class # 34340

MATH 7721, SPRING 2018
M-W-F 3:00 p.m., EC 243
A DAY-BY-DAY LIST OF TOPICS

[**KG**] stands for *Kähler Geometry from a Riemannian Perspective* at
`https://people.math.osu.edu/derdzinski.1/courses/7721/kg.pdf`

[**DG**] for *Differential Geometry* at
`https://people.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf`

[**PS**] for *Notes for Math 7721*: *Projective Spaces and Grassmannians* at
`https://people.math.osu.edu/derdzinski.1/courses/7721/ps.pdf`

[**FR**] for *Further References* at
`https://people.math.osu.edu/derdzinski.1/courses/7721/fr.pdf`

January 8: Almost-complex manifolds. Examples: complex vector spaces; open submanifolds; Cartesian products [**FR**]. Orientations in a real vector space of a positive finite dimension [**FR**]. Connectedness of the automorphism group (and of the set of all ordered bases) of a finite-dimensional complex vector space [**FR**]. The conclusion that finite-dimensional complex vector spaces and, consequently, almost-complex manifolds, are *canonically oriented.* The covariant 2-tensors

$$aJ = a(J\cdot\,,\,\cdot), \qquad Ja = -a(\,\cdot\,,J\cdot) \tag{1.1}$$

arising from a given covariant 2-tensor a on an almost-complex manifold. Hermitian (symmetric) 2-tensors a and skew-Hermitian 2-forms a, defined by requiring that

$$aJ = Ja. \tag{1.2}$$

Hermitian metrics on an almost-complex manifold, characterized, equivalently [**Homework #1**, Problems 1–2], by being real parts of complex-valued Hermitian fibre metrics; or, being those metrics which make J skew-adjoint at each point; or, finally, by being the metrics turning J, at each point, into a linear isometry of the tangent space. The fact that, when one uses a Hermitian metric g to identify any covariant 2-tensor a with the endomorphism A of the tangent bundle TM characterized by $a(v,\,\cdot\,) = g(Av,\,\cdot\,)$ for all tangent vector fields v, the 2-tensors aJ and Ja correspond in the same way to the composite bundle endomorphisms AJ and JA, and so

$$aJ = Ja \quad \text{if and only if} \quad [J,A] = 0, \tag{1.3}$$

$[\,,\,]$ being the commutator. The one-to-one J-correspondence between Hermitian 2-tensors and skew-Hermitian 2-forms. The *Kähler form*

$$\varOmega = gJ, \qquad \text{that is, } \varOmega = g(J\cdot\,,\,\cdot), \tag{1.4}$$

of the given Hermitian metric g on an almost-complex manifold. The equality, in which $(\;)^{\wedge m}$ and $\mathrm{d}g$ denote the mth exterior power [**FR**] and, respectively, the volume form of the oriented Riemannian manifold (M,g) (see below, under January 10):

$$\varOmega^{\wedge m} = m!\,\mathrm{d}g, \quad \text{for } m = \dim_{\mathbb{C}} M. \tag{1.5}$$

References: [**KG**]: Section 3 except formula (3.2), and Remark 4.1(iii) except the last sentence. **Homework #1**.

January 10: The relations $\det A = \pm 1$ for the transition matrix A between two orthonormal bases in a Euclidean n-space V or, in other words, any orthogonal $n \times n$ matrix [**FR**], as well as $\zeta(w_1, \ldots, w_n) = \delta\zeta(v_1, \ldots, v_n)$ whenever $v_1, \ldots, v_n, w_1, \ldots, w_n \in V$ satisfy the matrix equality $[w_1 \ldots w_n] = [v_1 \ldots v_n]A$ (that is, $w_j = A_j^k v_k$), where A is an arbitrary $n \times n$ matrix and δ denotes its determinant [**FR**]. The volume form of an oriented Euclidean space, or of an oriented Riemannian manifold [**FR**]. Tensor products and symmetric/exterior powers of finite-dimensional real/complex vector spaces [**FR**]. Proof of (1.5). Holomorphic mappings and biholomorphisms. Integrability of almost-complex structures. Complex manifolds.

References: [**KG**]: Section 7 (the first 4 lines and the final paragraph). **Homework #2**.

January 12: Kähler connections on almost-complex manifolds. Kähler metrics/manifolds. Examples: complex vector spaces with Hermitian inner products; open submanifolds; Cartesian products; oriented Riemannian surfaces (since an oriented Euclidean plane is, naturally, the same as a complex line with a Hermitian inner product). Almost-complex submanifolds. The Levi-Civita connection of a submanifold metric [**FR**]. The fact that almost-complex submanifolds of Kähler manifolds become Kähler manifolds when endowed with the submanifold metric. Locally symmetric Kähler manifolds, and a proof of their actually being Kähler manifolds.

References: [**KG**]: Section 4 (the first three paragraphs and the paragraph immediately following Remark 7.2). **Homework #3**.

January 17: The Stiefel manifold $\mathrm{St}_q V$ of q-frames, and the Grassmannian $\mathrm{Gr}_q V$ of q-planes in V, where V is a vector space of finite positive dimension n over the field $\mathbb{K}$ of real/complex numbers or quaternions, and $q \in \{0, 1, \ldots, n\}$. The projective space $\mathrm{P}V = \mathrm{Gr}_1 V$. The smooth projections $\pi : \mathrm{St}_q V \to \mathrm{Gr}_q V$ and $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$, holomorphic in the complex case ($\mathbb{K} = \mathbb{C}$). Holomorphicity, when $\mathbb{K} = \mathbb{C}$, of the mappings $\mathrm{P}V \to \mathrm{P}V$ induced by complex-linear automorphisms of V.

References: [**PS**]. **Homework #4**.

January 19: The normal quotient metric g arising when an isometric action of a Lie group G on a pseudo-Riemannian manifold (Σ, γ) has nondegenerate orbits and admits a smooth quotient manifold M. The case where, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and V, n, q as before, V is endowed with a bilinear/sesquilinear, symmetric/Hermitian, nondegenerate form $\langle\,,\rangle$ which is either positive definite ($\varepsilon = 1$) or has the sign pattern of q minuses and $n - q$ pluses ($\varepsilon = -1$), and the resulting open submanifold M of $\mathrm{Gr}_q V$ given by $M = \{L \in \mathrm{Gr}_q V :\ \varepsilon\langle\,,\rangle$ is positive definite on $L\}$. The normal quotient metric g on M for $q = 1$ and the submanifold metric γ on $\Sigma = \{x \in V :\ \langle x, x\rangle = \varepsilon\}$, with the group G of unit scalars in $\mathbb{K}$ acting via ordinary multplication. The terminology used for (M, g): a *constant-curvature real projective space* or a *complex projective space* with a *Fubini-Study metric* ($\varepsilon = 1$), and a *real/complex hyperbolic space* ($\varepsilon = -1$). The fact that such (M, g) are (globally) symmetric Riemannian (or, Kähler) manifolds.

References: [**PS**]; [**KG**, the paragraph following Remark 7.2]. **Homework #5**.

January 22: The generalization of the preceding construction to arbitrary q, using the submanifold Σ of $\mathrm{St}_q V$ consisting of all (anti)orthonormal q-frames, the orthogonal or unitary matrix group G, so that $G = \mathrm{O}(q)$ if $\mathbb{K} = \mathbb{R}$, and $G = \mathrm{U}(q)$ if $\mathbb{K} = \mathbb{C}$, while γ is the submanifold metric inherited by Σ from its ambient vector space, namely, the qth

Cartesian power V^q. The terminology used for (M,g): a *real/complex Grassmannian* with a *standard metric* ($\varepsilon = 1$) and, respectively, *its noncompact dual* ($\varepsilon = -1$). The observation that, again, all resulting (M,g) are symmetric Riemannian/Kähler manifolds.

References: [**PS**]; [**KG**, the paragraph following Remark 7.2]. **Homework #6**.

January 24: Nonsingular projective algebraic varieties as examples of Kähler manifolds. The Grassmannian $\mathrm{Gr}_q^+ V$ of oriented q-planes in a finite-dimensional real vector space V, with the two-to-one surjective projection $\mathrm{Gr}_q^+ V \to \mathrm{Gr}_q V$, including the special case of the sphere $\mathrm{Gr}_1^+ V$. Almost-Kähler metrics/manifolds, including Kähler metrics as a special case. Finite partitions of unity, oriented integration of compactly supported continuous top-degree differential forms, the Stokes theorem, and de Rham cohomology [**FR**], [**KG**, Section 2]. The *Kähler form* $\varOmega = gJ$ of the given almost-Kähler manifold (M,g), and its *Kähler* (cohomology) *class*

$$[\varOmega] \in H^2(M,\mathrm{I\!R}). \tag{7.1}$$

References: [**PS**]; [**KG**, Section 5 (the first paragraph, (iii) in Remark 4.1, the 4-line paragraph preceding Theorem 5.3, and Remark 5.1)]. **Homework #7**.

January 26: Positive and negative cohomology classes in $H^2(M,\mathrm{I\!R})$ on an almost complex manifold M, and the fact that, by (1.5), if M is compact, positivity implies being nonzero. Mutual exclusiveness of positivity/negativity/vanishing in $H^2(M,\mathrm{I\!R})$ for a compact almost-complex manifold M. Complex-linearity of $R(u,v)$ in Kähler manifolds:

$$[R(u,v),J] = 0, \tag{8.1}$$

where $[\ ,\ ]$ denotes the commutator, and u,v are any tangent vector fields; the latter is also the case in the following equality, for the *Ricci form* $\rho = rJ$ of an arbitrary Kähler manifold, r being its Ricci tensor:

$$\mathrm{tr}_{\mathrm{I\!R}}\, J[R(u,v)] = -2\rho(u,v) \tag{8.2}$$

(in coordinates, $R_{klp}{}^q J_q^p = -2\rho_{kl}$), easily implying, via the second Bianchi identity,

$$\text{Hermitian symmetry of } r \text{ and closedness } (d\rho = 0) \text{ of } \rho. \tag{8.3}$$

Proof of (8.2) based on the identity

$$\rho_{kl} = R_{pkl}{}^q J_q^p, \tag{8.4}$$

which arises, via contraction against g^{qs}, from the relation

$$R_{qlsp} J_k^p = R_{qlkp} J_s^p \tag{8.5}$$

that is, symmetry of $R_{qlsp} J_k^p$ in s,k, reflecting self-adjointness of the composite of two commuting skew-adjoint morphisms – namely, $R(u,v)$ and J in (8.1), cf. [**Homework #9**, Problem 1].

References: [**KG**, Section 5 (the text following the proof of Theorem 5.3); Section 4 (the two lines preceding formula (4.1) plus the first 11 lines following it, and (i)–(ii) in Remark 4.1); **Homework #8**.

January 29: Another consequence of (8.5), obtained by contracting it against J_r^k:

$$R(Ju, Jv) \;=\; R(u, v), \tag{9.1}$$

u, v being any vector fields. The first Chern class $c_1(\mathcal{E})$ of a complex vector bundle $\mathcal{E}$ and the formula for it using the curvature tensor R of any connection in $\mathcal{E}$:

$$2\pi c_1(\mathcal{E}) \;=\; [\operatorname{Im}\zeta], \quad \text{where} \;\; \zeta \;=\; \operatorname{tr}_{\mathbb{C}}[R(\,\cdot\,,\,\cdot\,)], \tag{9.2}$$

that is, $\zeta(u,v) = \operatorname{tr}_{\mathbb{C}}[R(u,v)]$ for all tangent vector fields u, v. The conclusion that

$$\text{(a)}\quad \operatorname{tr}_{\mathbb{C}}[R(\,\cdot\,,\,\cdot\,)] \;=\; i\rho, \qquad\qquad \text{(b)}\quad [\rho] \;=\; 2\pi c_1(M) \,\in\, H^2(M, \mathrm{I\!R}) \tag{9.3}$$

in any Kähler manifold (M, g), derived from (8.2), (9.2) and the equality

$$\operatorname{tr}_{\mathrm{I\!R}} A \;=\; 2\operatorname{Re}\operatorname{tr}_{\mathbb{C}} A, \tag{9.4}$$

valid whenever A is a complex-linear endomorphism of a finite-dimensional complex vector space [**Homework #9**, Problem 2]. The (complex-valued) *connection* 1*-forms* $\varGamma_a^b$ and *curvature* 2*-forms* R_a^b on U, representing a given connection ∇ in a complex vector bundle $\mathcal{E}$ over a manifold M, and its curvature tensor R, relative to a system e_a of local trivializing sections defined on an open set $U \subseteq M$, with

$$\nabla_v e_a = \varGamma_a^b(v) e_b, \qquad R(v,w) e_a = R_a^b(v,w) e_b, \tag{9.5}$$

v, w being arbitrary vector fields on U, so that, for ζ as in (9.2),

$$R_a^b \;=\; -d\varGamma_a^b \,+\, \varGamma_a^c \wedge \varGamma_c^b, \qquad \text{and} \;\; \zeta \;=\; R_a^a \;=\; -d\varGamma, \;\; \text{where} \;\; \varGamma = \varGamma_a^a. \tag{9.6}$$

The resulting local formula

$$\rho \;=\; i\, d\varGamma \tag{9.7}$$

obtained by combining (9.3.a) with the second part of (9.6) in the case where $\mathcal{E}$ is the tangent bundle of a Kähler manifold. The fact that, due to positivity of Kähler classes, $H^2(M, \mathrm{I\!R}) \neq \{0\}$ for any compact almost-complex manifold M admitting an almost-Kähler metric. An example of a compact complex manifold M with $H^2(M, \mathrm{I\!R}) = \{0\}$ (and hence with no almost-Kähler metric), provided by a Hopf manifold, that is, $M = S^1 \times \varSigma$, where S^1 denotes the circle of unit complex numbers, $\varSigma$ the unit sphere in a finite-dimensional complex vector space V carrying a fixed Hermitian norm $|\;|$, and the complex structure is uniquely characterized by the requirement that the locally diffeomorphic surjective mapping

$$V \smallsetminus \{0\} \,\ni\, x \,\mapsto\, (e^{i\theta\log|x|}, x/|x|) \,\in\, S^1 \times \varSigma \tag{9.8}$$

be holomorphic, θ being any nonzero real constant. (In other words, the local inverses of (9.8) form a coordinate atlas with transition mapping which are holomorphic, namely, constitute multiplications by positive constants.) The Betti numbers of spheres and the Künneth formula for $S^1 \times N$, derived from the Mayer-Vietoris sequence [**FR**].

