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COMPACT RICCI SOLITONS

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ABSTRACT. This is a self-contained exposition of results on compact Ricci solitons, with proofs phrased in the language of real Riemannian geometry.

CONTENTS

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Introduction

A *Ricci soliton* [63] is a Riemannian manifold (M,g) such that

$$\pounds_w g \,+\, \mathrm{r} \,=\, \lambda g \qquad \text{for a constant } \lambda \tag{0.1}$$

and for some C^∞ vector field w on M, where $\pounds_w$ is the Lie derivative and r denotes the Ricci tensor. If, in addition, g is a Kähler metric on a complex manifold M, we call (M,g) a *Kähler-Ricci soliton* (cf. [78], [17]). Metrics g with (0.1) are also of interest to physicists, who refer to them as *quasi-Einstein* [52], [33], [28].

Compact Ricci solitons are the fixed points of the Ricci flow $dg/dt = -2\mathrm{r}$ projected, from the space of metrics, onto its quotient modulo diffeomorphisms and scalings (see the end of Appendix U). Complete Ricci solitons in turn often arise as blow-up limits for the Ricci flow on compact manifolds [65], [107]. They also serve as model cases of various Harnack inequalities for the Ricci flow [64], which become equalitites when the flow consists of Ricci solitons. Finally, Kähler-Ricci solitons are natural candidates for 'optimal' Kähler metrics on compact complex manifolds whose first Chern class is positive, zero, or negative [111], and existing results establish this for compact complex *surfaces.* See the text following (1.7).

There is a vast literature on Ricci solitons, both compact and complete noncompact ones (see the bibliography). Its contents range from constructions of examples, through various structure theorems, to existence, uniqueness and classification results; the proofs use a variety of techniques, including the Ricci-flow approach and the continuity method for the complex Monge-Ampère equation.

This article is a presentation of known results on compact Ricci solitons, with all proofs phrased in the language of real Riemannian geometry. The reader need not be familiar with complex manifolds: Appendices F – L provide a self-contained exposition of relevant facts from Kähler geometry.

The discussion is limited to the compact case due to its relative simplicity. A comparable presentation of results on complete noncompact Ricci solitons does not seem possible in a text of this size.

For another such exposition, see the article [48] by Eminenti, La Nave and Mantegazza.

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1. Questions and answers

The most obvious examples of (0.1) are *Einstein solitons* (M,g), in which $\mathrm{r} = \lambda g$ and $\pounds_w g = 0$, that is, g is an Einstein metric with the Einstein constant λ and w is a Killing field for g. For obvious reasons (cf. the last line in Remark 4.5) this is automatically the case whenever (M,g) is a compact Ricci soliton and g is an Einstein metric, even if λ in (0.1) is not *assumed* to be the Einstein constant.

Koiso [78] and, independently, Cao [17] constructed examples of non-Einstein Kähler-Ricci solitons on simply connected compact complex manifolds in all even real dimensions $n \geq 4$. See §11. Each of their examples satisfies (0.1) with $\lambda > 0$ and a gradient vector field w, and has a nonconstant scalar curvature $\mathrm{s} > 0$.

Since the Riemannian product of two Ricci solitons with the same constant λ in (0.1) is a Ricci soliton, the Koiso-Cao examples have *trivial extensions* (their products with Einstein manifolds of positive scalar curvature), which shows that non-Kähler, non-Einstein compact Ricci solitons exist in all real dimensions $n \geq 6$.

One may ask whether the properties of the Koiso-Cao examples (and their trivial extensions), listed above, hold more generally. Namely, if (M, g) is a non-Einstein compact Ricci soliton of dimension n with the scalar curvature function s,

(1.1) Does it follow that $n \geq 4$? Must s be nonconstant? Must it be positive? Does the constant λ in (0.1) have to be positive?

All four questions have been answered in the affirmative: the first one by Hamilton [63] and Ivey [71], who showed that $n \neq 2$ and, respectively, $n \neq 3$, the second by Bourguignon [9], as a special case of a more general theorem, and the last two by Ivey [71], cf. also Friedan [52]. For details, see §4.

The motivations provided here for Questions (1.1) – (1.8) do not always reflect the historical chronology of events. For instance, Hamilton's proof [63] of the fact that $n \neq 2$ came before the Koiso-Cao examples [78], [17] used here to justify the question, and Bourguignon's result [9] even predates, by more than a decade, the very emergence of Ricci solitons as a subject of study.

Another question about compact Ricci solitons (M, g), suggested by the Koiso-Cao examples and their trivial extensions, is

(1.2) Can the vector field w with (0.1) always be replaced by a gradient, that is, must w be the sum of a gradient and a Killing vector field?

That the answer is 'yes' was shown by Perelman [101]. See §6.

