \nabla\,\mathrm{Im}\,\psi)$. For $\psi = Pw$, we have $\mathrm{Re}\,\psi = Lw$, $\mathrm{Im}\,\psi = -LJw$, and (8.11) equals e^{Pw} times

$$\begin{aligned}&\Delta Lw + 2\lambda\delta w - i\,[\Delta LJw + 2\lambda\delta(Jw)]\\ &+ |\nabla Lw|^2 + 2\lambda d_w Lw - [|\nabla LJw|^2 + 2\lambda d_{Jw} LJw]\\ &- 2i\,[g(\nabla Lw, \nabla LJw) + \lambda(d_{Jw}Lw + d_w LJw)].\end{aligned}$$

Each of the three lines is separately equal to zero, due to a part of Lemma 8.4: the first, by (b); the second, by (c); and the third, in view of (d). □

9. Kähler-Einstein metrics

On an arbitrary Riemannian manifold (M,g), we denote by D the operator sending any vector field w on M to the vector field Dw characterized by

$$\imath_{Dw}g = -\Delta\imath_w g - \imath_w \mathrm{r}, \quad \text{that is,} \quad (Dw)_j = -w_{j,k}{}^k - R_{jk}w^k. \tag{9.1}$$

Replacing $R_{jk}w^k$ by $w^k{}_{,jk} - w^k{}_{,kj}$ (cf. (1.11.b)), we get $(Dw)_j = -(w_{j,k} + w_{k,j})^k + w^k{}_{,kj}$. Rewritten with the aid of (1.5.a), this equality gives

$$\imath_{Dw}g = -\delta\pounds_w g + d\delta w, \tag{9.2}$$

while, applied to $w = \nabla\psi$ for a function $\psi : M \to \mathbf{R}$, it yields, again by (1.11.b), $(Dw)_j = -2\psi_{,kj}{}^k + \psi_{,k}{}^k{}_j = -2R_{jk}w^k - \psi_{,k}{}^k{}_j$, that is,

$$\imath_{Dw}g = -d\Delta\psi - 2\imath_w\mathrm{r} \quad \text{if} \quad w = \nabla\psi. \tag{9.3}$$

Also, for any vector field w on a Riemannian manifold,

$$\Delta\delta w = -\delta Dw - 2\delta\imath_w\mathrm{r}, \tag{9.4}$$

since, in local coordinates, (1.11.b) gives $w_{j,}{}^{jk}{}_k = w_{j,}{}^{kj}{}_k + (R_{jk}w^j)^{,k}$, while formula (1.10.f) (or, more precisely, its coordinate form, cf. the lines following (1.11)) yields $w_{j,}{}^{kj}{}_k = w_{j,}{}^k{}_{kj} = \delta\Delta w$ (and so (9.1) implies (9.4)).

Note that D is a second-order elliptic differential operator; it is also self-adjoint, in view of symmetry of r and the relation $-g(\Delta w, v) = \langle\nabla w, \nabla v\rangle - \delta[(\nabla w)^* v]$ (which has the local-coordinate form $-v^j w_{j,k}{}^k = v^{j,k}w_{j,k} - (v^j w_{j,k})^{,k}$). Applied to $v = w$, this last relation shows that, on a compact Riemannian manifold (M,g),

$$(Dw, w) = \|\nabla w\|^2 - \textstyle\int_M \mathrm{r}(w,w)\,\mathrm{d}g \tag{9.5}$$

for any vector field w on M. Here and below $(\ ,\ )$ stands for the L^2 inner product of functions and vector or tensor fields, while $\|\ \|$ is the corresponding L^2 norm.

Similarly, any function ϕ and vector field w on a compact Riemannian manifold satisfy the L^2 inner-product relations

$$2(\nabla w, \nabla d\phi) = (Dw - \nabla\delta w, \nabla\phi). \tag{9.6}$$

In fact, $2\phi^{,jk}w_{j,k} = \phi^{,jk}w_{j,k} + \phi^{,jk}w_{k,j}$ differs by a divergence from $-\phi^{,j}w_{j,k}{}^k + \phi^{,j}w_{k,j}{}^k$ which, in view of (9.1) and (1.11.b), equals $\phi^{,j}(Dw)_j - \phi^{,j}w^k, kj$.