References: [**KG**, Section 4 (the two lines before, and the first 11 lines after formula (4.1), plus Remark 4.1); Section 2; Section 6 (parts (i), (ii) of Remark 6.1 and the first two paragraphs following it)]. **Homework #9**.

January 31: The observation that, in a Euclidean space V,

$$2\,\mathrm{tr}\,BA \,=\, \mathrm{tr}\,B(A-A^*) \quad \text{if} \quad A,B \in \mathrm{End}\,V \quad \text{and} \quad B^* = -B. \tag{10.1}$$

The component formula for the (pointwise) adjoint A^* of any smooth bundle morphism $A : TM \to TM$ in a Riemannian manifold (M,g):

$$B = A^* \quad \text{has the components } \; B_j{}^k = A^k{}_j, \quad \text{that is, } \; B_j{}^k = A_p{}^q g^{pk} g_{qj}. \tag{10.2}$$

The local identities, valid for any smooth vector field v on a Kähler manifold (M,g),

$$\begin{array}{rl} \text{i)} & \mathrm{tr}\,(JA) \,=\, \delta(Jv),\\ \text{ii)} & \delta(JA^*) \,=\, \rho(v,\,\cdot\,),\\ \text{iii)} & \mathrm{tr}\,JAJA \,=\, -r(v,v) \,+\, (\mathrm{tr}\,JA)^2 \,+\, \delta[JAJv - (\mathrm{tr}\,JA)Jv],\\ \text{iv)} & \mathrm{tr}\,JAJA^* \,=\, -r(v,v) \,+\, \delta(JA^*Jv), \end{array} \tag{10.3}$$

with $A = \nabla v : TM \to TM$, and δ denoting the divergence of both vector fields and bundle endomorphisms of TM. The coordinate versions $J_q^p v^q{}_{,p} = (J_q^p v^q)_{,p}$ and

$$J_q^p v_{k,}{}^q{}_p \,=\, \rho_{lk} v^l \tag{10.4}$$

of (10.3.i) and (10.3.ii). Proofs of (10.3.i) – (10.3.iii). The observation that, by (10.4),

$$J_q^p v_{k,p}{}^q \,=\, -\rho_{lk} v^l, \tag{10.5}$$

since $\mathrm{tr}\,JA^* = -\mathrm{tr}\,JA$, for any bundle endomorphism of TM. (In fact, $\mathrm{tr}\,JA^* = \mathrm{tr}\,(JA^*)^* = \mathrm{tr}\,AJ^* = -\mathrm{tr}\,AJ = -\mathrm{tr}\,JA$.)

References: [**KG**, Section 4 (formulae (4.1.b) and (4.2), along with the three lines preceding formula (4.2) and six lines following it)]. **Homework #10**.

February 2: Proof of (10.3.iv). The notation $v \sim \xi$, or $\xi \sim v$, and $a \sim A$, or $A \sim a$, for a vector field v, a differential 1-form ξ, a twice-covariant tensor field a, and a bundle endomorphism A of TM, in a Riemannian manifold (M,g), meaning that

$$\xi \,=\, g(v,\,\cdot\,) \qquad \text{and} \qquad a \,=\, g(A\,\cdot\,,\,\cdot\,). \tag{11.1}$$

The observation that

$$\nabla\xi \,\sim\, \nabla v \qquad \text{if} \qquad \xi \,\sim\, v. \tag{11.2}$$

The seemingly-counterintuitive minus sign in the relation

$$\xi J \sim -Jv \qquad \text{whenever} \qquad \xi \sim v, \tag{11.3}$$

satisfied by vector fields v and 1-forms ξ on an almost-complex manifold M carrying a fixed Hermitian metric g, and immediate from skew-adjointness of J (which, for $\xi = g(v,\,\cdot\,)$, gives $\xi J = g(v, J\,\cdot\,) = -g(Jv,\,\cdot\,)$); here ξJ, also written as $J^*\xi$, is the composite bundle morphism $TM \to TM \to M \times \mathbb{R}$, of J followed by ξ. The operator $i\partial\overline{\partial}$ associated with any given almost-complex manifold M, sending each smooth function $f : M \to \mathbb{R}$ to the exact 2-form $i\partial\overline{\partial}f$ such that

$$2i\partial\overline{\partial}f \;=\; -d[(df)J]. \tag{11.4}$$

The expression for $i\partial\overline{\partial}$ in terms of any given torsionfree connection ∇ on M:

$$2i\partial\overline{\partial}f \;=\; aJ \,+\, Ja \,-\, (df)(dJ), \quad \text{for } \; a = \nabla df, \tag{11.5}$$

$(df)(dJ)$ being the composite in which dJ is the TM-valued 2-form assigning $[\nabla_{\!u}J]v - [\nabla_{\!v}J]u$ to vector fields u, v. The conclusion that, when the almost-complex manifold M admits a Kähler connection ∇ (a torsionfree one having $\nabla J = 0$), the operator $i\partial\overline{\partial}$ takes values in skew-Hermitian 2-forms – as A with $A \sim i\partial\overline{\partial}f$, being then, for any f, the anticommutator of J with B characterized by $B \sim \nabla df$, must commute with J, cf. [**Homework #11**, Problem 3]. The formula

$$\mathrm{tr}_g[(i\partial\overline{\partial}f)J] \;=\; -\Delta f, \tag{11.6}$$

for smooth functions f on Kähler manifolds. The equality

$$\sqrt{2}\,\|\zeta\| = \|\mathrm{tr}_g\,\zeta J\| \tag{11.7}$$

satisfied by any exact skew-Hermitian differential 2-form ζ on a compact Kähler manifold (M, g), where $\|\;\|$ denotes the L^2 norm, both for functions and bundle endomorphisms of TM, the latter based on the inner product $\langle\,,\rangle$ with

$$\langle A, B\rangle \;=\; \mathrm{tr}\,AB^* \tag{11.8}$$

in $\mathrm{End}\,V$, for any Euclidean space V. Proof of (11.7), consisting of the following steps. First, let $\zeta = d\xi$ for a 1-form ξ, and let $A = \nabla v$ for the vector field v with $v \sim \xi$. Thus,

$$\zeta \;\sim\; A - A^*. \tag{11.9}$$

Note that (10.1) for $B = J$ combined with (11.9) yields

$$\mathrm{tr}_g\,\zeta J \;=\; 2\,\mathrm{tr}\,JA. \tag{11.10}$$

Now $2\|\zeta\|^2$ equals, by (10.1) for $B = A - A^*$, the integral of $-4\,\mathrm{tr}\,(A - A^*)A$, and hence the integral of $4\,\mathrm{tr}\,J(A - A^*)JA$. (The 'skew-Hermitian' hypothesis means that $A - A^*$ commutes with J, and so $A - A^* = -J(A - A^*)J$.) The integral of $4\,\mathrm{tr}\,J(A - A^*)JA =$

$4(\operatorname{tr} JAJA - \operatorname{tr} JAJA^*)$ equals, however, that of $(2\operatorname{tr} JA)^2$, due to (10.3.iii) – (10.3.iv) and the divergence theorem. Therefore (11.8) follows from (11.10).

References: [**KG**, Section 4 (the three lines preceding Remark 4.1); formula (3.2) in Section 3; Section 4 (formula (4.3), part (d) of Lemma 4.2, and the first paragraph of its proof)]. **Homework #11**.

February 5: The conclusion, obvious from (11.8) that, for a compact Kähler manifold,

(12.1) the operator $\eta \mapsto \operatorname{tr}_g \eta J$ acting on exact skew-Hermitian 2-forms is injective.

The $\partial\overline{\partial}$ Lemma in any compact connected Kähler manifold the operator $i\partial\overline{\partial}$ sending smooth real-valued functions f to values to smooth exact skew-Hermitian 2-forms ζ is surjective, and its kernel consists of constant functions. Proof of the $\partial\overline{\partial}$ Lemma, with the claim about the kernel obvious [**FR**] from (11.6), and surjectivity immediate since the function $\operatorname{tr}_g \zeta J$, having the integral 0 due to (11.10), (10.3.i), and the divergence theorem, must equal $-\Delta f$ for some f, so that, from (11.6), $\operatorname{tr}_g \eta J = 0$ for $\eta = \zeta - i\partial\overline{\partial} f$ which, by (12.1), yields $\eta = 0$, as required. The ratio $\gamma : M \to (0,\infty)$ of the volume elements of two Riemannian metrics g and $\hat{g}$ on an oriented manifold M, characterized by the equality $\mathrm{d}\hat{g} = \gamma\,\mathrm{d}g$, and the observation that

$$\det_g \hat{g} \,=\, \gamma^2, \tag{12.2}$$

$\det_g \hat{g}$ meaning $\det \hat{H}$ for $\hat{H} = A : TM \to TM$ as in (11.1) with $a = \hat{g}$. The formula

$$\hat{\rho} \,=\, \rho \,-\, i\partial\overline{\partial} \log \gamma, \tag{12.3}$$

relating the Ricci forms ρ and $\hat{\rho}$ of two Kähler metrics $g, \hat{g}$ on the same almost-complex manifold, where γ denotes the ratio of their volume elements, that is, $\mathrm{d}\hat{g} = \gamma\,\mathrm{d}g$. Proof of (12.3), first part: from (9.5) – (9.7) we get $\rho = i\,d\mathit{\Gamma}$, where $\mathit{\Gamma} = \mathit{\Gamma}_a^a$ and, analogously, $\hat{\rho} = i\,d\hat{\mathit{\Gamma}}$, with $\hat{\mathit{\Gamma}} = \hat{\mathit{\Gamma}}_a^a$. At the same time, g and $\hat{g}$ are the real parts of (unique) Hermitian fibre metrics

$$h \,=\, g \,-\, i\mathit{\Omega} \tag{12.4}$$

and $\hat{h} = \hat{g} - i\hat{\mathit{\Omega}}$, cf. [**Homework #1**, Problem 2] and (1.4). The complex-linear endomorphism $\hat{H} : TU \to TU$ over the local-trivialization domain U characterized by

$$\hat{h} \,=\, h(\hat{H}\,\cdot\,,\,\cdot\,) \tag{12.5}$$

must be the same as $\hat{H}$ in the line following (12.2), as one sees applying Re to (12.5). Its complex-valued component functions $\hat{H}_a^b$ with $\hat{H}e_a = \hat{H}_a^b e_b$ satisfy the relation

$$\hat{h}_{ab} \,=\, \hat{H}_a^c h_{cb}, \tag{12.6}$$

obtained by evaluating (12.5) on the pair (e_a, e_b). Four functions on U, valued in complex $m \times m$ matrices (where $m = \dim_{\mathbb{C}} M$) now emerge: $\mathcal{G}$, $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$, having the entry in the

bth row and ath column equal to h_{ab}, $\hat{h}_{ab}$ and, respectively, $\hat{H}_a^b$, cf. (9.5). As the values of $\mathcal{G}$ and $\hat{\mathcal{G}}$ are Hermitian and positive definite, it follows, according to [**Homework #12**, Problem 1] and, respectively, (12.6), that

$$\mathcal{D} = \det \mathcal{G} \quad \text{and} \quad \hat{\mathcal{D}} = \det \hat{\mathcal{G}} \quad \text{are real and positive, while} \quad \mathcal{G}\hat{\mathcal{H}} = \hat{\mathcal{G}}. \tag{12.7}$$

Applying $\det$ to the equality $\mathcal{G}\hat{\mathcal{H}} = \hat{\mathcal{G}}$ in (12.7) we obtain

$$\det \hat{\mathcal{H}} = \hat{\mathcal{D}}/\mathcal{D} \qquad \text{and} \qquad \det \hat{\mathcal{H}} = \det_{\mathbb{C}} \hat{H} = \gamma, \tag{12.8}$$

where $\det \hat{\mathcal{H}} = \det_{\mathbb{C}} \hat{H}$ since $\hat{\mathcal{H}}$ is, for every point of $x \in U$, the matrix of $\hat{H}$ in the complex basis of $T_x M$ formed by the values of e_a at x, and $\det_{\mathbb{C}} \hat{H}$ also equals γ, the square root of $\det_{\mathbb{R}} \hat{H}$ – see (12.2) – as $\hat{H}$, being h-self-adjoint and positive due to (12.5), has in some complex orthonormal basis of $T_x M$ a diagonal matrix with positive real entries on the diagonal, while in the corresponding real orthonormal basis [**Homework #1**, Problem 6] each of these diagonal entries is repeated twice. The formula

$$(\det F)^{\cdot} = (\det F) \operatorname{tr} F^{-1} \dot{F}, \tag{12.9}$$

valid for any smooth curve $t \mapsto F = F(t)$ of linear automorphisms of a finite-dimensional real/complex vector space [**FR**], [formula (6.1) in **KG** and the paragraph following it].

References: [**KG**, Section 4 (parts (a) – (c) of Lemma 4.2 and the second paragraph of its proof; formula (6.2.a) along with the three lines before and five lines after it; part (iii) of Remark 6.1 and the four paragraphs following it)]. **Homework #12**.