On the other hand, since Ricci solitons form a generalization of Einstein manifolds, it is natural to ask whether the Myers and Bochner theorems for Einstein manifolds with positive Einstein constants remain valid for Ricci solitons. This amounts to the following questions about the fundamental group $\pi_1 M$ and the first Betti number $b_1(M)$ of a Ricci soliton (M, g) such that $\lambda > 0$ in (0.1):

(1.3) If M is compact, does $\pi_1 M$ have to be finite? Is it then at least true that $b_1(M) = 0$? Does completeness of (M, g) imply compactness of M?

That the answer to the first question in (1.3) is 'yes' was first shown by Li [84], then, independently, by Fernández-López and García-Río [50]. The same clearly follows for the second question, cf. [43]. For the third question, the answer is generally 'no' (as illustrated by the *Gaussian soliton*, in which w is the radial vector field on a Euclidean space; further counterexamples were found by Feldman, Ilmanen and Knopf [49, Theorem 1.5]). However, it is 'yes' under the additional assumption that $g(w, w)$ is bounded; this is, again, due to Fernández-López and García-Río [50]. More recently, Zhang [120] independently answered the first two questions in (1.3) using a more direct argument. See §5.

For an oriented compact Ricci soliton (M, g) in dimension $n = 4$, with the signature $\tau(M)$ and Euler characteristic $\chi(M)$, the following question arises in view of the Einstein case ([6], [68], [110]):

(1.4) If $n = 4$, must M satisfy the Thorpe inequality $3|\tau(M)| \leq 2\chi(M)$?

The answer is unknown, but as $b_1(M) = 0$ when $\lambda > 0$, while g must be Einstein if $\lambda \leq 0$ (cf. Questions (1.3) and (1.1)), a weaker conclusion is true: namely, Berger's inequality $\chi(M) > 0$ holds, if $n = 4$, not only for non-flat compact Einstein manifolds [6, Theorem 6.32], but also for all non-flat compact Ricci solitons.

Ma [88] showed that the answer to Question (1.4) is 'yes' under the additional assumption that the squared L^2 norm of the scalar curvature does not exceed $24\lambda^2$ times the volume of M. See Remark 5.14.

Returning to arbitrary dimensions $n \geq 2$, we have a further question: $n > 4$:

(1.5) Let (M, g) be a compact Ricci soliton, of any dimension n, with positive (or, nonnegative) curvature operator. Must (M, g) have constant sectional curvature (or, respectively, be a locally symmetric Einstein manifold)?

Question 1.5 has two separate motivations. First, the answer is 'yes' both if $n = 4$, by a result of Hamilton [62], and for $n \leq 3$, in view of what Hamilton [63] and Ivey [71] proved about the lowest dimensions (see the lines following Question 1.1). On the other hand, a 1974 theorem of Tachibana [109] provides an affirmative answer in the special case of Einstein manifolds of all dimensions. (In fact, Tachibana assumes only that (M, g) has harmonic curvature, which is generally much weaker than the Einstein condition.)

Böhm and Wilking [8] recently proved that, in the case of positive curvature operator, the answer to Question 1.5 is 'yes' in all dimensions (as conjectured by Hamilton). For details, see §7 and Appendix E.

The next two questions deal with Kähler-Ricci solitons. Namely, suppose that a compact Kähler manifold (M, g) satisfies (0.1) for some w and λ.

(1.6) Must w be holomorphic? If $\lambda \neq 0$, is g determined by the complex structure of M uniquely up to a complex automorphism and a scale factor?

The first of these questions is suggested by the Koiso-Cao examples, the second by uniqueness of Kähler-Einstein metrics (due to Calabi [16] for negative Einstein constants, to Bando and Mabuchi [3] for positive ones). The answer is 'yes' in both cases: holomorphicity of w appears to be a folklore result, while uniqueness of g was proved by Tian and Zhu [115]. See §8, Appendix M and §10.

For a compact complex manifold M with the first Chern class c_1, one may ask:

(1.7) Does M admit a Kähler-Ricci soliton whenever $c_1 < 0$, $c_1 = 0$ or $c_1 > 0$?

If $c_1 = 0$ or $c_1 < 0$, the answer is 'yes' and, in fact, a Kähler-*Einstein* metric exists on M, in view of Calabi's conjectures, proved by Aubin [2] and Yau [119]; however, a Kähler-Einstein metric need not exist when $c_1 > 0$, the simplest counterexamples being the compact complex surfaces obtained by blowing up one or two points in $\mathbf{C}\mathrm{P}^2$. (They admit no Kähler-Einstein metrics due to theorems of Lichnérowicz [85] and Matsushima [92].) Still, both surfaces do admit Kähler-Ricci solitons: the one-point blow-up appears among the Koiso-Cao examples [78], [17], and for the other surface this follows from a recent result of Wang and Zhu [116] (see §11).

