# NOTES FOR MATH 7721: PROJECTIVE SPACES AND GRASSMANNIANS

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[**DG**] stands for *Differential Geometry* at
`https://people.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf`
[**KG**] for *Kähler Geometry from a Riemannian Perspective* at
`https://people.math.osu.edu/derdzinski.1/courses/7721/kg.pdf`

## 1. The manifold structures

Let $V$ be a vector space of positive dimension $n < \infty$ over the scalar field $\mathrm{I\!K}$, where $\mathrm{I\!K}$ is $\mathrm{I\!R}, \mathbb{C}$ or $\mathrm{I\!H}$ and, in the last (quaternionic) case, we mean a *left* vector space. By the *projective space* of $V$ one means the set

$$\mathrm{P}V = \{L : L \text{ is a 1-dimensional vector subspace of } V\}, \tag{1.1}$$

and a surjective *projection mapping* $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ is defined by

$$\pi(x) = \mathrm{I\!K}x. \tag{1.2}$$

The set $\mathrm{P}V$ carries a natural manifold structure provided by the atlas

$$\{(U_f, \varphi_f) : f \in V^* \smallsetminus \{0\}\} \tag{1.3}$$

indexed by all nonzero linear functionals on $V$, where

$$U_f = \{L \in P(V) : L \text{ is not contained in } \mathrm{Ker}\, f\} \tag{1.4}$$

(instead of 'is not contained in $\mathrm{Ker}\, f$' one could also write '$f(L) = \mathrm{I\!K}$' or, equivalently, '$f$ maps $L$ isomorphically onto $\mathrm{I\!K}$'), and $\varphi_f : U_f \to f^{-1}(1)$ sends each $L \in U_f$ onto its unique intersection point with $f^{-1}(1)$. Also, $f^{-1}(1)$ is a coset of $\mathrm{Ker}\, f$, which makes it an affine space with the translation vector space $\mathrm{Ker}\, f$, and

$$\begin{array}{l} \varphi_f : U_f \to f^{-1}(1) \quad \text{is a bijection with the inverse } \pi : f^{-1}(1) \to U_f \text{ and} \\ \varphi_f(\mathrm{I\!K}x) = x/f(x) \quad \text{whenever } L = \mathrm{I\!K}x \in U_f \text{ (that is, } x \in V \smallsetminus \mathrm{Ker}\, f). \end{array} \tag{1.5}$$

Compatibility of any two charts in (1.3) now follows since, for $f, h \in V^* \smallsetminus \{0\}$, the set $\varphi_f(U_f \cap U_h) = A_f \smallsetminus \mathrm{Ker}\, h$ is open in $f^{-1}(1)$ (due to closedness of $\mathrm{Ker}\, h$ in the ambient space $V$), while $(\varphi_f \circ \varphi_h^{-1})(x) = x/f(x)$ as a consequence of (1.5). (For the meaning of compatibility, see [**DG**, Section 1].)

**Lemma 1.1.** *The atlas* (1.3) *satisfies the Hausdorff and countability axioms, cf.* [**DG**, Section 1 and 14]*, and so it actually turns* $\mathrm{P}V$ *into a smooth manifold which, in addition, is compact.*

*Proof.* See Problem 1 in **Homework #3**. □

**Lemma 1.2.** *Every linear automorphism of* $V$*, acting in an obvious manner on* $\mathrm{P}V$*, constitutes a smooth diffeomorphism. The projection* $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ *is smooth as well.*

*Proof.* Let $A : V \to V$ be a linear automorphism. Using the same symbol for $A : \mathrm{P}V \to \mathrm{P}V$, we obtain, from (1.5), the rational (and hence smooth) chart representations $(\varphi_f \circ A \circ \varphi_h^{-1})(x) = Ax/f(Ax)$. On the other hand, the chart representations of $\pi$ are identity mappings, cf. the first line of (1.5). □

**Lemma 1.3.** *If* $\mathbb{K} = \mathbb{C}$*, the projective space* $\mathrm{P}V$ *carries a unique structure of a complex manifold such that all chart mappings* $\varphi_f$ *are biholomorphisms. In addition, the projection* $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ *is then also holomorphic.*

*Proof.* This is immediate since the transition mappings $\varphi_f \circ \varphi_h^{-1}$, being rational, are holomorphic. For the claim about $\pi$, see the proof of Lemma 1.2. □

When $V = \mathbb{K}^n$, rather than $\mathrm{P}V$ one writes $\mathbb{K}\mathrm{P}^{n-1}$ and speaks of the *real, complex or quaternionic projective space* of dimension $n-1$ over the respective field, where the latter the real/complex dimension $n-1$ or (for $\mathbb{K} = \mathbb{H}$) the real dimension $4(n-1)$. The 1-dimensional subspace $L \in P(V)$ spanned by a nonzero vector $(x^1, \dots, x^n)$ in $\mathbb{K}^n$ is then denoted by $[x^1, \dots, x^n] \in P(V)$, and one refers to $x^1, \dots, x^n$ as *homogeneous coordinates* of $L = [x^1, \dots, x^n]$.