February 7: Three proofs of (12.9): one using the cofactor expansion of the determinant, another based on evaluating the differential of det at the identity, and the third one involving the two actions of F on the top exterior power of the given vector space. Proof of (12.3), second part: applying (12.9) to $F = \mathcal{G}$ or $F = \hat{\mathcal{G}}$ along any integral curve $t \mapsto x(t) \in M$ of any smooth local vector field w tangent to M, one gets, from (12.7),

$$d_w \log \mathcal{D} = \operatorname{tr}_{\mathbb{C}}(\mathcal{G}^{-1} d_w \mathcal{G}), \qquad d_w \log \hat{\mathcal{D}} = \operatorname{tr}_{\mathbb{C}}(\hat{\mathcal{G}}^{-1} d_w \hat{\mathcal{G}}). \tag{13.1}$$

Next, as $\nabla h = 0$ by (12.4), the Leibniz rule gives

$$d_w h_{ab} = \Gamma_a^c(w) h_{cb} + \overline{\Gamma_b^c(w) h_{ca}} \tag{13.2}$$

as well as its analog for $\hat{h}$ and the Levi-Civita connection $\hat{\nabla}$ of $\hat{g}$ or, in matrix form,

$$d_w \mathcal{G} = \mathcal{G}\mathcal{T} + (\mathcal{G}\mathcal{T})^*, \qquad d_w \hat{\mathcal{G}} = \hat{\mathcal{G}}\hat{\mathcal{T}} + (\hat{\mathcal{G}}\hat{\mathcal{T}})^*, \tag{13.3}$$

* denoting the conjugate transpose, and $\mathcal{T}$ or $\hat{\mathcal{T}}$ the matrix-valued function with the entry in the bth row and ath column equal to $\Gamma_a^b(w)$ or, respectively, the analogous expression $\hat{\Gamma}_a^b(w)$ corresponding to $\hat{\nabla}$. Now (13.1) and (13.3) give

$$d_w \log \mathcal{D} = \operatorname{tr}(\mathcal{G}^{-1} d_w \mathcal{G}) = \operatorname{tr}(\mathcal{T} + \mathcal{T}^*) = 2 \operatorname{Re} \operatorname{tr} \mathcal{T} = 2 \operatorname{Re} \Gamma(w), \tag{13.4}$$

for $\mathit{\Gamma} = \mathit{\Gamma}_a^a$ as in (9.6). Hence $d\log\mathcal{D} = 2\,\mathrm{Re}\,\mathit{\Gamma}$ and, analogously, $d\log\hat{\mathcal{D}} = 2\,\mathrm{Re}\,\hat{\mathit{\Gamma}}$, so that (12.8) and (9.7) yield

$$(13.5) \qquad d\log\gamma = 2\,\mathrm{Re}\,(\hat{\mathit{\Gamma}} - \mathit{\Gamma}), \qquad \rho - \hat{\rho} = i\,d(\mathit{\Gamma} - \hat{\mathit{\Gamma}}) = d[i(\mathit{\Gamma} - \hat{\mathit{\Gamma}})].$$

References: [**KG**, Section 6 (the long paragraph before Theorem 6.2)]. **Homework #13.**

February 9: Proof of (12.3), third part: $\mathit{\Gamma} - \hat{\mathit{\Gamma}}$ is, at every point $x \in U$, a *complex-linear* mapping $T_xM \to \mathbb{C}$. (In fact, so is $\mathit{\Gamma}_a^b - \hat{\mathit{\Gamma}}_a^b$ for each pair of indices a, b, since $\hat{\nabla}_v w - \nabla_v w$ depends on v, w symmetrically and complex-bilinearly: symmetry follows as both connections are torsionfree, while $\mathbb{C}$-linearity in v is immediate from symmetry and $\mathbb{C}$-linearity in w, the latter being due to the relations $\nabla J = \hat{\nabla} J = 0$.) Therefore, since ρ and $\hat{\rho}$ are real-valued, (13.5) gives $\rho - \hat{\rho} = d[(\mathit{\Gamma} - \hat{\mathit{\Gamma}})J] = \mathrm{Re}\,d[(\mathit{\Gamma} - \hat{\mathit{\Gamma}})J] = d\,\mathrm{Re}\,[(\mathit{\Gamma} - \hat{\mathit{\Gamma}})J] = -d[(d\log\gamma)J]/2 = i\partial\overline{\partial}\,\log\gamma$, with the last equality due to (11.4), which proves (12.3). Einstein and Kähler-Einstein metrics/manifolds. Ricci-flatness.

References: [**KG**, Section 6 (the long paragraph before Theorem 6.2)]. **Homework #14.**

February 12: Positivity/negativity of $c_1(M)$ for any compact almost-complex manifold M carrying a non-Ricci-flat Kähler-Einstein metric, the sign being the same as that of the Einstein constant λ. The Calabi conjecture (for $c_1 < 0$, proved independently by Aubin and Yau): *every compact almost-complex manifold M with $c_1(M) < 0$, admitting a Kähler metric, also admits a Kähler-Einstein metric.* The Calabi conjecture (for $c_1 = 0$, proved by Yau): *if ρ is a closed skew-Hermitian 2-form on a compact almost-complex manifold M admitting a Kähler metric, and $[\rho] = 2\pi c_1(M)$, then every positive cohomology class in $H^2(M, \mathrm{I\!R})$ contains the Kähler form of a Kähler metric for which ρ the Ricci form.* The Goldberg conjecture (still open): *a compact almost-Kähler Einstein manifold is necessarily a Kähler manifold.* The inequalities

$$(15.1) \qquad a \le b, \quad a < b, \quad a \ge b, \quad a > b$$

for twice-covariant symmetric tensors or tensor fields a, b, meaning that $a - b$ is positive semidefinite (or definite), or negative semidefinite (or definite), at the given point, or at every point. Uniqueness in the Calabi conjectures (proved by Calabi, with the normalization $\lambda = -1$ for the Einstein constant λ). Proof of uniqueness in the first Calabi conjecture.

References: [**KG**, Section 5 (the paragraph preceding Lemma 5.2); Section 6 (part (a) of Theorem 6.2 and its proof)]. **Homework #15.**

February 13: Proof of the uniqueness assertion in the second Calabi conjecture.

References: [**KG**, Section 6 (the long paragraph before Theorem 6.2)]. **Homework #16.**

February 14:

$$(17.2) \qquad \pounds_u g = g(A\cdot\,,\,\cdot) \quad \text{for} \quad A = B + B^* \quad \text{and} \quad B = \nabla u,$$

$$(17.3) \qquad (\pounds_u g)_{jk} = u_{j,k} + u_{k,j}\,.$$

$$(17.4) \qquad \pounds_v J \,=\, [J, \nabla v].$$

The Lie-bracket-versus-commutator relation

$$(17.5) \qquad \pounds_{[v,w]} \,=\, [\pounds_v, \pounds_w],$$

The divergence of a smooth real-linear endomorphism A of tangent bundle of a manifold M endowed with a connection, defined to be the 1-form $\eta = \delta A$ on M given by

$$(17.8) \qquad \eta_j \,=\, A^k_{j,k},$$

so that, due to (10.2), in the case of the Levi-Civita connection of a Riemannian metric g, for the (pointwise) adjoint A^* of A and $\xi = \delta A^*$, one has

$$(17.9) \qquad \xi_j \,=\, g_{jl} A^{l,k}_k.$$

References: [**KG**, Section 6 (the long paragraph before Theorem 6.2)]. **Homework #17**.

February 16: The Lie subalgebra $\mathfrak{i}(M,g)$ (or, $\mathfrak{h}(M)$) of $\mathfrak{X}M$ for a Riemannian (or, almost-complex) manifold (M,g) (or, M), consisting of all Killing (or, respectively, holomorphic) vector fields, $\mathfrak{X}M$ being the Lie algebra of all smooth vector fields on M. The Bochner identity [**FR**]

$$(18.1) \qquad R_{jk} v^k \,=\, v^k{}_{,jk} \,-\, v^k{}_{,kj}, \quad \text{that is,} \quad r(\,\cdot\,, v) \,=\, \delta\nabla v \,-\, d\delta v$$

valid whenever v is a smooth vector field on a manifold endowed with a torsionfree connection ∇ (the Ricci tensor r of which need not be symmetric). The linear differential operator $D : \mathfrak{X}M \to \mathfrak{X}M$, associated with an arbitrary Riemannian manifold (M,g), and defined by

$$(18.2) \qquad Dw \,=\, -\Delta w \,-\, rw, \quad \text{that is,} \quad (Dw)_j \,=\, -w_{j,k}{}^k \,-\, R_{jk} w^k,$$

where the second formula is the local-coordinate version after index lowering. The observation that (18.2) and (18.1) give $(Dw)_j = -w_{j,k}{}^k - w_{k,j}{}^k + w^k{}_{k,j} = -(w_{j,k} + w_{k,j})^{,k} + w^k{}_{k,j}$ or, equivalently,

$$(18.3) \qquad g(Dw, \cdot\,) \,=\, -\delta \pounds_w g \,+\, d\delta w$$

for any smooth vector field w on a Riemannian manifold (M,g). The identity

$$(18.4) \qquad |\pounds_w g|^2 \,=\, 2(w_{j,k} \,+\, w_{k,j}) w^{j,k}$$

arising as a trivial consequence of symmetry of $w_{j,k} + w_{k,j}$ in j,k. The equality

$$(18.5) \qquad 2(Du,u) \,=\, \|\pounds_u g\|^2 \,-\, 2\|\delta u\|^2$$

satisfied by all compactly supported smooth vector fields u on any Riemannian manifold (M,g). The proof of (18.5) based on integration by parts:

$$\text{letting} \quad \approx \quad \text{always mean `differs by a divergence'} \tag{18.6}$$

we see that (18.4) yields $|\pounds_u g|^2 \approx -2(u_{j,k} + u_{k,j})^{,k} u^j$, while this last expression is the inner product of u and the vector field corresponding via g to $-2\delta\pounds_u g$, or – by (18.3) – of u and the vector field $2(Du - \nabla\delta u)$, and so $|\pounds_u g|^2 \approx 2[g(Du,u) + (\delta u)^2]$, due to the obvious relation $-g(\nabla\delta u, u) = -u^k{}_{,kj}u^j \approx u^k{}_{,k}u^j{}_{,j} = (\delta u)^2$. A further integral formula:

$$(Dv, v) \,=\, \|\nabla v\|^2 \,-\, \int_M r(v,v)\,\mathrm{d}g, \tag{18.7}$$

where v is, again, a compactly supported smooth vector field on a Riemannian manifold (M,g), and the claim is, again, obvious from (18.2) via integration by parts. The equality

$$\delta[J, \nabla v]^* \,=\, -g(JDv,\,\cdot\,) \tag{18.8}$$

valid for any smooth vector field v on a Kähler manifold, and its consequence

$$2(Dv, v) \,=\, \|\pounds_v J\|^2 \tag{18.9}$$

in the case where v is compactly supported (both established below), $\|\;\|$ and $(\,,\,)$ denoting the L^2 norm and L^2 inner product. The conclusion – immediate from (18.9), (18.8) and (18.3) – that, in a compact Kähler manifold (M,g),

$$D \,\geq\, 0 \quad \text{ and } \quad \mathfrak{i}(M,g) \,\subseteq\, \mathfrak{h}(M) \,=\, \mathrm{Ker}\,D, \tag{18.10}$$

or, equivalently, the operator D is nonnegative, and its kernel consists precisely of all holomorphic vector fields, while all Killing fields are holomorphic. Proof of (18.8) in local coordinates: the kth component of the left-hand side is $g_{kl}(J^l_p v^p{}_{,q} - v^l{}_{,p}J^p_q)^{,q} = g_{kl}J^l_p v^p{}_{,q}{}^{,q} - J^p_q v_{k,p}{}^q$ while, by (10.5), $-J^p_q v_{k,p}{}^q$ equals the the kth component of $\rho(v,\,\cdot\,) = r(Jv,\,\cdot\,) = g(rJv,\,\cdot\,) = g(Jrv,\,\cdot\,)$ due to Hermitian symmetry of r, so that (18.2) yields (18.8). Proof of (18.9): by (17.5), for $A = \nabla v$ one has $\pounds_v J = [J,A]$, which gives $|\pounds_v J|^2 = \mathrm{tr}\,[J,A][J,A]^* \,=\, 2\,\mathrm{tr}\,JAJA^* + 2\,\mathrm{tr}\,AA^*$, and so, as $\mathrm{tr}\,AA^* \,=\, |\nabla v|^2$, (18.9) follows via integration, in view of (10.3.iv) and the divergence theorem, combined with (18.7).

References: [**KG**, Section 9 (formulae (9.1), (9.2), (9.5), parts (i) and (iii) in Remark 9.1, Lemma 9.2)]. **Homework #18**.

February 19: Examples of Einstein manifolds: vector spaces with constant metrics (which are flat, hence Ricci-flat); suitable Riemannian products (with Einstein factors of the same Einstein constant); Riemannian surfaces of constant Gaussian curvature. The Einstein condition as a consequence of irreducibility of the local isotropy representation at every point (or just at one point, in the locally homogeneous case); further examples of Einstein manifolds provided, for this last reason, by complex projective spaces with the Fubini-Study metrics, complex hyperbolic spaces, and standard spheres.

Bochner's integral formula [**FR**] valid whenever v is a compactly supported smooth vector field on a Riemannian manifold (M,g):

$$(19.2)\qquad \int_M r(v,v) \,=\, \|\delta v\|^2 \,-\, \int_M \mathrm{tr}\,(\nabla v)^2\,\mathrm{d}g,$$

and its version for gradients $v=\nabla f$ of compactly supported smooth functions:

$$(19.3)\qquad \int_M r(v,v) \,=\, \|\Delta f\|^2 \,-\, \|\nabla v\|^2,$$

leading to the equality

$$(19.4)\qquad \tau\|v\|^2 \,=\, (Dv,v) \,+\, 2\int_M r(v,v)\,\mathrm{d}g \quad \text{if } \ v=\nabla f \ \text{ and } \ \Delta f=-\tau f,$$

f being here a smooth function on a compact Riemannian manifold (M,g). Proof of (19.4): as $\tau\|v\|^2=-\tau(f,\Delta f)=\|\Delta f\|^2$, (19.3) gives $\tau\|v\|^2=\|\nabla v\|^2+\int_M r(v,v)\,\mathrm{d}g$, and (18.7) yields (19.4). A trivial consequence of (19.4) and the first part of (18.10): in any compact Kähler manifold (M,g) such that $r\geq\lambda g$, cf. (15.1), and $\lambda\in(0,\infty)$, one has

$$(19.5)\qquad \tau \,\geq\, 2\lambda \ \text{ for every nonzero eigenvalue } \ \tau \ \text{ of } \ -\Delta.$$

The Lichnerowicz inequality $\tau\geq n\lambda/(n-1)$ (a conclusion analogous to, but weaker than (19.5)), valid [**FR**] whenever (M,g) above is only assumed to be a compact *Riemannian* manifold, of (real) dimension $n\geq 2$.

References: [**KG**, Section 9 (Lemma 9.2, formula (9.9), and the first part of Theorem 9.4 along with the first line of its proof)]. **Homework #19**.

February 21:

$$(20.9)\qquad D\nabla f \,=\, -\nabla\Delta f \,-\, 2r\nabla f.$$

References: [**KG**, Section 9 (Lemma 9.2, formula (9.9), and the first part of Theorem 9.4 along with the first line of its proof)]. **Homework #20**.