The answer to question (1.7) is 'yes' in complex dimension 2, but 'no' in general; in the former case, this follows from combined results of Aubin [2], Yau [119], Koiso [78], Wang and Zhu [116], and Tian [111]. In the latter, counterexamples were found by Tian [111], in the form of compact complex manifolds with $c_1 > 0$ admitting neither a Kähler-Einstein metric nor a nontrivial holomorphic vector field. (As to why this precludes the existence of a Kähler-Ricci soliton, see Proposition 8.2.)

The questions on our final list, still open, are motivated by the scarcity of known examples. They have been raised by Gang Tian in various talks, especially for

$n = 4$. Here (M,g) is assumed to be a compact n-dimensional Ricci soliton:

(1.8) If $n = 4$ and g is not Einstein, must some finite Riemannian covering space of (M,g) be a Kähler manifold? If $n = 5$, is (M,g) necessarily an Einstein manifold? More generally, does the Riemannian universal covering of (M,g) have to be a Kähler manifold, an Einstein manifold, or a Riemannian product with one Kähler and one Einstein factor?

2. Preliminaries

Let (M,g) be a Riemannian manifold of dimension n. We always assume that M is connected and all functions, vector and tensor fields under considerations are C^∞ differentiable. The symbols $\nabla, R, \mathrm{r}, \mathrm{s}$ denote the Levi-Civita connection, curvature tensor, Ricci tensor and scalar curvature of g. Thus,

$$R(u,v)w = \nabla_v\nabla_u w - \nabla_u\nabla_v w + \nabla_{[u,v]}w \quad \text{for vector fields } u,v,w \tag{2.1}$$

and $\mathrm{r}(u,w) = \mathrm{tr}\,[v \mapsto R(u,v)w]$ for vectors $u,v,w \in T_xM$ at any point $x \in M$. Given vector fields u,v, we denote by $R(u,v)$ the vector-bundle morphism

$$R(u,v) : TM \to TM\,, \qquad \text{acting on vector fields by} \quad w \mapsto R(u,v)w. \tag{2.2}$$

Remark 2.1. The metric g will often be used to identify twice-covariant tensors a on M with bundle morphisms $A : TM \to TM$ by requiring that $g(Av,w) = a(v,w)$ for all vector fields v,w. Symmetry/skew-symmetry of a amounts to self-adjointness/skew-adjointness of A. We denote by $\langle\,,\rangle$ the inner product of twice-covariant tensors, so that $\langle a,b\rangle = \langle A,B\rangle$ for A,B related to a,b as above, with $\langle A,B\rangle = \mathrm{tr}\,AB^*$, where A^* is the (pointwise) adjoint of A. The symbols $|\,|$ and tr_g will stand for the corresponding norm and the g-trace. Thus, $\mathrm{tr}_g a = \langle g,a\rangle$ and

$$\text{i) } \mathrm{s} = \langle g,\mathrm{r}\rangle = \mathrm{tr}_g\mathrm{r}\,, \quad \text{ii) } |\mathrm{r}|^2 = |\mathrm{e}|^2 + \mathrm{s}^2/n\,, \quad \text{where iii) } \mathrm{e} = \mathrm{r} - \mathrm{s}g/n\,. \tag{2.3}$$

One calls e the *Einstein tensor* of g, or its *traceless Ricci tensor*.

Remark 2.2. The curvature tensor of (M,g) gives rise to two bundle morphisms $\hat{R} : [T^*M]^{\wedge 2} \to [T^*M]^{\wedge 2}$ and $\mathring{R} : [T^*M]^{\odot 2} \to [T^*M]^{\odot 2}$, known as the *curvature operators* acting on exterior 2-forms ω and, respectively, twice-covariant symmetric tensors a, and uniquely characterized by $[\hat{R}(\xi\wedge\eta)](w,w') = g(R(u,v)w,w')$, $[\mathring{R}(\xi\odot\xi)](w,w') = g(R(u,w)u,w')$ for $x \in M$, $u,v,w,w' \in T_xM$ and $\xi = \imath_u g$, $\eta = \imath_v g$. In local coordinates, $2(\hat{R}\omega)_{jk} = \omega^{lm}R_{jklm}$ and $(\mathring{R}a)_{jl} = a^{km}R_{jklm}$. (See [6, Defn. 1.131(b)], [12].) Our conventions about $\xi\wedge\eta$ and $\xi\odot\xi$ are

$$(\xi\wedge\eta)(w,w') = \xi(w)\eta(w') - \eta(w)\xi(w'), \quad (\xi\odot\xi)(w,w') = \xi(w)\xi(w'). \tag{2.4}$$