Remark 9.1. Our discussion of the operator D, defined by (9.1) on a Riemannian manifold (M,g), deals mainly with the case where M is compact. In many cases, however, one has $Dw=0$ for purely local reasons:

(i) $Dw=0$ if w is a Killing field;
(ii) $Dw=0$ if w satisfies the soliton equation (**??**);
(iii) $Dw=0$ if (M,g) is a Kähler manifold and w is holomorphic;
(iv) $Dw=-\nabla(\varDelta\psi+2\lambda\psi)$ whenever (M,g) is an Einstein manifold with the Einstein constant λ and $w=\nabla\psi$ for a function $\psi:M\to\mathbf{R}$. Thus, we then have $Dw=0$ if $w=\nabla\psi$ and $\varDelta\psi=-2\lambda\psi$.

Namely, (i) follows from (9.2), as the equality $(\nabla w)^*=-\nabla w$ gives $\delta w=0$. That (**??**) yields $Dw=0$ is clear from (**??**.ii) and (1.13.ii). Next, if g is a Kähler metric and w is holomorphic, $[J,\nabla w]=0$ (see Remark 7.1(b)), so that $J_p^k w^p{}_{,q}{}^q = J_q^p w^k{}_{,p}{}^q$, which, by (4.1.a), equals $-J_p^k R_l^p w^l$, proving (iii). Finally, (iv) is immediate from (9.3).

About the relation between D and the Hodge Laplacian, see Remark 9.8 below.

Lemma 9.2. *On any compact Kähler manifold (M,g), the operator D with* (9.1) *is nonnegative, and its kernel consists of all holomorphic vector fields. In addition, for every C^2 vector field w on M, the L^2 norm of $\pounds_w J$ is given by*

$$\|\pounds_w J\|^2 \,=\, 2(Dw,w)\,. \tag{9.7}$$

In fact, for any vector field w on M, setting $A=\nabla w$ we have $\pounds_w J=[J,A]$ (see Remark 7.1(b)), and so $|\pounds_w J|^2 = \operatorname{tr}[J,A][J,A]^* = 2\operatorname{tr}JAJA^* + 2\operatorname{tr}AA^*$. As $\operatorname{tr}AA^*=|\nabla w|^2$, we now obtain (9.7) by integration, using (9.5), (4.2.ii) and (1.14). Our assertion then follows from Remark 9.1(iii).

Remark 9.3. Inspired by Lemma 9.2, one might define the space of "holomorphic" vector fields on any compact Riemannian manifold (M,g) to be the kernel of D. However, as observed by Yano [9], for any C^2 vector field w on a compact Riemannian manifold (M,g), we have

$$2(Dw,w) \;=\; \|\pounds_w g\|^2 \,-\, 2\|\delta w\|^2, \tag{9.8}$$

since $|\pounds_w g|^2=(w_{j,k}+w_{k,j})(w^{j,k}+w^{k,j})=2(w_{j,k}+w_{k,j})w^{j,k}$, which differs from $-2(w_{j,k}+w_{k,j})^{,k}w^j$ by a divergence, and so (9.8) follows from (9.2) by integration.

Thus, nonnegativity of D fails in general: examples with $(Dw,w)<0$ are non-Killing conformal vector fields w in dimensions $n>2$, for which $n\pounds_w g=2(\delta w)g$, and so (9.8) gives $n(Dw,w)=(2-n)\|\delta w\|^2<0$. Further such examples arise from Remark 9.1(iv): for instance, on a sphere S^n of constant curvature K, choosing an eigenfunction ψ of $-\varDelta$ for the lowest positive eigenvalue nK, and noting that $\lambda=(n-1)K$, we get $Dw=(2-n)w$ for $w=\nabla\psi$.

On the other hand, D provides a characterization of Killing fields w on compact Riemannian manifolds by a pair of scalar equations: $Dw=0$ and $\delta w=0$. This is clear from (9.8) and Remark 9.1(i).