February 23: The fact that, whenever ζ is a smooth bivector field (twice-contravariant tensor field, skew-symmetric at every point) on a manifold M with a fixed torsionfree connection admitting, locally, a parallel volume form, one has

$$(21.1)\qquad \delta\delta\zeta \,=\, 0 \qquad \text{(in coordinates,} \quad \zeta^{jk}{}_{,jk}=0),$$

and its proof via integration by parts: $f\zeta^{jk}{}_{,jk} \,\approx\, -f_{,k}\zeta^{jk}{}_{,j} \,\approx\, -f_{,kj}\zeta^{jk} \,=\, 0$, with the convention (18.6), whenever f is a smooth function compactly supported in an open set forming the domain of a parallel volume form; here $f_{,kj}\zeta^{jk}\,=\,0$ due to symmetry of the Hessian of f. An alternative proof of (21.1), in [**Homework #21**, Problems 3–4], with the local existence of parallel volume forms replaced by the equivalent requirement of symmetry of the Ricci tensor. The conclusion that any smooth vector field w on a Riemannian manifold satisfies the relation

$$w_{j,k}{}^{kj} = w_{j,k}{}^{jk}, \tag{21.2}$$

obvious from (21.1) applied to $\zeta^{jk} = w^{j,k} - w^{k,j}$, as well as the identity

$$\delta(Dw + 2rw) = -\Delta\delta w, \tag{21.3}$$

which follows since (18.2) and (18.1) give $(Dw + 2rw)_j = -w_{j,k}{}^k - R_{jk}w^k + 2R_{jk}w^k = -w_{j,k}{}^k + R_{jk}w^k = -w_{j,k}{}^k + w^k{}_{j,k} - w^k{}_{k,j}$, and so $(Dw + 2rw)_j{}^{,j} = -w^k{}_{k,j}{}^j = -\Delta\delta w$, the vanishing of $-w_{j,k}{}^{kj} + w^k{}_{j,k}{}^j$ being nothing else than (21.2). An obvious corollary: in an Einstein manifold with the Einstein constant λ,

$$\Delta\phi = -2\lambda\phi \qquad \text{whenever} \quad w \in \operatorname{Ker} D \quad \text{and} \quad \phi = \delta w. \tag{21.4}$$

References: [**KG**, Section 9 (Lemma 9.2, formula (9.9), and the first part of Theorem 9.4 along with the first line of its proof)]. **Homework #21**.

February 26: Matsushima's theorem (the general Riemannian version): *for any compact Einstein manifold, one has the L^2-orthogonal decomposition*

$$\operatorname{Ker} D = \mathfrak{k} \oplus \mathfrak{p}, \tag{22.4}$$

$\mathfrak{k} = \mathfrak{i}(M, g)$ *and* $\mathfrak{p}$ *denoting, respectively, the Lie algebra of all Killing fields and the space of all gradient vector fields in* $\operatorname{Ker} D$. *In the case where the Einstein constant* λ *is nonzero, the* $\mathfrak{k}$ *component* u *and* $\mathfrak{p}$ *component* v *of any* $w \in \operatorname{Ker} D$ *are given by*

$$u = w + \frac{1}{2\lambda}\nabla\phi \quad \text{and} \quad v = -\frac{1}{2\lambda}\nabla\phi, \quad \text{with} \quad \phi = \delta w. \tag{22.5}$$

Also, $\mathfrak{k} = \mathfrak{p} = \operatorname{Ker} D = \{0\}$ *if* $\lambda < 0$. *Finally, when* $\lambda = 0$, *the space* $\mathfrak{p}$ *is again trivial, and* $\operatorname{Ker} D = \mathfrak{k}$ *consists of all parallel vector fields.* Proof of Matsushima's theorem: the claim about the case $\lambda \le 0$ is obvious from (18.7) and (18.2), since the (18.1) gives $r(v, \cdot\,) = 0$ for any parallel vector field v.

References: [**KG**, Section 9 (Lemma 9.2, formula (9.9), and the first part of Theorem 9.4 along with the first line of its proof)]. **Homework #22**.

February 27: Three trivial observations. First, whenever A, B, J are linear endomorphisms of a Euclidean space such that $J^2 = -\mathrm{Id}$ and $J^* = -J$,

$$\begin{aligned} &\text{if } [J, A] = 0 \text{ and } A^* = A, \text{ while } B = JA, \text{ then } [J, B] = 0 \text{ and } B^* = -B,\\ &\text{if } [J, B] = 0 \text{ and } B^* = -B, \text{ while } A = -JB, \text{ then } [J, A] = 0 \text{ and } A^* = A. \end{aligned} \tag{23.1}$$

Second, given smooth vector fields v, u on a Riemannian manifold,

$$v \text{ is a local gradient if and only if } A^* = A, \text{ where } A = \nabla v, \tag{23.2}$$

(by (11.9), as the local-gradient property of v obviously means closedness of the 1-form ξ with $\xi \sim v$); at the same time, from (17.2),

(23.3) $\quad u$ is a Killing field if and only if $B^* = -B$, where $B = \nabla u$.

Third, for smooth vector fields v, u on a Kähler manifold, since J is parallel,

(23.4) $\quad$ if $u = Jv$, then $A = \nabla v$ and $B = \nabla u$ are related by $B = JA$.

The consequence that, for J acting on vector fields in any Kähler manifold (M, g),

(23.5) $\quad J$ maps $\mathfrak{h}(M)$ isomorphically onto itself, and it also maps the space of local gradients in $\mathfrak{h}(M)$ isomorphically onto the space of Killing fields in $\mathfrak{h}(M)$.

which is immediate if one uses (23.1) – (23.4) for $A = \nabla v$ and $B = \nabla u$, noting that holomorphicity of v (or, of u) amounts to the equality $[J, A] = 0$ (or, $[J, B] = 0$), cf. (17.4). The fact that, whenever f is a smooth function on a Kähler manifold,

(23.7) $\quad \nabla f$ is holomorphic if and only if the Hessian ∇df is Hermitian,

due to (1.2) – (1.3) applied to $v = \nabla f$, with $a = \nabla df$ and $A = \nabla v$, cf. (11.2) for $\xi = f$. The Kähler case of Matsushima's theorem, in which (22.5) – according to (18.10) – reads

$$\mathfrak{h}(M) \,=\, \mathfrak{k} \oplus \mathfrak{p}, \tag{23.8}$$

$\mathfrak{k} = \mathfrak{i}(M, g)$ still being the real Lie algebra of all Killing fields, and $\mathfrak{p}$ now (also) the space of all holomorphic gradients while, in addition,

(23.9) $\quad \mathfrak{p} \,=\, J\mathfrak{k}$ if the Einstein constant λ is nonzero.

Proof of (23.9); That J maps $\mathfrak{p}$ into $\mathfrak{k}$ is clear from the second claim in (23.5). Next, if $u \in \mathfrak{k}$ and the first claim in (23.5) gives $Ju \in \mathfrak{h}(M)$ and so, by (23.5) and (23.8), Ju is a local gradient and $Ju = w - v$ with $v \in \mathfrak{p}$ and $w \in \mathfrak{k}$. According to (23.2) – (23.4), in the resulting equality $\nabla[Ju] = \nabla w - \nabla v$ the first and last terms are self-adjoint, the remaining term both skew-adjoint and self-adjoint, and hence $\nabla w = 0$. Now (18.1) with $r = \lambda g$ gives $0 = rw = \lambda w$, and so $w = 0$, proving that $Ju = -v \in \mathfrak{p}$. Thus, J maps $\mathfrak{k}$ into $\mathfrak{p}$, as required. The relation

$$u_{q,jk} = R_{qjk}{}^{s} u_s + a_{qj,k} + a_{qk,j} - a_{jk,q}, \quad \text{where} \quad a_{j,k} = (u_{j,k} + u_{k,j})/2 \tag{23.10}$$

valid for any smooth 1-form u on a manifold carrying a fixed torsionfree connection, and its proof based on rewriting the difference $2[u_{q,jk} - (a_{qj,k} + a_{qk,j} - a_{jk,q})]$ with the aid of three applications of the Ricci identity $u_{q,jk} - u_{q,kj} = R_{kjq}{}^{s} u_s$ for 1-forms, followed by the use of the first Bianchi identity. An obvious special case of (23.10):

$$u_{j,kl} \,=\, R_{jkl}{}^{s} u_s \tag{23.11}$$

whenever u is a Killing field on a Riemannian manifold, so that one then has

$$\nabla_w B = R(u, w), \quad \text{where} \quad B = \nabla u \quad \text{and } w \text{ is any smooth vector field.} \tag{23.12}$$

The resulting system of linear equations, satisfied by a Killing field u and $B = \nabla u$ along any smooth curve $t \mapsto x(t)$:

$$\nabla_{\dot{x}} u = B\dot{x}, \qquad \nabla_{\dot{x}} B = R(u, \dot{x}). \tag{23.13}$$

The conclusion that, due to uniqueness of solutions for (23.13), given a connected Riemannian manifold and a point $x \in M$,

(23.14) the linear operator $\mathfrak{i}(M, g) \ni u \mapsto (u_x, [\nabla u]_x) \in T_x M \times \mathfrak{so}(T_x M)$ is injective.

The resulting dimension estimates, with $n = \dim M$ and, respectively, $m = \dim_{\mathbb{C}} M$:

$$\dim \mathfrak{i}(M, g) \le \frac{n(n+1)}{2}, \qquad \dim \mathfrak{i}(M, g) \le m(m+2), \tag{23.15}$$

the first valid for any connected Riemannian manifold, the second for any compact connected Kähler manifold, the improved estimate being due to the fact that, by (18.10), the injective operator $u \mapsto (u_x, [\nabla u]_x)$ takes values in $T_x M \times \mathfrak{u}(T_x M)$. Complexifications and real forms of Lie algebras. The conclusion – from (23.9) and (23.15) – that, in any compact non-Ricci-flat Kähler-Einstein manifold (M, g), the Lie algebra $\mathfrak{k} = \mathfrak{i}(M, g)$ of Killing vector fields is a real form of $\mathfrak{h}(M)$, and so

$$\dim_{\mathbb{R}} \mathfrak{i}(M, g) = \dim_{\mathbb{C}} \mathfrak{h}(M) \le m(m+2), \tag{23.16}$$

as a consequence of (23.15). The Lie algebra $\operatorname{Der}\mathfrak{k}$ of all derivations of a Lie algebra $\mathfrak{k}$ (linear endomorphisms of $\mathfrak{k}$ obeying the Leibniz rule) and the Ad representation of $\mathfrak{k}$, that is, that Lie-algebra homomorphism $\operatorname{Ad} : \mathfrak{k} \to \operatorname{Der}\mathfrak{k}$ with

$$\operatorname{Ad} u = [u, \cdot\,], \tag{23.17}$$

where the $\operatorname{Der}\mathfrak{k}$-valuedness and the homomorphic property both amount to the Jacobi identity. Compact (real) Lie algebras $\mathfrak{k}$, defined by requiring the existence of a Euclidean inner product making $\operatorname{Ad} u$ skew-adjoint for all $u \in \mathfrak{k}$. Compactness of $\mathfrak{i}(M, g)$ for a compact Riemannian manifold (M, g), the required condition being provided by the L^2 inner product, due to the equality, satisfied whenever u, v are Killing fields:

$$g([u, v], v) = \delta w, \qquad \text{with} \quad w = -[g(v, u)]v. \tag{23.18}$$

Proof of (23.18): first, on a Riemannian manifold,

(23.19) $\delta(fv) = d_v f + f\delta v$ for any vector field v and function f, both smooth.

(A trivial exercise.) Now $g([u, v], v) = g(\nabla_u v, v) - g(\nabla_v u, v)$. As ∇v is skew-adjoint, and g symmetric, this equals $-g(\nabla_v v, u) - g(v, \nabla_v u) = -d_v[g(v, u)]$. (We choose to ignore the fact that $g(v, \nabla_v u) = 0$.) By (23.19), the last expression is nothing else than δw in (23.18).

The corollary that $\mathfrak{h}(M)$ *has a compact real form whenever the given compact almost-complex manifold M admits a non-Ricci-flat Kähler-Einstein metric* (or, equivalently – by (9.3.b) – M admits a Kähler-Einstein metric and $c_1(M) \neq 0$). The *Jacobi equation*

$$\nabla_{\dot{x}} \nabla_{\dot{x}} u \,=\, R(u, \dot{x})\dot{x} \tag{23.20}$$

satisfied, in view of (23.13), by any Killing field on a Riemannian manifold, along any geodesic. An alternative version

$$u^j{}_{,kl} \,=\, u^s R_{slk}{}^j \tag{23.21}$$

of (23.11), of interest since it makes sense for a smooth vector field u on a manifold endowed with a torsionfree connection, and is known to hold if and only if the local flow of u preserves the connection.

References: [**KG**, Section 9 (Lemma 9.2, formula (9.9), and the first part of Theorem 9.4 along with the first line of its proof)]. **Homework #23**.