We let $\pounds_w$ stand for the Lie derivative in the direction of a vector field w on M. Thus, $\pounds_w f$ for a function f coincides with the directional derivative $d_w f$. Given a twice-covariant symmetric tensor a, the usual expression $(\pounds_w a)(u,v) = d_w[a(u,v)] - a([w,u],v) - a(u,[w,v])$ for vector fields u,v can be rewritten as $(\pounds_w a)(u,v) = (\nabla_w a)(u,v) + a(\nabla_u w,v) + a(u,\nabla_v w)$, that is,

$$\pounds_w a \;=\; \nabla_w a \;+\; a\nabla w \;+\; (\nabla w)^* a\,, \tag{2.5}$$

the two multiplications by a on the right-hand side being the compositions with A that corresponds to a as in Remark 2.1. Also, with ∇f denoting the g-gradient of a function f,

$$\text{a) } \pounds_w g = \nabla w + (\nabla w)^*, \quad \text{b) } \pounds_w g = 2\nabla df \text{ if } w = \nabla f. \tag{2.6}$$

(In fact, (a) follows from (2.5), and implies (b).) Here ∇w is treated as a vector-bundle morphism $TM \to TM$ sending any vector (or vector field) v to $\nabla_v w$, while $(\nabla w)^* : TM \to TM$ stands for its (pointwise) adjoint, and $a = \pounds_w g$ is identified with $A = \nabla w + (\nabla w)^*$ as in Remark 2.1. For a vector field w and a twice-covariant symmetric tensor a, we have

$$\text{(2.7)}\quad \text{i)}\ \delta w = \operatorname{tr}\nabla w\,, \quad \text{ii)}\ 2\delta\imath_w a \,=\, 2\imath_w\delta a \,+\, \langle a, \pounds_w g\rangle\,, \quad \text{iii)}\ \langle g, \pounds_w g\rangle \,=\, 2\delta w\,.$$

Here (i) defines the divergence operator δ, (iii) is obvious from (ii) (or (2.6.a)), and (ii) follows from (2.6.a) via the local-coordinate calculation $2(w^j a_{jk})^{,k} = 2w^j a_{jk}{}^{,k} + (w^{j,k} + w^{k,j})a_{jk}$. Next, for a vector field w and a function f,

$$\text{(2.8)}\qquad \text{a)}\ \ d_w f \,=\delta(fw) \,-\, f\delta w, \quad \text{where} \quad \text{b)}\ \ d_w f = \imath_w df = g(w, \nabla f)\,.$$

We can also apply δ to vector-bundle morphisms $A : TM \to TM$, such as ∇w, resulting in the 1-form δA that sends any vector field v to the function

$$\text{(2.9)}\qquad (\delta A)v \;=\; \delta(Av) \;-\; \operatorname{tr}(A\nabla v)\,,$$

the "product" of A and ∇v being the composite. We then further extend δ to twice-covariant symmetric tensors a by setting $\delta a = \delta A$, where A corresponding to a as in Remark 2.1. Given such a (an example of which is the Ricci tensor r), and a vector field v, we define the 1-form $\imath_v a$ by the usual formula $\imath_v a = a(v, \cdot\,)$. Thus, $v \mapsto \imath_v g$ is the "index-lowering" isomorphism $TM \to T^*M$. The relations

$$\text{(2.10)}\quad \text{i)}\ \imath_v g \,=\, df \ \text{ if } \ v = \nabla f, \quad \text{ii)}\ 2\imath_v a \,=\, dQ \ \text{ if } \ v = \nabla f,\ \ Q = |v|^2 \text{ and } a = \nabla df\,,$$

valid for any function $f : M \to \mathbf{R}$, follow since $d_w f = g(w, v)$ for all vectors w, while $2f^{,j} f_{,jk} = [f^{,j} f_{,j}]_{,k}$ in local coordinates. The divergence $\delta\xi$ of a 1-form ξ is given by $\delta\xi = \delta v$ for the vector field v with $\xi = \imath_v g$. Now δ may be applied twice in a row to a bundle morphism $A : TM \to TM$ such as ∇w or $(\nabla w)^*$. In addition, $\delta\xi$ has an obvious generalization to once-contravariant tensor fields on (M, g), with any number of covariant arguments, and

$$\text{(2.11)}\qquad \begin{array}{llll} \text{a)} & d\nabla w \,=\, -R(\,\cdot\,,\cdot\,)w\,, & \text{b)} & \imath_w \mathrm{r} \,=\, \delta\nabla w \,-\, d\delta w\,,\,, \\ \text{c)} & 2\delta\mathrm{r} \,=\, d\mathrm{s}\,, & \text{d)} & \delta R \,=\, -d\mathrm{r}\,, \\ \text{e)} & \langle \mathrm{r}, \pounds_w g\rangle \,=\, 2\delta\imath_w\mathrm{r} \,-\, d_w\mathrm{s}\,, & \text{f)} & \delta\delta\nabla w \,=\, \delta\delta(\nabla w)^* \end{array}$$

for any vector field w. Equalities (2.11.a) – (2.11.d) have the local-coordinate forms

$$\text{(2.12)}\qquad \begin{array}{llll} \text{a)} & w^j{}_{,kl} - w^j{}_{,lk} \,=\, R_{kls}{}^j w^s\,, & \text{b)} & R_{kl}w^k \,=\, w^k{}_{,lk} \,-\, w^k{}_{,kl}\,, \\ \text{c)} & 2R_j{}^k{}_{,k} \,=\, \mathrm{s}_j & \text{d)} & R_{jkl}{}^s{}_{,s} \,=\, R_{jl,k} - R_{kl,j}\,. \end{array}$$