For a function $\psi:M\to\mathbf{R}$ on a compact Riemannian manifold (M,g),

$$\mu\|w\|^2 \,=\, (Dw,w) \,+\, 2\textstyle\int_M \mathrm{r}(w,w)\,\mathrm{d}g \quad\text{if}\quad w=\nabla\psi \ \text{ and } \ \varDelta\psi=-\mu\psi. \tag{9.9}$$

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[9] cf. S. Kobayashi, *Transformation Groups in Differential Geometry*, Ergebnisse, vol. **70**, Springer-Verlag, Berlin, 1972, p. 93

In fact, by (1.19.a), $\mu\|w\|^2 = -\mu(\psi, \Delta\psi) = \|\Delta\psi\|^2$. Bochner's formula (1.21), with $\varphi = \psi$, thus yields $\mu\|w\|^2 = \|\nabla w\|^2 + \int_M \mathrm{r}(w,w)\,\mathrm{d}g$, and (9.5) gives (9.9).

In the following theorem, the inequality $\mathrm{r} \geq \lambda g$ means that $\mathrm{r} - \lambda g$ is positive semidefinite at every point, r being, as usual, the Ricci tensor; in other words, λ is assumed to be a lower bound on the Ricci curvature.

Theorem 9.4. *Let (M,g) be a compact Kähler manifold such that*

$$\mathrm{r} \;\geq\; \lambda g \quad \text{with a constant } \lambda > 0\,. \tag{9.10}$$

Then $\mu \geq 2\lambda$ for every positive eigenvalue μ of $-\Delta$.

If, in addition, $\mathrm{r} = \lambda g$, that is, g is a Kähler-Einstein metric with the Einstein constant $\lambda > 0$, then the assignment $\psi \mapsto \nabla\psi$ defines a linear isomorphism between the space of all functions $\psi : M \to \mathbf{R}$ with $\Delta\psi = -2\lambda\psi$ and the space of all holomorphic gradient vector fields on M.

Proof. That $\mu \geq 2\lambda$ is obvious from (9.9) and Lemma 9.2. Now let $\mathrm{r} = \lambda g$. If $\psi : M \to \mathbf{R}$ and $\Delta\psi = -2\lambda\psi$, (9.9) with $\mu = 2\lambda$ gives $(Dw, w) = 0$ for $w = \nabla\psi$, and so, by (9.7), w is a holomorphic gradient. Thus, the operator $\psi \mapsto \nabla\psi$ is valued in the required space, and it is also injective, as ψ can be constant only if $\psi = 0$. Finally, let w be any holomorphic gradient, so that $w = \nabla\psi$ for some $\psi : M \to \mathbf{R}$. Since $Dw = 0$ (see Remark 9.1(iii)), assertion (iv) in Remark 9.1 shows that $\Delta\psi + 2\lambda\psi$ is constant and, adding a constant to ψ, we may assume that $\Delta\psi = -2\lambda\psi$, as required. □

A weaker form of Theorem 9.4 holds when (M,g), rather than being Kähler, is just assumed to be a compact Riemannian manifold of any real dimension n. Condition (9.10) then implies the *Lichnérowicz inequality* $\mu \geq (n-1)^{-1}n\lambda$ for every positive eigenvalue μ of $-\Delta$. (Proof: if $\Delta\psi = -\mu\psi$ and $\mu\|\psi\| > 0$, the Schwarz inequality $(\Delta\psi)^2 = \langle g, \nabla d\psi\rangle^2 \leq n|\nabla d\psi|^2$ implies, for $w = \nabla\psi$, that $(\delta w)^2 - \operatorname{tr}(\nabla w)^2 = (\Delta\psi)^2 - |\nabla d\psi|^2 \leq (n-1)(\Delta\psi)^2/n$, and so (1.21) gives $(n-1)^{-1}n\lambda\|w\|^2 \leq (n-1)^{-1}n\int_M \mathrm{r}(w,w)\,\mathrm{d}g \leq \|\Delta\psi\|^2$. Since $\mu\|\psi\|^2 = -(\psi, \Delta\psi) = \|w\|^2$ by (1.19.a), we now get $(n-1)^{-1}n\lambda\mu\|\psi\|^2 = (n-1)^{-1}n\lambda\|w\|^2 \leq \|\Delta\psi\|^2 = \mu^2\|\psi\|^2$, as required.)