February 28: The trivial fact that, on a connected Riemannian manifold, a local gradient can at the same time be a Killing field only if it is parallel. A generalization – given a smooth vector field v and Riemannian metrics $g, \hat{g}$ on a connected manifold, if v is a local gradient relative to g (meaning: closedness of the 1-form $g(v, \cdot\,)$), and also a Killing field for $\hat{g}$, then v must either vanish identically, or be nonzero everywhere. (Proof: if $v = 0$ at a point z, let $A = [\nabla v]_z = [\hat{\nabla} v]_z$ for the Levi-Civita connections $\nabla, \hat{\nabla}$ of g and g, with the equality arising from the independence of A of the connection used; by (23.2) – (23.3), A is g_z-self-adjoint – and hence diagonalizable – as well as $\hat{g}_z$-skew-adjoint, and so any of its real eigenvalues equals zero; thus, $A = 0$ and, consequently, (23.14) gives $v = 0$ on M.) *Maximality of the real form* $\mathfrak{k} = \mathfrak{i}(M, g)$ *in* (23.8) *among the Lie algebras* $\hat{\mathfrak{k}} = \mathfrak{i}(M, \hat{g})$ *of Killing fields for all Kähler metrics* $\hat{g}$ on the given compact almost-complex manifold M carrying our fixed non-Ricci-flat Kähler-Einstein metric. (Proof of maximality: if $\hat{\mathfrak{k}}$ contained $\mathfrak{k}$ as a proper subspace, being – by (18.10) – itself contained in $\mathfrak{h}(M)$, it would, due to dimensional reasons, nontrivially intersect the other summand $\mathfrak{p}$ in (23.8), that is, a nontrivial gradient relative to g would at the same time be a Killing field for $\hat{g}$, which contradicts the preceding observation since, on a compact manifold, a gradient must vanish somewhere.) The linear vector field $x \mapsto Ax$ on a finite-dimensional real/complex vector space V, associated with any given linear endomorphism $A \in \mathrm{End}\, V$, and its local flow $t \mapsto e^{tA}$, defined as usual [**FR**]. The first-order linear ordinary differential equation

$$\dot{\Psi} \,=\, A\Psi \tag{24.1}$$

with $(\ )^{\boldsymbol{\cdot}} = d/dt$, satisfied when $t \mapsto \Psi = \Psi(t)$ takes values in V (or, in $\mathrm{End}\, V$) and $\Psi(t)$ is equal to one of the expressions $e^{tA}x$, $e^{tA}(x+y) - e^{tA}x - e^{tA}y$, $e^{tA}(cx) - ce^{tA}x$, for any fixed $x, y \in V$ and $c \in \mathrm{I\!R}$ (or, respectively, $\Psi = Ae^{tA} - e^{tA}A$). The resulting completeness of the vector field A, as linearity leads to global solutions [**FR**]. Linearity of the flow transformations e^{tA}, and the commutation relation $Ae^{tA} = e^{tA}A$, both derived from the uniqueness of solutions (as they involve the initial value 0 at $t = 0$). Projectability (the existence of push-forwards) for vector fields under smooth mappings between manifolds [**FR**]. Uniqueness and smoothness of the push-forward under a surjective submersion, if the original vector field is smooth and projectable, immediate from the rank theorem [**FR**].

The observation that any linear vector field A on V is projectable under $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ (namely, for two π-preimages $x, cx \in V \smallsetminus \{0\}$ of a point in $\mathrm{P}V$, the vectors of A at x and cx are the velocitites at $t = 0$ of the curves $t \mapsto e^{tA}x$ and $t \mapsto e^{tA}(cx) = ce^{tA}x$, which both project onto the same curve in $\mathrm{P}V$). Holomorphicity of the projected vector fields $(d\pi)A$ on $\mathrm{P}V$, for all $A \in \mathrm{End}\,V$, due to the fact that they exist and are smooth (according to the last two sentences), while the mappings $\mathrm{P}V \to \mathrm{P}V$ induced by complex-linear automorphisms of V, such as e^{tA}, are holomorphic [**FR**]. Injectivity of the push-forward operator $d\pi$ on the space $\mathrm{End}_0 V \subseteq \mathrm{End}\,V$ of all traceless endomorphisms (the kernel of $d\pi$ clearly being the span of the identity), implying that, for $m = \dim_{\mathbb{C}} M$, by (23.16), $m(m+2) = \dim_{\mathrm{I\!R}} \mathrm{End}_0 \le \dim_{\mathbb{C}} \mathfrak{h}(M) \le m(m+2)$, which yields

(24.2) the complex-linear isomorphism $d\pi : \mathrm{End}_0 V \to \mathfrak{h}(M)$, where $M = \mathrm{P}V$.

If $A \in \mathrm{End}_0 V$ is nonzero, so is $(d\pi)A \in \mathfrak{h}(M)$, while $(d\pi)A = 0$ somewhere, since A has eigenvectors; thus, as a consequence of two lines following (22.5),

(24.3) Fubini-Study metrics are Einstein metrics with positive Einstein constants.

(Their Einstein property was established in the discussion preceding (19.2).)

References: [**DG**], [**PS**], as listed in [**FR**]. **Homework #24**.

March 1: A description of $\mathrm{Ker}\,(\Delta + 2\lambda)$ for the complex projective space $\mathrm{P}V$ with the Fubini-Study metric, associated with the given Hermitian inner product $\langle\,,\rangle$ in the vector space V: *namely,* $\mathrm{Ker}\,(\Delta + 2\lambda)$ *consists of all functions*

$$(25.1) \qquad \mathbb{C}x \mapsto \frac{\langle Ax, x\rangle}{\langle x, x\rangle},$$

where A ranges over traceless self-adjoint complex-linear endomorphisms of V. Proof of the italicized statement:

The equality, in which w is any tangent vector field:

$$(25.2) \qquad 2\nabla_{\!w}\zeta \,=\, \zeta_w \,+\, J\zeta_w J$$

valid whenever torsionfree connections $\nabla, \hat\nabla$ and a Hermitian twice-covariant tensor field h on an almost-complex manifold M satisfy the conditions $\hat\nabla J = 0$ and $\nabla h = 0$, for the skew-Hermitian 2-form $\zeta = hJ$ and the 2-form $\zeta_w = (d\zeta)(w,\cdot\,,\cdot)$. A corollary, immediate when one applies the above assertion to $h = g$ and the Levi-Civita connection ∇ of g: *on an almost-complex manifold admitting a Kähler connection – that is, a torsionfree one making J parallel – every almost-Kähler metric g is necessarily a Kähler metric.*

References: [**KG**, Lemma 5.2 in Section 5]. **Homework #25**.

March 2: Proof of (25.2).

References: [**KG**, Proof of Lemma 5.2 in Section 5]. **Homework #26**.

March 5: The existence, on any compact connected Riemannian manifold, of a smooth function f, unique up to an additive constant, such that

$$(27.1) \qquad \Delta f \,+\, s \,=\, s_{\mathrm{avg}}$$

(or, equivalently, $\Delta f + s$ is constant), where s_{avg} denotes the average value of the scalar curvature s. The resulting linear differential operator L sending each smooth vector field v to the a smooth function Lv, with

$$Lv = \delta v - d_v f. \tag{27.2}$$

The Futaki invariant of a given compact (connected) Kähler manifold (M,g), defined to be the real-linear functional

$$\boldsymbol{F} : \mathfrak{h}(M) \to \mathrm{I\!R}, \quad \text{given by} \quad \boldsymbol{F}v = (s_{\mathrm{avg}})^m \int_M d_v f \, \mathrm{d}g, \quad \text{where } m = \dim_{\mathbb{C}} M, \tag{27.3}$$

for f as in (27.1). The observation that, due to the presence of the factor $(s_{\mathrm{avg}})^m$,

$$\boldsymbol{F} \text{ remains unchanged when } g \text{ is rescaled.} \tag{27.4}$$

The existence on a compact almost-complex manifold M with $c_1(M) > 0$ or $c_1(M) < 0$, admitting a Kähler metric, of a further Kähler metric g such that

$$i\partial\overline{\partial} f + \rho = \lambda\Omega \text{ for some smooth function } f : M \to \mathrm{I\!R} \text{ and some } \lambda \in \mathrm{I\!R} \smallsetminus \{0\}, \tag{27.5}$$

the sign of λ being characterized by $\lambda c_1(M) > 0$ (and its actual value by $\lambda[\Omega] = 2\pi c_1(M)$). Proof of this claim, arising as an obvious consequence of the definition of positivity/negativity in $H^2(M, \mathrm{I\!R})$, formula (9.3.b), the italicized statement at the end of the paragraph following (25.2), and the $\partial\overline{\partial}$ Lemma (see February 5): namely, for any fixed $\lambda \neq 0$, such g is just any Kähler metric with $\lambda[\Omega] = 2\pi c_1(M)$. The fact that on any Kähler manifold (M,g), whether compact or not,

$$\text{condition (27.5) implies } \Delta f + s = m\lambda/2, \text{ where } m = \dim_{\mathbb{C}} M, \tag{27.6}$$

as one sees "multiplying" (27.5) from the right by J, then applying tr_g and using (11.6) along with the definitions of Ω and ρ preceding (7.1) and, respectively, (8.2).

References: [**KG**, Lemma 5.2 in Section 5]. **Homework #27**.

March 21: The fact that, whenever a smooth function $f : M \to \mathrm{I\!R}$ on a Kähler manifold (M,g) and a real constant λ satisfy the condition

$$i\partial\overline{\partial} f + \rho = \lambda\Omega, \tag{28.1}$$

then for every smooth vector field v on M, its divergence δv, and the function $Lv = \delta v - d_v f$ one has the identity

$$\nabla Lv - J\nabla LJv = -2\lambda v - JES, \quad \text{where } S = \pounds_v J, \tag{28.2}$$

E being the linear differential operator which sends any smooth endomorphism S of TM to the vector field ES having the components $(ES)^j = g^{jk}(ES)_k$ with $(ES)_k$ given by

$$(ES)_k = S_{k,p}^p - f_{,p} S_k^p, \qquad \text{that is,} \quad g(ES, \cdot) = \delta S - (df)S, \tag{28.3}$$

cf. (17.8). The result of applying δ to (28.2): under the above assumptions,

$$\Delta Lv \,=\, -2\lambda\delta v \,-\, \delta(JES), \tag{28.4}$$

as one sees noting that

$$\delta(Jw) \,=\, 0 \quad \text{if } \ w \ \text{ is a local gradient}, \tag{28.5}$$

since $\delta(Jw) = (J_l^k w^l)_{,k} = J_l^k w^l{}_{,k} = \mathrm{tr}\,(J\nabla w)$ must then vanish due to skew-adjointness of J and self-adjointness of ∇w. Proof of (28.2), based on the relations

$$\begin{aligned} &\text{a) } v^k{}_{,pq} = v^k{}_{,qp} + R_{pqs}{}^k v^s, && \text{b) } J_k^p v^k{}_{,q} = J_q^k v^p{}_{,k} + S_q^p, \\ &\text{c) } J_l^q J_k^p R_{pqs}{}^k v^s = R_{kls}{}^k v^s = -R_{lp}v^p, && \text{d) } J_l^q J_q^k = -\delta_l^k, \\ &\text{e) } v^k{}_{,kl} - v^p{}_{,lp} = -R_{lp}v^p, && \text{f) } (\nabla df)J + J(\nabla df) = 2(\lambda\Omega - \rho) \end{aligned} \tag{28.6}$$

due, respectively, to: the Ricci identity [**KG**, formula (1.11.a)]; the fact that, in view of (17.4), $S = \pounds_v J$ equals $[J, \nabla v]$; (9.1) followed by the definition of the Ricci tensor; the equality $J^2 = -\mathrm{Id}$; the Bochner identity (18.1); and (11.5) (in which $dJ = 0$ as $\nabla J = 0$) combined with (28.1). In local coordinates, the component $[g(\nabla Lv - J\nabla LJv,\,\cdot\,)]_l$ of the left-hand side of (28.2) equals

$$v^k{}_{,kl} + J_l^q J_k^p v^k{}_{,pq} - v^k{}_{,l}f_{,k} - v^k f_{,kl} - J_l^q J_k^p v^k f_{,pq} - J_l^q J_k^p v^k{}_{,q}f_{,p}, \tag{28.7}$$

the first (or, third and fourth) term(s) representing the differential of δv (or, of $-d_v f$), the second (or, the last two) similarly corresponding to the analogous contributions from $-J\nabla LJv$. Note the signs, consistent with (11.3). Rewriting the second term of (28.7) via (28.6.a), then using (28.6.c) along with the relation $J_k^p v^k{}_{,qp} = J_q^k v^p{}_{,kp} + S_{q,p}^p$ (obvious from (28.6.b) since $\nabla J = 0$) and, in turn, applying (28.6.d), we see that the sum of the first two terms is $v^k{}_{,kl} - v^p{}_{,lp} - R_{lp}v^p + J_l^q S_{q,p}^p$ which, from (28.6.e), amounts to $-2R_{lp}v^p + J_l^q S_{q,p}^p$.

Next, the fourth and fifth terms of (28.7) add up to $2R_{lp}v^p - 2\lambda v_l$. Namely, with $u = Jv$, their sum is the lth component of $-[(\nabla df)(v,\,\cdot\,) + (\nabla df)(Jv, J\,\cdot\,)] = (\nabla df)(Ju,\,\cdot\,) - (\nabla df)(u, J\,\cdot\,) = [(\nabla df)J + J(\nabla df)](u,\,\cdot\,)$ which, according to (28.6.f), is nothing else than $2(\lambda\Omega - \rho)(u,\,\cdot\,) = 2(\lambda g - r)(Ju,\,\cdot\,) = 2(r - \lambda g)(v,\,\cdot\,)$, as required.

Finally, the last term of (28.7), successively rewritten with the aid of (28.6.b) and (28.6.d), becomes $v^k{}_{,l}f_{,k} - J_l^q f_{,p}S_q^p$. Consequently, (28.7) equals

$$(-2R_{lp}v^p + J_l^q S_{q,p}^p) \,-\, v^k{}_{,l}f_{,k} \,+\, (2R_{lp}v^p - 2\lambda v_l) \,+\, (v^k{}_{,l}f_{,k} - J_l^q f_{,p}S_q^p),$$

that is, $-2\lambda v_l + J_l^q(S_{q,p}^p - f_{,p}S_q^p)$, and (28.2) follows, with the minus sign due to (11.3).

Time-dependent objects (such as differential operators, including connections, and tensor fields, including Riemannian metrics and functions) on a given manifold, defined to be objects that depend smoothly on a time parameter t ranging over a fixed interval, in the sense that their local-coordinate components C^∞ are functions of the coordinates and t. The convention that those objects' dependence on t is usually suppressed in the notation, and $(\ )^{\boldsymbol\cdot}$ stands for d/dt. The relations

$$\text{i) } (\mathrm{d}g)^{\boldsymbol\cdot} = \varphi\,\mathrm{d}g, \qquad \text{ii) } \mathrm{tr}_g\,\dot g = 2\varphi, \qquad \text{iii) } \dot\delta = d\varphi \tag{28.8}$$

satisfied by any time-dependent Riemannian metric g on an oriented manifold, its volume form $\mathrm{d}g$, and its associated divergence operator δ, (28.8.i) being the ***definition*** of a time-dependent function φ, and (28.8.iii) reading $(\delta v)^{\cdot} = d_v\varphi$ for every time-independent smooth vector field v (its time-independence meaning that $\dot v = 0$). The case of a time-dependent Kähler metric g on an almost-complex manifold that is ***fixed*** (and so $\dot J = 0$), with

$$\text{(28.9)} \qquad \varphi = \Delta\chi \qquad \text{and} \qquad \dot\rho = -i\partial\overline{\partial}\Delta\chi \qquad \text{whenever} \qquad \dot\Omega = 2i\partial\overline{\partial}\chi$$

for a time-dependent function χ, where φ is characterized by (28.8.i). Proofs of (28.8.ii) – (28.8.iii): applying (12.9) to the contracted version $2\Gamma_{jk}^{j} = g^{jl}\partial_k g_{jl}$ of the Christoffel-symbol formula $2\Gamma_{jk}^{l} = g^{ls}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})$ and, respectively, noting that $\mathrm{d}g$ has the component function $(\det[g_{jl}])^{1/2}$, we get

$$\text{(28.10)} \qquad \text{a) } 2\Gamma_{jk}^{j} = \partial_k \log\det[g_{jl}], \qquad \text{b) } 2\varphi = (\log\det[g_{jl}])^{\cdot}.$$

Thus, (28.10.b) and (12.9) yield $2\varphi = (\log\det[g_{jl}])^{\cdot} = g^{jl}\dot g_{jl} = \mathrm{tr}_g\,\dot g$. Next, applying d/dt to $2\delta v = 2\partial_j v^j + 2\Gamma_{jk}^{j}v^k = 2\partial_j v^j + v^k\partial_k \log\det[g_{jl}]$, cf. (28.10.a), and switching d/dt with ∂_k, we obtain (28.8.iii) from (28.10.b).