The first three of them are known as the *Ricci identity*, the *Bochner* (or or *Weitzenböck*) *formula*, and the *Bianchi identity for the Ricci tensor.* To justify (2.11), note that (2.11.a) is, essentially, the definition of the curvature tensor R, (2.11.b), (2.11.d) and (2.11.c) are immediate if one applies a contraction to (2.11.a), the second Bianchi identity for R and, respectively, (2.11.d), while (2.11.e) follows from (2.7.ii) and (2.11.c). Finally, (2.11.f) is obvious since $\delta^2 = 0$ for the divergence operator δ acting on differential forms; namely, being skew-adjoint, $\nabla w - (\nabla w)^*$ corresponds, as in Remark 2.1, to a 2-form. Here is a direct local-coordinate verification of (2.11.f): $\delta\delta\nabla w - \delta\delta(\nabla w)^* \,=\, w^j{}_{,kj}{}^k - w^j{}_{,k}{}^k{}_j \,=\, 0$, immediate from (2.12.a) and symmetry of the Ricci tensor.

For functions $f : M \to \mathbf{R}$, (2.11.b) gives

$$\text{(2.13)}\qquad \imath_v\mathrm{r} \,=\, \delta a \,-\, dY \ \ \text{if} \ \ v = \nabla f,\ \ a = \nabla df \ \ \text{and} \ \ Y = \varDelta f.$$

The symbol $\varDelta$ will also stand for the 'rough Laplacian' acting on arbitrary tensors A, so that $\varDelta A$ is obtained from the second covariant derivative of A by g-contraction applied to the differentiation arguments. Thus, for a function f we have $\varDelta f = \delta\xi$, with the 1-form $\xi = df$, while, for any vector field w,

$$\text{(2.14)} \qquad \text{i)}\ \varDelta f \,=\, \delta\nabla f \,=\, \mathrm{tr}_g\nabla df \,=\, \langle g, \nabla df\rangle, \qquad \text{ii)}\ \varDelta\imath_w g \,=\, \delta(\nabla w)^*.$$

Relation (2.14.ii) is easily verified in local coordinates, using (2.5) and the Ricci identity (2.12.a).

We denote by $\mathrm{d}g$ and $\mathrm{V} = \int_M \mathrm{d}g \in (0,\infty]$ the volume element of g and the total volume of M relative to g. If M is compact, $f_{\max}$ and $f_{\min}$ stand for the extrema of a function $f : M \to \mathbf{R}$, while $f_{\mathrm{avg}} = \mathrm{V}^{-1}\int_M f\,\mathrm{d}g$ is its average value. We will repeatedly use the *divergence theorem*:

$$\text{(2.15)} \qquad \int_M \delta w\,\mathrm{d}g \,=\, 0 \quad \text{for any compactly supported vector field } w.$$

Given a function $f : M \to \mathbf{R}$ on a compact Riemannian manifold (M, g),

$$\text{(2.16)} \qquad f_{\mathrm{avg}} = 0 \ \text{ if and only if } \ f = \varDelta\phi \ \text{ for some } \ \phi : M \to \mathbf{R}.$$

Recall that a function is, by definition, C^∞-differentiable.

The 'if' part of (2.16) is obvious from (2.14.i) and (2.15). The 'only if' claim in (2.16) is one of the very few facts from analysis that are esed in this exposition.

From (2.15) and (2.8.a) it follows that, for a function f and a vector field w,

$$\text{(2.17)} \qquad \int_M f\delta w\,\mathrm{d}g \,=\, -\int_M d_w f\,\mathrm{d}g \ \text{ if } M \text{ is compact.}$$