The following is an obvious consequence of Theorem 9.4:

Corollary 9.5. *In a compact Kähler-Einstein manifold (M,g), with a positive Einstein constant λ,*

(i) *$\mu \geq 2\lambda$ for every positive eigenvalue μ of $-\Delta$,*
(ii) *2λ is an eigenvalue of $-\Delta$ if and only if M admits a nontrivial holomorphic gradient vector field.*

The assertion of Corollary 9.5(ii) remains valid even if the word 'gradient' is dropped, as one easily sees using Theorem 9.6(d) below, due to Matsushima[10], along with (9.11).

Theorem 9.6. *Given a compact Einstein manifold (M,g), let $\lambda, \mathfrak{h}, \mathfrak{g}$ and $\mathfrak{p}$ be the Einstein constant of g, the kernel of the operator D given by* (9.1)*, the Lie algebra of all Killing fields on (M,g) and, respectively, the space of all gradient vector fields w on M with $Dw = 0$. Then we have an L^2-orthogonal decomposition*

$$\mathfrak{h} \;=\; \mathfrak{g} \oplus \mathfrak{p}\,. \tag{9.11}$$

[10] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kaehlérienne*, Nagoya Math. J. **11** (1957) 145-150

In particular, $\mathfrak{g} \subset \mathfrak{h}$. *Furthermore,*

(a) $\mathfrak{h} = \mathfrak{g} = \mathfrak{p} = \{0\}$ *if* $\lambda < 0$,

(b) $\mathfrak{p} = \{0\}$ *and* $\mathfrak{h} = \mathfrak{g}$ *is the space of all parallel vector fields, if* $\lambda = 0$.

(c) *In the case where* $\lambda > 0$, *the* $\mathfrak{g}$ *and* $\mathfrak{p}$ *components of any* $w \in \mathfrak{h}$, *relative to the decomposition* (9.11), *are* $w + (2\lambda)^{-1}\nabla\delta w$ *and* $-(2\lambda)^{-1}\nabla\delta w$, *while* $\mathfrak{p}$ *consists of the gradients of all functions* $\psi : M \to \mathbf{R}$ *with* $\Delta\psi = -2\lambda\psi$.

(d) *If, in addition,* (M, g) *is a Kähler manifold and* $\lambda \neq 0$, *then* $\mathfrak{h}$ *coincides with the space* $\mathfrak{h}(M)$ *of all holomorphic vector fields on* M, *and* $\mathfrak{p} = J\mathfrak{g}$.