Proof of (28.9): as $\dot g J = \dot\Omega = 2i\partial\overline{\partial}\chi$, we have $\dot g = -2(i\partial\overline{\partial}\chi)J$. Hence, by (28.8.ii) and (11.6), $2\varphi = \mathrm{tr}_g\,\dot g = 2\Delta\chi$, that is, $\varphi = \Delta\chi$. Finally, let us set $\hat g = g(\hat t)$ for a fixed value $\hat t$ of t, so that $\mathrm{d}\hat g = \gamma\,\mathrm{d}g$ with a time-dependent positive function γ, and (12.3), with time-independent left-hand side, gives $\dot\rho = i\partial\overline{\partial}(\log\gamma)^{\cdot}$. At the same time, from (28.8.i), $0 = (\mathrm{d}\hat g)^{\cdot} = (\gamma\,\mathrm{d}g)^{\cdot} = (\dot\gamma + \varphi\gamma)\,\mathrm{d}g$, so that $(\log\gamma)^{\cdot} = -\varphi = -\Delta\chi$, and our last claim follows. References: [**KG**, in Section 8: the paragraph following Lemma 8.3, the two paragraphs surrounding formula (8.10), and parts (i), (ii) of Lemma 8.7]. **Homework #28**.

March 23: A trivial consequence of (28.9):

$$\text{(29.1)} \qquad (i\partial\overline{\partial}f + \rho - \lambda\Omega)^{\cdot} = i\partial\overline{\partial}(\dot f - \Delta\chi - 2\lambda\chi)$$

whenever time-dependent functions χ, f and a time-dependent Kähler metric g on a fixed almost-complex manifold satisfy the condition $\dot\Omega = 2i\partial\overline{\partial}\chi$, while λ is a time-independent real constant. Futaki's theorem, stating that, ***on a compact almost-complex manifold M with $c_1(M) > 0$ or $c_1(M) < 0$, admitting a Kähler metric, the Futaki invariant (27.3) does not depend on the choice of a Kähler metric g having the property (27.5).*** Proof of Futaki's theorem: let two time-independent Kähler metrics $\tilde g, \hat g$, and functions $\tilde f, \hat f$ have the property that $i\partial\overline{\partial}\tilde f + \tilde\rho = \lambda\tilde\Omega$ and $i\partial\overline{\partial}\hat f + \hat\rho = \lambda\hat\Omega$, for the same real constant λ (which may be achieved by rescaling $\tilde g$). Thus, $\lambda(\hat\Omega - \tilde\Omega) = i\partial\overline{\partial}(\hat f - \tilde f - \log\gamma)$, with γ as in (12.3) for $\tilde g$ rather than g. The condition $\dot\Omega = 2i\partial\overline{\partial}\chi$, where $\chi = (\hat f - f - \log\gamma)/\lambda$, now clearly holds for the time-dependent Kähler metric defined to be the line segment $[0,1] \ni t \mapsto g = g(t)$ joining $\tilde g$ to $\hat g$. Denoting by f the unique time-dependent function having $\dot f = \Delta\chi + 2\lambda\chi$ and $f(0) = \tilde f$, we see that, by (29.1), $i\partial\overline{\partial}f + \rho = \lambda\Omega$ for every t. Thus, up to an additive constant, $\hat f = f(1)$ (see the claim about the kernel in the $\partial\overline{\partial}$ Lemma, February 5), and so we just need to show that the Futaki invariant $\boldsymbol{F} = \boldsymbol{F}(t)$ of the metric $g(t)$ is the same for all $t \in [0,1]$. To achieve this, we note that (27.6) gives $s_{\mathrm{avg}} = m\lambda/2$, which does not depend on t, and so, applying d/dt to $(s_{\mathrm{avg}})^{-m}\boldsymbol{F}v$, where $v \in \mathfrak{h}(M)$, we obtain the integral over M of $d_v\dot f\,\mathrm{d}g + d_v f\,(\mathrm{d}g)^{\boldsymbol\cdot} = (d_v\dot f + \varphi\,d_v f)\,\mathrm{d}g = [d_v(\Delta\chi + 2\lambda\chi) + (\Delta\chi)\,d_v f]\,\mathrm{d}g$. Cf. (28.8.i) and (28.9). Formula (23.19) and its obvious consequence

$$\psi\Delta\phi \approx \phi\Delta\psi \tag{29.2}$$

for any smooth functions ψ,ϕ, with the convention (18.6), which arises since, by (23.19), $\psi\Delta\phi = \psi\delta(\nabla\phi) \approx g(\nabla\psi,\nabla\phi)$ (and the last expression is symmetric in ψ,ϕ), now yield $d_v(\Delta\chi + 2\lambda\chi) \approx -(\Delta\chi + 2\lambda\chi)\delta v \approx -(\chi\Delta\delta v + 2\lambda\chi\delta v)$ and $(\Delta\chi)\,d_v f \approx \chi\Delta(d_v f)$. Thus, due to the divergence theorem, integrating $[d_v(\Delta\chi + 2\lambda\chi) + (\Delta\chi)\,d_v f]\,\mathrm{d}g$ over M we get the L^2 inner product of $-\chi$ and the function $\Delta(\delta v - d_v f) + 2\lambda\delta v = \Delta Lv + 2\lambda\delta v = 0$, the last two equalities being immediate from (27.2) and (28.4) with $S = \pounds_v J = 0$.

Ricci solitons, defined to be Riemannian manifolds (M,g) such that

$$\pounds_w g + r = \lambda g \quad \text{or, equivalently,} \quad w_{j,k} + w_{k,j} + R_{jk} = \lambda g_{jk} \tag{29.3}$$

for some smooth vector field w on M and some real constant λ. The role of Hamilton's Ricci-flow equation

$$\dot g = -2r \tag{29.4}$$

in Perelman's proof of the three-dimensional Poincaré conjecture. The interpretation of compact Ricci solitons as the fixed points of the Ricci flow projected, from the space of metrics, onto its quotient modulo diffeomorphisms and scalings.

References: [**KG**, Lemma 5.2 in Section 5]. **Homework #29**.

March 26: Kähler-Ricci solitons, by which one means those Ricci solitons (M,g) which at the same time are Kähler manifolds for some almost-complex structure J on M. The observation that tr_g applied to (29.3) yields

$$2\delta w + s = n\lambda, \quad \text{where } n = \dim_{\mathbb{R}} M. \tag{30.1}$$

The conclusion – obtained by integrating (30.1) and using the divergence formula – that when M is compact and oriented, λ is uniquely determined by g (being equal to $1/n$ times the average scalar curvature), which in turn makes w unique up to the addition of a Killing field. Gradient Ricci solitons: the Ricci solitons with w in (29.3) which is a gradient, $w = \nabla f/2$, that is, the Riemannian manifolds (M,g) satisfying the condition

$$\nabla df + r = \lambda g \quad \text{or, equivalently,} \quad f_{,jk} + R_{jk} = \lambda g_{jk} \ \text{ for a constant } \lambda \tag{30.2}$$

and some smooth function f. The result of Perelman, which we will not prove or use:

(30.3) every compact Ricci soliton is a gradient Ricci soliton.

Einstein manifolds as the simplest examples of (gradient) Ricci solitons. A corollary:

$$Dw = 0 \quad \text{whenever } \pounds_w g + r = \lambda g \text{ with a constant } \lambda, \tag{30.4}$$

which follows since taking the differentials of both sides of (30.1) and subtracting the result of applying 2δ to (29.3) we obtain $2g(Dw,\,\cdot\,)=0$ from (30.1), (18.3) and the equality $2\delta r=ds$ known as the Bianchi identity for the Ricci tensor [**FR**]. The immediate consequence that, by (18.10),

(30.5) $\quad w$ in (29.3) is holomorphic for any compact Kähler-Ricci soliton (M,g).

The general local fact that, given a Kähler manifold (M,g) and $v\in\mathfrak{h}(M)$,

$$\text{(30.6)}\qquad \pounds_v g \quad \text{is Hermitian, and} \quad (\pounds_v g)J\,=\,d[g(Jv,\,\cdot\,)].$$

(Thus, besides being, due to the Hermitian property of $\pounds_v g$, a – necessarily skew-Hermitian – differential 2-form, $(\pounds_v g)J$ must then in addition be exact.) Proof of (30.6): for u,A,B as in (23.4), $\pounds_v g\,\sim\,A+A^*$ by (17.2), with the notation of the lines preceding (11.1), and so $(\pounds_v g)J\,\sim\,(A+A^*)J=B-B^*$, while $B-B^*\,\sim\,d[g(Jv,\,\cdot\,)]$ due to (11.9). An immediate corollary of (30.6): in the case where M is compact, the $\partial\overline{\partial}$ Lemma (see February 5) then also gives

$$\text{(30.7)}\qquad (\pounds_v g)J\,=\,i\partial\overline{\partial}f \quad \text{for some smooth function } f.$$

The conclusion that, in view of (30.5) and (30.7), *in every compact Kähler-Ricci soliton, multiplying* (29.3) *from the right by* J *one obtains* (27.5), and so, just as in the Kähler-Einstein case, $c_1(M)$ *is positive, negative or zero for any compact almost-complex manifold* M *admitting a Kähler-Ricci soliton metric*, the sign being the same as that of the soliton constant λ in (29.3). More on complexifications of real vector spaces, with an example of the latter (or, former) provided by the spaces

$$\text{(30.8)}\qquad \mathcal{F}M \quad \text{and its complexification} \quad \mathcal{F}_{\mathbb{C}}M,$$

consisting of all real-valued (or, complex-valued) smooth functions on a manifold M. The real-part and conjugation operators in a complexified space. The fact that *a complex-linear operator from a complex space into a complexification is uniquely determined by its real part* (which may be any real-linear operator). An application of this last fact resulting in the differential operator P, sending smooth vector fields v on a compact Kähler manifold to smooth complex-valued functions, and defined to be the unique complex-linear operator having the real part L with (27.2) or, explicitly,

$$\text{(30.9)}\qquad Pv\;=\;Lv\;-\;iLJv.$$

The Tian-Zhu invariant $\boldsymbol{T}$ of a compact Kähler manifold (M,g), defined to be the function

$$\text{(30.10)}\qquad \boldsymbol{T}:\mathfrak{h}(M)\to\mathbb{C},\quad \text{given by}\quad \boldsymbol{T}(v)\,=\,(s_{\mathrm{avg}})^m\int_M e^{Pv}\,\mathrm{d}g,\quad \text{where } m=\dim_{\mathbb{C}}M,$$

The (redundant) assumption that $\dim\mathfrak{h}(M)<\infty$, leading to the formula

$$\text{(30.11)}\qquad d\boldsymbol{T}_v u\,=\,(s_{\mathrm{avg}})^m\int_M e^{Pv}Pu\,\mathrm{d}g$$

for the differential of $\boldsymbol{T}$ at any $v \in \mathfrak{h}(M)$ and the observation that, in view of (30.11) and complex-linearity of P, *the function* $\boldsymbol{T} : \mathfrak{h}(M) \to \mathbb{C}$ *is holomorphic,* while (30.11) evaluated at $v = 0$ reads, as a consequence of the divergence theorem,

$$\boldsymbol{F} \,=\, -\mathrm{Re}\, d\boldsymbol{T}_0, \tag{30.12}$$

$\boldsymbol{F}$ being the Futaki invariant (27.3).

References: [**KG**, Section 8]. **Homework #30**.