For instance, given a function f on a compact Riemannian manifold (M, g),

$$\text{(2.18)} \qquad \int_M d_u f\,\mathrm{d}g \,=\, 0 \quad \text{if } \ u \ \text{ is a Killing field,}$$

since $\delta u = 0$. If $w = \nabla\phi$ is the gradient of a function ϕ, (2.17) becomes

$$\text{(2.19)} \qquad \int_M f\varDelta\phi\,\mathrm{d}g \,=\, -\int_M g(\nabla f, \nabla\phi)\,\mathrm{d}g = \int_M \phi\varDelta f\,\mathrm{d}g \ \text{ if } M \text{ is compact,}$$

which, applied to $\phi = f$, shows that

$$\text{(2.20)} \qquad \begin{array}{ll} \text{a)} & \int_M f\varDelta f\,\mathrm{d}g \,=\, -\int_M |\nabla f|^2\mathrm{d}g \ \text{ if } M \text{ is compact, and so} \\ \text{b)} & \text{a function } f : M \to \mathbf{R} \text{ is constant if } M \text{ is compact and } \varDelta f \ge 0. \end{array}$$

(Namely, as $\int_M \varDelta f\,\mathrm{d}g = 0$ by (2.15), the inequality $\varDelta f \ge 0$ yields $\varDelta f = 0$.) Another consequence of (2.15) is *Bochner's integral formula*

$$\text{(2.21)} \qquad \int_M \mathrm{r}(w,w)\,\mathrm{d}g \,=\, \int_M (\delta w)^2\mathrm{d}g \,-\, \int_M \mathrm{tr}\,(\nabla w)^2\,\mathrm{d}g,$$

valid for all compactly supported vector fields w on a Riemannian manifold (M, g) (and easily derived from (2.12.b)). An important special case of (2.21) arises when $w = \nabla\varphi$ is the gradient of a function:

$$\text{(2.22)} \qquad \int_M \mathrm{r}(\nabla f, \nabla f)\,\mathrm{d}g \,=\, \int_M (\varDelta f)^2\mathrm{d}g \,-\, \int_M |\nabla df|^2\,\mathrm{d}g.$$

In the case of oriented manifolds, (2.15) may be restated as the *Stokes formula* (which we need only in Appendix H): on an oriented n-dimensional manifold M,

$$\text{(2.23)} \qquad \int_M d\eta \,=\, 0 \quad \text{for any compactly supported } (n-1)\text{-form } \eta.$$

In fact, as M is oriented, we may treat the volume element $\mathrm{d}g$ of any fixed metric g as a positive differential n-form, and then $d\eta = (\delta w)\,\mathrm{d}g$ for the unique vector field w corresponding to η under the Hodge-star isomorphism $TM \to [T^*M]^{\wedge(n-1)}$

(in the sense that $\eta = \imath_w \,\mathrm{d}g$). The exterior derivative of a 1-form ξ or 2-form ζ acts on vector fields u, v, w by

$$\text{(2.24)}\qquad \begin{aligned} &\text{a)} \quad (d\xi)(u,v) \,=\, d_u[\xi(v)] \,-\, d_v[\xi(u)] \,-\, \xi([u,v])\,,\\ &\text{b)} \quad (d\xi)(u,v) \,=\, [\nabla_{\!u}\xi](v) \,-\, [\nabla_{\!v}\xi](u)\,,\\ &\text{c)} \quad (d\zeta)(u,v,w) \,=\, [\nabla_{\!u}\zeta](v,w) \,+\, [\nabla_{\!v}\zeta](w,u) \,+\, [\nabla_{\!w}\zeta](u,v)\,. \end{aligned}$$

Here (a) expresses our convention about $d\xi$, while (b) and (c) easily follow from the Leibniz rule, for any torsionfree connection ∇, such as the Levi-Civita connection of a Riemannian metric.

In any Riemannian manifold, for the four-times covariant tensor fields A, B with $A_{jklm} = R_j{}^p{}_l{}^s R_{pksm} - R_k{}^p{}_l{}^s R_{pjsm}$ and $B_{jklm} = R_{jk}{}^{sp} R_{splm}$, we have

$$\text{(2.25)}\quad R_{jklm,}{}^s{}_s = R_{jkl}{}^s{}_{,sm} - R_{jkm}{}^s{}_{,sl} + R_{jkl}{}^s R_{sm} - R_{jkm}{}^s R_{sl} - 2A_{jkml} - B_{jklm}\,.$$

In fact, by the second Bianchi identity, $R_{jklm,}{}^s{}_s = R_{jkl}{}^s{}_{,ms} - R_{jkm}{}^s{}_{,ls}$. The Ricci identity for R (analogous to (2.12.a) for vector fields w) now shows that $R_{jklm,}{}^s{}_s$ is the difference of the expression $R_{jkl}{}^s{}_{,sm} + R_{smj}{}^p R_{pkl}{}^s + R_{smk}{}^p R_{jpl}{}^s + R_{sml}{}^p R_{jkp}{}^s R_{jkl}{}^s R_{sm}$ and its version with l, m switched, which yields (2.25).