**Generalization to Grassmannians.** In addition to $V, n, \mathbb{K}$ as above, let us also fix an integer $q$ with $0 \le q \le n$, set

$$\mathrm{Gr}_q V = \{L : L \text{ is a } q\text{-dimensional vector subspace of } V\}, \tag{1.6}$$

and define a surjective *projection mapping* $\pi : \mathrm{St}_q V \to \mathrm{Gr}_q V$ by

$$\pi(\mathbf{x}) = \mathrm{Span}\,\mathbf{x} \quad \text{for } \mathbf{x} = (x_1, \dots, x_q) \in \mathrm{St}_q V, \tag{1.7}$$

where $\mathrm{St}_q V$ denotes the *Stiefel manifold* formed by all *$q$-frames* (that is, linearly independent ordered $q$-tuples of vectors) in $V$. (Thus, $\mathrm{St}_q V$ is an open subset of the $q$th Cartesian power $V^q$.) One calls $\mathrm{Gr}_q V$ the *Grassmannian of $q$-planes* in $V$. The set $\mathrm{Gr}_q V$ carries a natural manifold structure provided by the atlas

$$\{(U_f, \varphi_f) : f \in V^* \smallsetminus \{0\}\}, \qquad \text{with } U_f = \{L \in P(V) : f(L) = \mathbb{K}^q\}, \tag{1.8}$$

indexed by all surjective linear operators $f : V \to \mathbb{K}^q$. (Instead of '$f(L) = \mathbb{K}^q$' one may equivalently write '$f$ maps $L$ isomorphically onto $\mathbb{K}^q$'). The chart mappings

$$\varphi_f : U_f \to f^{-1}(e_1) \times \ldots \times f^{-1}(e_q)$$

with $e_1, \dots, e_q$ denoting the standard basis of $\mathbb{K}^q$, are slightly more complicated: $\varphi_f$ sends each $L \in U_f$ onto the unique ordered $q$-tuple $\mathbf{x} = (x_1, \dots, x_q)$ of vectors in $L$ such that $f(x_a) = e_a$ for $a = 1, \dots, q$. In other words, using the inverse $f_L^{-1}$ of the restriction isomorphism $f_L : L \to \mathbb{K}^q$, we have $\varphi_f(L) = (f_L^{-1}(e_1), \dots, f_L^{-1}(e_q))$. Note that $f^{-1}(e_1) \times \ldots \times f^{-1}(e_q)$ a coset, in $V^q$, of the $q$th Cartesian power of $\mathrm{Ker}\, f$, and hence an affine subspace of $V^q$.

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## 2. The locally symmetric metrics

**Lemma 2.1.** *Given a Lie group $G$ and a smooth isometric left action of $G$ on a pseudo-Riemannian manifold $(\Sigma, \gamma)$, along with a manifold $M$ and a surjective submersion $\pi : \Sigma \to M$ for which the $\pi$-preimages of points in $M$ are nondegenerate submanifolds of $(\Sigma, \gamma)$ and coincide with the orbits of the $G$ action, there exists a unique pseudo-Riemannian metric $g$ on $M$ such that $\pi^*g$ and $\gamma$ have the same restriction to the $\gamma$-orthogonal complement $\mathcal{H}$ of the vertical distribution $\mathcal{V} = \mathrm{Ker}\, d\pi$ of $\pi$.*

*Furthermore, under the identification, provided by $\pi$, between $M$ and the set $\Sigma/G$ of all $G$ orbits, every isometry of $(\Sigma, \gamma)$ commuting with the $G$ action leads to an obvious bijection $\Sigma/G \to \Sigma/G$, and hence $M \to M$, which is then a smooth isometry of $(M, g)$ onto itself.*

*Proof.* □

**Generalization to Grassmannians.** Irreducible (globally) symmetric Riemannian manifolds come in pairs: one compact, and one not. The latter is usually called the *noncompact dual* of the former.