March 28: The Lie algebra $\mathcal{X}M$ of all smooth vector fields on a manifold M, and the associative algebras $\mathcal{F}M, \mathcal{F}_{\mathbf{c}}M$ appearing in (30.8). The case of a compact Kähler manifold (M, g) and a fixed smooth function f on M, leading to three differential operators

$$\partial : \mathcal{F}_{\mathbf{c}}M \to \mathcal{X}M, \qquad P : \mathcal{X}M \to \mathcal{F}_{\mathbf{c}}M, \qquad \Theta : \mathcal{F}_{\mathbf{c}}M \to \mathcal{F}_{\mathbf{c}}M, \tag{31.1}$$

all of them complex-linear, namely, the complex-gradient operator ∂ given by

$$\partial\psi \,=\, \nabla\mathrm{Re}\psi \,+\, J\nabla\mathrm{Im}\psi, \tag{31.2}$$

P with (30.9), that is, $Pv = Lv - iLJv$, for L as in (27.2), and

$$\Theta \,=\, \Delta \,-\, d_u \,-\, i\,d_{Ju}, \quad \text{where} \;\; u = \nabla f, \tag{31.3}$$

while $\Delta, d_u, d_{Ju} : \mathcal{F}M \to \mathcal{F}M$ are complex-linearly extended to $\mathcal{F}_{\mathbf{c}}M$. The relation

$$P\partial\psi \,=\, \Theta\psi \;\; \text{for any} \;\; (M,g), f \;\; \text{as above and any} \;\; \psi \in \mathcal{F}_{\mathbf{c}}M, \tag{31.4}$$

trivially derived from the definitions of ∂ and P. Futaki's theorem:

$$\begin{aligned}&\text{if a compact Kähler manifold } (M,g) \text{ satisfies (27.5), then, for } f \text{ as in (27.5),}\\ &\partial \text{ maps } \mathrm{Ker}\,(\Theta + 2\lambda) \text{ isomorphically onto } \mathfrak{h}(M)\text{, with the inverse } -P/(2\lambda).\end{aligned} \tag{31.5}$$

The conclusion, in any Kähler manifold (M, g), stating that

$$v \,\mapsto\, ES, \quad \text{where} \;\; S = \pounds_v J, \quad \text{is a complex-linear operator} \;\; \mathcal{X}M \to \mathcal{X}M, tag 31.6$$

for E as in (28.3), due to the following antilinearity relations:

$$\pounds_{Jv} J \,=\, -(\pounds_v J)J, \qquad E(SJ) \,=\, -JES. \tag{31.7}$$

The first of them is immediate from (17.4) and (23.4), the second from the obvious local-coordinate equality $[E(SJ)]_k = (S^p_{l,p} - f_{,p}S^p_l)J^l_k$, cf. (28.3), resulting in the minus sign consistent with (11.3). The observation that, given smooth functions real-valued functions ψ, ϕ on a Kähler manifold, one (or both) compactly supported, we have

$$(\nabla\psi, J\nabla\phi) \,=\, 0, \tag{31.8}$$

$(\,,\,)$ being the L^2 inner product. In fact, with the convention (18.6), $\psi^{,k} J_k^l \phi_{,l} \approx -\psi J_k^l \phi_{,l}{}^k$, which equals zero due to skew-adjointness of J and self-adjointness of ∇w for $w = \nabla\phi$, cf. (23.2), and so (31.8) follows from the divergence theorem. The conclusion, immediate from (31.8) and (31.2), that on any compact (connected) Kähler manifold,

(31.9) $\quad$ $\mathrm{Ker}\,\partial$ consists precisely of all constant complex-valued functions.

References: [**KG**, Section 8]. **Homework #30**.

March 30: Before proving (31.5), it is useful to establish the relations

(32.1) $\quad$ a) $(\Theta + 2\lambda)\psi = -iPES$ if $\psi = Pv$, $\qquad$ b) $\partial Pv = -2\lambda v - JES$,

valid whenever $i\partial\overline{\partial} f + \rho = \lambda\varOmega$ for a real constant λ and a smooth function f on a Kähler manifold (M, g), whether compact or not, where, with any $v \in \mathcal{X}M$ and $\psi \in \mathcal{F}_{\mathbf{c}}M$, one sets $S = \pounds_v J$, and the operators P, E, ∂, Θ are given by (30.9), (28.3) and (31.2) – (31.3). First, (32.1.b) is nothing else than (28.2). As for (32.1.a), complex-linearity of P allows us to rewrite its right-hand side as $-PJES$. At the same time, due to the complex-linear dependence of both sides on v (immediate from (32.1.b) as well as, separately, from (31.6)), and the italicized statement following (30.8), it suffices to verify equality between the real parts of both sides, that is, between $\mathrm{Re}\,[(\Theta + 2\lambda)Pv]$ and $-\mathrm{Re}\,PJES$. This last equality in turn reads $\Delta Lv + 2\lambda\delta v + \delta(JES) = g(u, \nabla Lv - J\nabla LJv + 2\lambda v + JES)$, with $u = \nabla f$, both sides of which equal zero according to (28.4) and (28.2). An obvious consequence of (32.1.a) – (32.1.b) with $S = 0$: for a Kähler manifold (M, g),

(32.2) $\quad$ P maps $\mathfrak{h}(M)$ injectively into $\mathrm{Ker}\,(\Theta + 2\lambda)$ whenever (27.5) holds.

The μ-adjoint $\varPi^*$ of a real/complex linear operator $\varPi$, characterized by

(32.3) $$(\varPi\psi, \phi) \;=\; (\psi, \varPi^*\phi),$$

where μ is a fixed smooth positive volume form on an oriented manifold M and $(\,,\,)$ denotes the L^2 inner product of compactly supported smooth sections of any given real/complex vector bundle over M, associated with μ and any fixed Riemannian/Hermitian fibre metric in the bundle, while $\varPi$ sends compactly supported smooth sections of one such bundle to analogous sections of the other. Uniqueness of the μ-adjoint when it exists. Some obvious facts: generally, for composites of operators as above,

(32.4) $\quad$ $(\varPi\varLambda)^* = \varLambda^*\varPi^*$, $\quad$ both $\varPi\varPi^*$ and $\varPi^*\varPi$ are self-adjoint, while $\varPi^{**} = \varPi$,

and a trivial observation, of interest only when M is compact:

(32.5) $\quad$ $\varPi^*\varPi$ is self-adjoint, nonnegative, and $\mathrm{Ker}\,\varPi^*\varPi = \mathrm{Ker}\,\varPi$.

The Cauchy-Riemann operator H sending each smooth vector field v on an almost-complex manifold to the endomorphism $\pounds_v J$ of its tangent bundle; in the Kähler case,

(32.6) $$Hv \;=\; \pounds_v J, \qquad \text{that is,} \quad Hv \;=\; [J, \nabla v],$$

cf. (17.4). The dg-adjoint H^* of the Cauchy-Riemann operator of a Kähler manifold (M,g), sending any smooth real-linear endomorphisms S of TM to $H^*S \in \mathcal{X}M$ given by

$$g(H^*S,\,\cdot\,)\;=\;\delta[J,S^*] \tag{32.7}$$

with δ as in (17.8). Justification of (32.7) based on first observing that any such S has

$$[J,S]^*\;=\;[J,S^*] \tag{32.8}$$

and so, for the fibre metric $\langle\,,\rangle$ in TM given by (11.8), any S as above, and $v\in\mathcal{X}M$, we obtain, from (32.6) and (32.8), $\langle Hv,S\rangle=\operatorname{tr}S(Hv)^*=\operatorname{tr}S[J,\nabla v]^*=\operatorname{tr}S[J,(\nabla v)^*]$. With the convention (18.6), one thus gets, in local coordinates, $\langle Hv,S\rangle=S_l^k(J_p^l v_k{}^{,p}-v_p{}^{,l}J_k^p)\approx S_l^{k,l}J_k^p v_p-S_l^{k,p}J_p^l v_k=v_k(J_l^k S_p^{l,p}-S_l^{k,p}J_p^l)=v_k[J,S]_p^{k,p}=\xi(v)$, the 1-form ξ being the divergence of $[J,S]^*$, defined as in (17.9). Now (32.8) yields $\xi=\delta[J,S^*]$, and (32.7) follows via integration by parts (that is, from the divergence theorem).

The easily-verified fact that, if Π has the μ-adjoint Π^* and f is a smooth function on M, then Π has the $e^{-f}\mu$-adjoint Π_f^* with

$$\Pi_f^*\phi\;=\;e^f\Pi^*(e^{-f}\phi). \tag{32.9}$$

The e^{-f}-weighted L^2 inner product $(\,,)_f$, related to $(\,,)$, the original one, by

$$(\,\cdot\,,\phi)_f\;=\;(\,\cdot\,,e^{-f}\phi), \tag{32.10}$$

so that Π_f^* in (32.9) is at the same time the adjoint of Π relative to $(\,,)_f$:

$$(\Pi\psi,\phi)_f\;=\;(\psi,\Pi_f^*\phi)_f\,. \tag{32.11}$$

References: [**KG**, Section 8]. **Homework #32**.

April 2: The easily-verified relations which hold in any Riemannian manifold with a fixed smooth function f, the divergence δ being applied to any smooth vector field v or, respectively, any smooth endomorphism S of the tangent bundle, cf. (17.8):

$$Lv\;=\;e^f\delta(e^{-f}v),\qquad ES\;=\;e^f\delta(e^{-f}S), \tag{33.1}$$

for L,E given by (27.2) and (28.3). The dg-adjoint and e^{-f}dg-adjoint of the gradient operator $\nabla:\mathcal{F}M\to\mathcal{X}M$ of a Riemannian manifold (M,g), where f is a fixed smooth function on M, as well as the e^{-f}dg-adjoint of the complex-gradient operator ∂, cf, (31.1), in the case of a Kähler manifold, given by

$$\text{a)}\;\;\delta=-\nabla^*,\qquad\text{b)}\;\;L=-\nabla_f^*,\qquad\text{c)}\;\;P=-\partial_f^*. \tag{33.2}$$

δ,L,P being the divergence $\mathcal{X}M\to\mathcal{F}M$ and the operators with (27.2), (28.3). Proof of (33.2): the claims (33.2.a) and (33.2.b) are obvious from (23.19), via the divergence theorem and, respectively, from (33.2.a) combined with (32.9).

(33.3) $\Theta = -\partial_f^*\partial$ is $(\,,\,)_f$-self-adjoint, nonpositive, and $\operatorname{Ker}\Theta = \mathbb{C}$,

$\mathbb{C}$ being here the space of constant complex-valued functions on M. The conclusion that, for any compact almost-complex manifold M,

(33.4) $\mathfrak{h}(M) = \{0\}$ if $c_1(M) < 0$ and M admits a Kähler metric,

as $c_1(M) < 0$ yields $\lambda < 0$ in (27.5); thus, due to nonpositivity of Θ, cf. (33.3), -2λ cannot be an eigenvalue of Θ. Hence $\operatorname{Ker}(\Theta + 2\lambda) = \{0\}$, and (32.2) gives (33.4).

(33.5) $JA + AJ$ commutes, $[J, A]$ anticommutes with J,

(33.6) if S is self-adjoint, so are $[J, S]$ and $[J, JS]$

(33.7) $H_f^* Hv = 2JES^*$, where $S = \pounds_v J$,

(33.8) $(\pounds_v J)^* = \pounds_v J$ whenever $v = \partial\psi$ for any $\psi \in \mathcal{F}_{\mathbf{c}}M$

Corollary: in a Kähler manifold (M, g), if $i\partial\overline{\partial}f + \rho = \lambda\varOmega$, with $f \in \mathcal{F}M$ and $\lambda \in \mathrm{I\!R}$, then

(33.9) $H_f^* H\partial\psi = -2\partial[(\Theta + 2\lambda)\psi]$ for all $\psi \in \mathcal{F}_{\mathbf{c}}M$

Proof of Futaki's theorem (31.5): due to (32.2) and (33.9), the latter combined with (32.5), P and ∂ constitute complex-linear operators $\mathfrak{h}(M) \to \operatorname{Ker}(\Theta + 2\lambda)$ and, respectively, $\operatorname{Ker}(\Theta{+}2\lambda) \to \mathfrak{h}(M)$ while, by (31.4) and (32.1.b) with $S = 0$, both resulting compositions $P\partial$ and ∂P equal -2λ times the identity.

References: [**KG**, Section 8]. **Homework #33.**

April 4: The vector space, arising for a compact Kähler manifold (M, g) with (27.5),

(34.1) $\mathcal{B} = \operatorname{Ker}(\Theta + 2\lambda) \cap \operatorname{Ker} d_{Ju} = \{\psi \in \mathcal{F}_{\mathbf{c}}M : (\Theta + 2\lambda)\operatorname{Re}\psi = (\Theta + 2\lambda)\operatorname{Im}\psi = 0\}$,

with $u = \nabla f$ and the second equality due to (31.3). The fact that, if we denote by $\mathfrak{k} = \mathfrak{i}(M, g)$ the real Lie algebra of all Killing fields, and by $\mathfrak{p}$ the space of all holomorphic gradients of real-valued functions, then

(34.2) ∂ maps $\mathcal{B}$ isomorphically onto $\mathfrak{k} \oplus \mathfrak{p}$, while $\mathfrak{p} = J\mathfrak{k}$

and, under the isomorphism ∂ in (34.2),

(34.3) $\mathfrak{p}$ and $\mathfrak{k}$ correspond to the summands $\operatorname{Re}\mathcal{B}$ and $i\operatorname{Re}\mathcal{B}$ of $\mathcal{B}$,

$\mathrm{Re}\,\mathcal{B}$ being the space of all real-valued functions in $\mathcal{B}$ or, equivalently, real parts of all elements of $\mathcal{B}$. Proof of (34.2) – (34.3) reduced – in view of injectivity of ∂ on $\mathrm{Ker}\,(\Theta+2\lambda)$, cf. (31.4) – to showing that the ∂-images of $\mathrm{Re}\,\mathcal{B}$ and $i\,\mathrm{Re}\,\mathcal{B}$ are precisely $\mathfrak{p}$ and $\mathfrak{k}$.

Bochner's integral formula (19.2) for any compactly supported smooth vector field v on a Riemannian manifold (M,g), rewritten as

$$(34.4) \qquad (rv,v) \,=\, \|\delta v\|^2 \,-\, (A,A^*), \quad \text{where } \ A=\nabla v,$$

$(\,,)$ and $\|\;\|$ being the L^2 inner product and L^2 norm. The generalization of (34.4) involving an arbitrary fixed smooth function f on M, the e^{-f}-weighted L^2 inner product $(\,,)_f$, the corresponding e^{-f}-weighted L^2 norm $\|\;\|_f$, the "f-modified Ricci tensor" $h = \nabla df + r$, and the operator L given by (27.2):

$$(34.5) \qquad (hv,v)_f \,=\, \|Lv\|_f^2 \,-\, (A,A^*)_f, \quad \text{where } \ h=\nabla df \ \text{ and, again, } \ A=\nabla v.$$

Proof of (34.5): in local coordinates, (18.1) gives $e^{-f}h(v,v) = e^{-f}(f_{,jk}+R_{jk})v^jv^k = e^{-f}(f_{,jk}v^jv^k + v^k{}_{,jk}v^j - v^k{}_{,kj}v^j)$ and so, with the convention (18.6), one has $e^{-f}h(v,v) \approx e^{-f}(f_{,j}f_{,k}v^jv^k - f_{,j}v^j{}_{,k}v^k - f_{,j}v^jv^k{}_{,k} - v^k{}_{,j}v^j{}_{,k} + f_{,k}v^k{}_{,j}v^j + v^k{}_{,k}v^j{}_{,j} - f_{,j}v^k{}_{,k}v^j)$. As $v^k{}_{,k} = \delta v$ and $f_{,j}v^j = d_vf$, while $Lv = \delta v - d_vf$, the sum of the first, third and last two terms in parentheses equals $(Lv)^2$. At the same time, the second term cancels the fifth one, and the fourth term, $-\operatorname{tr} A^2$, is the inner product $-\langle A,A^*\rangle$, cf. (11.8), so that (34.5) follows.