Finally, if $a = \pounds_w g$ for a vector field w on a Riemannian manifold (M, g), then

$$\text{(2.26)}\qquad 2w_{j,kl} \,=\, 2R_{jkl}{}^p w_p \,+\, a_{jl,k} \,+\, a_{jk,l} \,-\, a_{kl,j}\,.$$

In fact, $a_{jk} = w_{k,j} + w_{j,k}$ (cf. (2.6.a)), while the Ricci identity (2.12.a) gives $a_{jl,k} + a_{jk,l} - a_{kl,j} = 2w_{j,kl} + R_{kjl}{}^s w_s + R_{klj}{}^s w_s + R_{ljk}{}^s w_s$, and so (2.26) follows, since $R_{klj}{}^s w_s + R_{ljk}{}^s w_s = R_{kjl}{}^s w_s$ by the first Bianchi identity.

Remark 2.3. The facts from analysis used in this text are
if $\int_M f\phi \,\mathrm{d}g = 0$, then $\phi = \Delta f$ for some f

3. Basic properties of Ricci solitons

Throughout this section, except for Lemma 3.1, (M, g) is assumed to be a Ricci soliton of dimension n, so that (0.1) holds for some fixed w and λ, while $R, \mathrm{r}, \mathrm{s}$ denote the curvature tensor, Ricci tensor and scalar curvature. First, we have

$$\text{(3.1)}\qquad \begin{aligned} &\text{a)} \quad 2\nabla_{\!w} R \,=\, \Delta R \,+\, 2A \,+\, B \,-\, 2\lambda R \quad \text{for } A, B \text{ as in (2.25)},\\ &\text{b)} \quad 2\nabla_{\!w}\mathrm{r} \,=\, \Delta\mathrm{r} \,-\, 2\mathring{R}\mathrm{r} \,-\, 2\lambda\mathrm{r},\\ &\text{c)} \quad 2d_w\mathrm{s} \,=\, \Delta\mathrm{s} \,+\, 2|\mathrm{r}|^2 \,-\, 2\lambda\mathrm{s}\,. \end{aligned}$$

It suffices to verify (3.1.a), as the other two identities then are obtained by successive contractions. In view of (2.25), proving (3.1.a) amounts to showing that, as a consequence of (0.1), the expression $R_{jkl}{}^s{}_{,sm} - R_{jkm}{}^s{}_{,sl} + R_{jkl}{}^s R_{sm} - R_{jkm}{}^s R_{sl}$ appearing in (2.25) equals $2w^p R_{jklm,p} - 2\lambda R_{jklm}$. To this end, first note that $w^p R_{jklm,p} = w^p (R_{jklp,m} + R_{jkmp,l})$ in view of the first Bianchi identity, so that the Ricci identity (2.12.a) gives $w^p R_{jklm,p} = w_{p,l} R_{jkm}{}^p - w_{p,m} R_{jkl}{}^p$ and, using formula (2.26) (in which $\nabla a = -\nabla\mathrm{r}$ by (0.1)), we obtain our claim as a consequence of the second Bianchi identity and (0.1).

Next, applying $\langle g, \cdot\rangle$ to (0.1), we obtain, from (2.7.iii) and (2.15),

$$\text{(3.2)}\qquad \text{i)} \quad 2\delta w \,+\, \mathrm{s} \,=\, n\lambda, \qquad \text{so that} \qquad \text{ii)} \quad \mathrm{s}_{\mathrm{avg}} \,=\, n\lambda \;\text{ if } M \text{ is compact},$$

so that (3.1.c) combined with (3.2.ii) and (2.3.ii) yields

$$\text{(3.3)}\qquad 2d_w\mathrm{s} \,=\, \Delta\mathrm{s} \,+\, 2|\mathrm{e}|^2 \,+\, 2(\mathrm{s} - \mathrm{s}_{\mathrm{avg}})\mathrm{s}/n \quad \text{if } M \text{ is compact.}$$

By (0.1), (2.7.iii) and (2.11.e), $|\pounds_w g|^2 = \langle \lambda g - \mathrm{r}, \pounds_w g\rangle = 2\delta(\lambda w - \imath_w \mathrm{r}) + d_w \mathrm{s}$. Thus,

$$\int_M d_w \mathrm{s}\,\mathrm{d}g \;=\; \int_M |\pounds_w g|^2 \mathrm{d}g \quad \text{if} \quad M \quad \text{is compact,} \tag{3.4}$$

by (2.15). Also, δ applied to both sides of (0.1) yields, by (2.6.a) and (2.11.c),

$$\text{i)}\;\; d\mathrm{s}/2 = -\delta\nabla w - \delta(\nabla w)^*, \qquad \text{ii)}\;\; \imath_w \mathrm{r} \,=\, -\delta(\nabla w)^*, \tag{3.5}$$

where (3.5.ii) follows from (2.11.b) and (3.5.i), as $d\mathrm{s} = -2d\delta w$ by (3.2.i).