Proof. That $\mathfrak{g} \subset \mathfrak{h}$ is obvious from from Remark 9.1(i), while L^2-orthogonality of the spaces $\mathfrak{g}$ and $\mathfrak{p}$ follows from formula (1.17), stating that Killing fields are L^2-orthogonal to gradients. Next, (a) and (b) are immediate from (9.5), and (9.11) is trivially satisfied when $\lambda \le 0$. Let us therefore suppose that $\lambda > 0$. We claim that $u = 2\lambda w + \nabla\delta w$ is a Killing field whenever $w \in \mathfrak{h}$. In fact, $|\pounds_u g|^2 = 2u^{j,k}(u_{j,k} + u_{k,j})$ (cf. the line following (9.8)), and so, since the same holds for w rather than u, we get $|\pounds_u g|^2/4 = 2\lambda^2 w^{j,k}(w_{j,k} + w_{k,j}) + 4\lambda w^{j,k} w^l{}_{,ljk} + w_{p,}{}^{pjk} w^l{}_{,ljk}$, that is, $|\pounds_u g|^2/4 = \lambda^2 |\pounds_w g|^2 + 4\lambda\langle\nabla w, \nabla d\phi\rangle + |\nabla d\phi|^2$, and so $\|\pounds_u g\|^2/4 = \lambda^2\|\pounds_w g\|^2 + 4\lambda(\nabla w, \nabla d\phi) + \|\nabla d\phi\|^2$, where $\phi = \delta w$. Relation (9.4) with $Dw = 0$ and $\mathrm{r} = \lambda g$ gives $\Delta\phi = -2\lambda\phi$. (From now on, ϕ stands for δw.) Thus, (1.21) with $\mathrm{r} = \lambda g$ and $\Delta\phi = -2\lambda\phi$ implies that $\|\nabla d\phi\|^2 = 4\lambda^2\|\phi\|^2 - \lambda\|\nabla\phi\|^2$, that is, $\|\nabla d\phi\|^2 = 2\lambda^2\|\phi\|^2$ (since $\|\nabla\phi\|^2 = 2\lambda\|\phi\|^2$ by (1.19.a)). Also, by (9.8) with $Dw = 0$, we have $\|\pounds_w g\|^2 = 2\|\phi\|^2$. Next, (9.6) with $Dw = 0$ and $\phi = \delta w$ reads $2(\nabla w, \nabla d\phi) = -\|\nabla\phi\|^2 = -2\lambda^2\|\phi\|^2$. Combining these equalities, we see that $\|\pounds_u g\|^2 = 0$, as required. Thus, (9.11) holds also when $\lambda > 0$, and each $w \in \mathfrak{h}$ has the $\mathfrak{g}$ and $\mathfrak{p}$ components described in (c). Also, if $w = \nabla\psi \in \mathfrak{p}$, Remark 9.1(iv) with $Dw = 0$ shows that $\Delta\psi + 2\lambda\psi$ is constant, and hence may be assumed equal to 0. This proves (c).

Finally, under the assumptions of (d), $\mathfrak{h} = \mathfrak{h}(M)$ by Lemma 9.2, and $J\mathfrak{p} \subset \mathfrak{g}$ (that is, $\mathfrak{p} \subset J\mathfrak{g}$) in view of Remark 7.1(d). Conversely, $J\mathfrak{g} \subset \mathfrak{p}$. In fact, for any $u \in \mathfrak{g}$, (9.11) gives $Ju = w + v$ with $w \in \mathfrak{g}$ and $v \in \mathfrak{p}$, while Ju is, locally, a gradient (Remark 7.1(d)). Thus, ∇w is both self-adjoint and skew-adjoint at every point, that is, $\nabla w = 0$, and (1.11.b) yields $w = 0$, as $\mathrm{r} = \lambda g$ and $\lambda \neq 0$. Hence $Ju = v \in \mathfrak{p}$, which completes the proof. □

Corollary 9.7. *For any compact Kähler-Einstein manifold* (M, g), *the identity component* $\mathrm{Isom}^{\mathrm{o}}(M, g)$ *of the isometry group of* (M, g) *is a maximal compact connected Lie subgroup of the biholomorphism group* $\mathrm{Aut}(M)$.

Proof. Suppose, on the contrary, that there exists a vector field $w \in \mathfrak{h}$ such that $w \notin \mathfrak{g}$ and w belongs to the Lie algebra, containing $\mathfrak{g}$, of a compact Lie group G of biholomorphisms of M. Replacing w by its $\mathfrak{p}$ component relative to the decomposition (9.11), we may assume that $w = \nabla\psi$ for some $\psi : M \to \mathbf{R}$. Thus, $d_w\psi = |w|^2$ is nonnegative everywhere and positive somewhere in M. Hence $\int_M d_w\psi \, \mathrm{d}g' > 0$ for any fixed G-invariant Riemannian metric g' on M, which contradicts (1.17), as w is a Killing field on (M, g'). □

Remark 9.8. If (M, g) is an Einstein manifold and λ is its Einstein constant, then $D = H - 2\lambda$, where $H = -d\delta - \delta d$ is the Hodge Laplacian acting on 1-forms ξ (identified with vector fields w, so that $\xi = \imath_w g$). Thus, $Dw = 0$ if and only if $Hw = 2\lambda w$. Note that the decomposition of w in Theorem 9.6 coincides with the Hodge decomposition of the eigenform $\xi = \imath_w g$ of the Hodge Laplacian.