References: [**KG**, Section 8]. **Homework #33**.

April 6: A generalization of (19.5): given a compact Kähler manifold (M,g) and a smooth function f on M such that

$$(35.1) \qquad h \,-\, JhJ \,\geq\, 2\lambda g, \quad \text{where } \ h=\nabla df,$$

with a positive constant λ, the operator Θ defined by (31.3) satisfies the inequality

$$(35.2) \qquad \tau \,\geq\, 2\lambda \ \text{ for every nonzero eigenvalue } \ \tau \ \text{ of } -\Theta.$$

(Context: since r is Hermitian, (35.2) with constant f amounts to (19.5); while, by (33.3), $-\Theta$ is $(\,,)_f$-self-adjoint and nonnegative, which causes its nonzero eigenvalues to be real and positive, and we also know that its kernel consists of constants.) To prove (35.2), we add to (34.5) its version for Jv rather than v, with the left-hand side of the latter formed by the $e^{-f}\mathrm{d}g$-integral of $h(Jv,Jv)$, noting that (35.1) gives $h(v,v)+h(Jv,Jv) \geq 2\lambda g(v,v)$, cf. (1.1), while, from (23.4), $\nabla u = JA$ whenever $u = Jv$ and $A=\nabla v$. Thus,

$$(35.3) \qquad 2\lambda\|v\|_f^2 \,\leq\, \|Pv\|_f^2 \,-\, [(A,A^*)_f \,+\, (JA,(JA)^*)_f],$$

since $\|Lv\|_f^2 + \|LJv\|_f^2 = \|Pv\|_f^2$ due to the equality $|Lv|^2 + |LJv|^2 = |Pv|^2$ of the $e^{-f}\mathrm{d}g$-integrands, immediate from (30.9). At the same time, setting $A=\nabla v$ and $S = \pounds_vJ$ we get $S=[J,A]$ according to (17.4), and so $\operatorname{tr} S^2 = \operatorname{tr}[(JA-AJ)(JA-AJ)] = 2\operatorname{tr}[A^2+(JA)^2]$, which is twice the $e^{-f}\mathrm{d}g$-integrand of $(A,A^*)_f \,+\, (JA,(JA)^*)_f$, and so (35.3) becomes

$$(35.4) \qquad 2\lambda\|v\|_f^2 \,\leq\, \|Pv\|_f^2 \,-\, (S,S^*)_f/2.$$

If, in addition, $v = \partial\psi$ for some $\psi \in \mathcal{F}_{\mathbf{c}}M$, one has, from (33.8), $S^* = S$, and (35.4) along with (31.4) imply that

$$2\lambda\|\partial\psi\|_f^2 \le \|P\partial\psi\|_f^2 - \|S\|_f^2/2 \le \|P\partial\psi\|_f^2 = \|\Theta\psi\|_f^2. \tag{35.5}$$

Let ψ now be an eigenfunction of $-\Theta$ for a nonzero eigenvalue τ. By (35.5) and (33.3),

$$2\lambda\|\partial\psi\|_f^2 \le \|\Theta\psi\|_f^2 = (\Theta\psi,\Theta\psi)_f = -\tau(\psi,\Theta\psi)_f = \tau(\psi,\partial_f^*\partial\psi)_f = \tau(\partial\psi,\partial\psi)_f = \tau\|\partial\psi\|_f^2,$$

which proves (35.2): ψ is nonconstant as $\tau \ne 0$, and so $\|\partial\psi\| > 0$, cf. (31.9). An immediate consequence of (32.2) and (35.2):

$$\begin{array}{l}\text{whenever a compact Kähler manifold } (M,g) \text{ with } \mathfrak{h}(M) \ne \{0\}\\ \text{satisfies } (27.5),\ 2\lambda \text{ is the lowest nonzero eigenvalue of } -\Theta.\end{array} \tag{35.6}$$

References: [**KG**, Section 8]. **Homework #33**.

April 9: The observation that, for a Killing field w on a Riemannian manifold,

$$\pounds_w \text{ commutes with } \Delta \text{ and with the gradient operator } \nabla, \tag{36.1}$$

and, in the case of a holomorphic Killing field w on a Kähler manifold,

$$\pounds_w \text{ commutes with the complex-gradient operator } \partial. \tag{36.2}$$

the first claim in (23.5)

$$\begin{array}{l}-i\,d_{Ju}\colon \mathrm{Ker}\,(\Theta + 2\lambda) \to \mathrm{Ker}\,(\Theta + 2\lambda) \text{ corresponds under the isomorphism}\\ \partial : \mathrm{Ker}\,(\Theta + 2\lambda) \to \mathfrak{h}(M) \text{ in (31.5) to } \mathrm{Ad}\,u = [u,\,\cdot\,] : \mathfrak{h}(M) \to \mathfrak{h}(M).\end{array} \tag{36.3}$$

$$\mathfrak{h}(M) = \bigoplus_\tau \mathfrak{h}_\tau, \quad \text{with } \mathfrak{h}_\tau = \mathrm{Ker}\,(\mathrm{Ad}\,u - \tau) \text{ whenever } \tau \in \mathrm{I\!R}, \tag{36.4}$$

τ in in the direct-sum decomposition ranging over the spectrum of $\mathrm{Ad}\,u$. The fact that $\mathrm{Ad}\,u$ is, for any Lie algebra $\mathfrak{h}$ and any $u \in \mathfrak{h}$, a derivation of $\mathfrak{h}$, cf. the Lie-algebra homomorphism (23.17), and so

$$[\mathfrak{h}_\tau, \mathfrak{h}_\sigma] \subseteq \mathfrak{h}_{\tau+\sigma} \quad \text{if } \tau,\sigma \in \mathrm{I\!R}. \tag{36.5}$$

A consequence of (30.2): in any gradient Ricci soliton, with f, λ satisfying (30.2),

$$\Delta f - g(\nabla f, \nabla f) + 2\lambda f \quad \text{is constant}. \tag{36.6}$$

Proof of (36.6): one observes that, for $Y = \Delta f$ and $Q = g(u,u)$, where $u = \nabla f$,

$$\begin{array}{rl}\text{a)} & dY + ds = 0,\\ \text{b)} & 2r(u,\,\cdot\,) + 2dY + ds = 0,\\ \text{c)} & dQ + 2r(u,\,\cdot\,) - 2\lambda\, df = 0,\end{array} \tag{36.7}$$

and then forms the linear combination of these three equations with the coefficients $-1, 1$ and -1, getting $d(Y - Q + 2\lambda f) = 0$. The three equations (36.7) are in turn obtained applying, to (30.2), tr_g followed by d, or twice the divergence operator δ, or – respectively – the operation $2a(u,\,\cdot\,)$, with a denoting the difference of the two sides in (30.2). More precisely, (18.1) for $v = u = \nabla f$ yields $\delta\nabla df = r(u,\,\cdot\,) + dY$, while $2\delta r = ds$ from the Bianchi identity for the Ricci tensor [**FR**], and $2(\nabla df)(u,\,\cdot\,) = dQ$

Gradient Ricci solitons: the Ricci solitons with w in (29.3) which is a gradient, $w = \nabla f/2$, that is, the Riemannian manifolds (M, g) satisfying the condition

$$\nabla df + r = \lambda g \quad \text{or, equivalently,} \quad f_{,jk} + R_{jk} = \lambda g_{jk} \quad \text{for a constant } \lambda \tag{30.2}$$

and some smooth function f.

References: [**KG**, Section 8]. **Homework #33**.

April 11: The version of (28.2) for Jv rather than v, which reads, due to (31.6),

$$\nabla LJv + J\nabla Lv = -2\lambda Jv + ES, \tag{37.1}$$

Two consequences of the same assumptions as in (28.2): with $S = \pounds_v J$,

$$\begin{aligned} &|\nabla Lv|^2 + 2\lambda d_v Lv + g(\nabla Lv, JES) = |\nabla LJv|^2 + 2\lambda d_{Jv} LJv - g(\nabla LJv, ES),\\ &2[g(\nabla Lv, \nabla LJv) + \lambda(d_{Jv}Lv + d_v LJv)] = g(\nabla Lv + J\nabla LJv, ES). \end{aligned} \tag{37.2}$$

Namely, the first left-hand side equals $g(\nabla Lv, \nabla Lv + 2\lambda v + JES) = g(\nabla Lv, J\nabla LJv)$, cf. (28.2), the (very obvious) invariance of which under the replacement of v with Jv amounts, by (37.1), to the first line in (37.2). Similarly, the remaining left-hand side in (37.2), $g(\nabla Lv, \nabla LJv + 2\lambda Jv) + g(\nabla LJv, \nabla Lv + 2\lambda v)$, rewritten via (28.2) and (37.1), becomes $g(\nabla Lv, J\nabla Lv + ES) + g(\nabla LJv, J\nabla LJv - JES)$, that is, the right-hand side, since J is skew-adjoint (and so $g(\nabla Lv, J\nabla Lv) = g(\nabla LJv, J\nabla LJv) = 0$). The relation

$$\Delta e^\psi = [\Delta\psi + g(\nabla\psi, \nabla\psi)]e^\psi, \tag{37.3}$$

valid for any smooth complex-valued function on a Riemannian manifold, with

$$g(\nabla\psi, \nabla\psi) = |\nabla \mathrm{Re}\psi|^2 - |\nabla \mathrm{Im}\psi|^2 + 2ig(\nabla\mathrm{Re}\psi, \nabla\mathrm{Im}\psi), \tag{37.4}$$

that is, g has been extended complex-bilinearly to ***complex vector fields*** (sections of the complexified tangent bundle), immediate when the ψ is real-valued, but also easily verified in the general complex-valued case. The equalities

$$\dot{L} = -2\lambda d\chi, \qquad (Pv)^{\boldsymbol\cdot} = 2i\lambda d_{Jv}\chi - 2\lambda d_v\chi, \tag{37.5}$$

Tian and Zhu's theorem:

and, since $(\mathrm{d}g)^{\boldsymbol\cdot} = (\Delta\chi)\,\mathrm{d}g$, we get $(s_{\mathrm{avg}})^{-m}\dot{\boldsymbol{T}}(v) = \int_M (2i\lambda d_{Jv}\chi - 2\lambda d_v\chi + \Delta\chi)e^{Pv}\,\mathrm{d}g$ from (...). Integrating by parts we see that this is equal to the L^2 inner product of χ and

$$\Delta e^{Pv} + 2\lambda(d_v e^{Pv} - i d_{Jv} e^{Pv}) + 2\lambda[\delta v - i\delta(Jv)]e^{Pv}. \tag{37.7}$$

To prove that (37.7) vanishes for every holomorphic vector field v, we use apply (37.3), with (37.4), to $\psi = Pv$, so that $\mathrm{Re}\,\psi = Lv$ and $\mathrm{Im}\,\psi = -LJv$, while and (37.7) equals e^{Pv} times

$$\begin{aligned}\Delta Lv\,+\,2\lambda\delta v\,-\,i\,[\Delta LJv\,+\,2\lambda\delta(Jv)]\\ +\,|\nabla Lv|^2\,+\,2\lambda d_v Lv\,-\,[|\nabla LJv|^2+2\lambda d_{Jv}LJv]\\ -\,2i\,[g(\nabla Lv,\nabla LJv)\,+\,\lambda(d_{Jv}Lv\,+\,d_v LJv)].\end{aligned}$$

Each of the three lines above is separately equal to zero, due to and, respectively,

Holomorphic functions $\phi : M \to \mathbb{C}$ on an almost-complex manifold M, characterized by $(d\phi)J = i\,d\phi$, as they are nothing else than holomorphic mappings from M into $\mathbb{C}$. The algebra of holomorphic functions. Holomorphicity of multiplicative inverses.

References: [**KG**, Section 8]. **Homework #33**.

April 13: Vector bundles over manifolds, local sections, local trivializations and their compatibility, in the sense of regularity of transition functions [**DG**, pp. 57-58]. Holomorphic complex vector bundles over almost-complex manifolds, defined analogously to smooth real/complex vector bundles over smooth manifolds [**DG**, p. 58], just with holomorphicity of transition functions rather than their smoothness. The tautological line bundle $\mathcal{T}$ over any complex projective space $\mathrm{P}V$, cf. [**DG**, p. 59]. Product bundles. Operations on holomorphic vector bundles: direct sum, Hom, the dual. Holomorphic vector-bundle morphisms, defined to be holomorphic sections of the Hom bundle, isomorphisms, and holomorphically trivial bundles. Subbundles and quotient bundles in the holomorphic category. The determinant bundle. The tangent bundle and canonical bundle of a complex manifold.

References: [**KG**, Section 8]. **Homework #33**.

April 16: The total space of a vector bundle [**DG**, p. 66]. The natural structure of a complex manifold on the total space of a holomorphic complex vector bundle over a complex manifold. Holomorphic sections as complex submanifolds of the total space, including the zero section, always identified with the base manifold. The tautological line bundle $\mathcal{T}$ over a complex projective space $\mathrm{P}V$ as a subbundle of the product bundle $\mathcal{P} = \mathrm{P}V \times V$, for any finite-dimensional complex vector space V. The natural isomorphic identifications

The natural biholomorphic identification

$$\mathcal{T}^* \,=\, \mathrm{P}(V\times\mathbb{C}) \smallsetminus \{\{0\}\times\mathbb{C}\}$$

for the tautological line bundle $\mathcal{T}$ over $\mathrm{P}V$, obtained by assigning the graph of ξ to any pair (Λ, ξ) with $\Lambda \in \mathrm{P}V$.

References: [**KG**, Section 8]. **Homework #33**.

April 18:

Tensor products

$$V\otimes\mathrm{I\!K} \,=\, V, \qquad \mathrm{Hom}\,(V,W) \,=\, V^*\otimes W.$$

$$T[\mathrm{P}V] \;=\; [\mathrm{Hom}\,(\mathcal{T},\mathcal{P}/\mathcal{T}).$$

$$\mathcal{T}^{\otimes(m+1)} \,=\, \det T^*[\mathrm{P}V].$$

References: [**KG**, Section 8]. **Homework #33**.

April 20:

References: [**KG**, Section 8]. **Homework #33**.

April 23:

References: [**KG**, Section 8]. **Homework #33**.