Again, let (M, g) be a Ricci soliton, so that (0.1) holds for some w and λ. In the open subset of M given by $\mathrm{s} \neq 0$, using (3.1.b) and (3.1.c), we obtain, for the vector field v with $2v = w - \nabla \log|\mathrm{s}|$,

$$\Delta|b|^2 \;=\; d_v|b|^2 \,+\, 2|\nabla b|^2 \,+\, 4\mathrm{s}|b|^4 - 4\langle b, \mathring{R} b\rangle, \quad \text{where} \;\; b = \mathrm{r}/\mathrm{s}\,. \tag{3.6}$$

Lemma 3.1. *Let* r *and* s *be the Ricci tensor and scalar curvature of a Riemannian manifold* (M, g) *with* $\dim M = n \geq 3$*. If* $\mathrm{s} \neq 0$ *everywhere in* M *and the tensor field* $b = \mathrm{r}/\mathrm{s}$ *is parallel, then* s *is constant.*

Proof. As $\nabla b = 0$, any given eigenvalue μ of b is constant on M, and the corresponding eigenspace distribution $\mathcal{E}_\mu = \mathrm{Ker}(b - \mu g)$ is tangent to a factor manifold in a local Riemannian-product decomposition of g. Let r', s' and n' be the Ricci tensor, scalar curvature and dimension of this factor manifold. Since $b = \mu g$ on $\mathcal{E}_\mu$ and $b = \mathrm{r}/\mathrm{s}$, while $\mathrm{r}' = \mathrm{r}$ on $\mathcal{E}_\mu$, we thus have $n'\mu\mathrm{s}g = n'\mathrm{s}b = n'\mathrm{r} = n'\mathrm{r}' = \mathrm{s}'g$ on $\mathcal{E}_\mu$, that is, $\mathrm{s}' = n'\mu\mathrm{s}$. As s' is constant along all other factor manifolds, so is s. Hence s must be constant on M unless $b = \mathrm{r}/\mathrm{s}$ has just one eigenvalue. In the latter case, however, s is constant by Schur's lemma.. □

4. Answers to Questions (1.1)

The answer to each of the questions (1.1) is affirmative, and the details are provided by Theorem 4.4 below. We begin with some lemmas.

Lemma 4.1. *Let* $\mathbf{v} = \mathbf{e}_1 + \ldots + \mathbf{e}_n$*, where* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *is the standard orthonormal basis for the inner product* $\langle\,,\rangle$ *of* $\mathbf{R}^n$*, and let* $\Phi : \mathbf{R}^n \to \mathbf{R}$ *be the function sending* $\mathbf{x} = (x_1, \ldots, x_n)$ *to* $x_1^3 + \ldots + x_n^3$*. For any* $\mathbf{x} \in \mathbf{R}^n$ *such that* $\langle \mathbf{v}, \mathbf{x}\rangle = 0$ *we then have the inequality* $n(n-1)\,[\Phi(\mathbf{x})]^2 \leq (n-2)^2\langle \mathbf{x}, \mathbf{x}\rangle^3$*, which is strict except when* $\mathbf{x}$ *equals a scalar times* $\mathbf{v} - n\mathbf{e}_j$ *for some* $j = 1, \ldots, n$*.*

Proof. It suffices to maximize $\Phi(\mathbf{x})$, subject to the constraints $\langle \mathbf{v}, \mathbf{x}\rangle = 0$ and $\langle \mathbf{x}, \mathbf{x}\rangle = 1$, using Lagrange multipliers; we then find that, for any critical point of Φ restricted to the unit sphere in $\mathbf{v}^\perp$, the n components x_j of $\mathbf{x}$ represent just two different values, as all components satisfy the same quadratic equation. The constraint equations show that the two values must be $\sqrt{l/(kn)}$, occurring k times, and $-\sqrt{k/(ln)}$, occurring l times, where k, l are postive integers and $k + l = n$. For such points, replacing (l, k) by $(l+1, k-1)$ increases the value of Φ, so that the maximum is attained only when $k = 1$ and $l = n - 1$. □

In the next two lemmas, s, r are the scalar curvature and Ricci tensor of a given Riemannian manifold (M, g) of any dimension $n \geq 1$. If $n \geq 3$, one also defines its *Schouten* and *Weyl* tensors S and W to be the twice and, respectively, four times covariant tensor fields $S = \mathrm{r} - (2n-2)^{-1}\mathrm{s}g$ and $W = R - (n-2)^{-1}g \wedge S$, where $\wedge$ is the exterior product of 1-forms valued in 1-forms, obtained from the valuewise multiplication also provided by $\wedge$ (thus, producing a 2-form valued in 2-forms). In local coordinates, $W_{jklm} = R_{jklm} - (n-2)^{-1}(g_{jl}S_{km} + g_{km}S_{jl} - g_{kl}S_{jm} - g_{jm}S_{kl})$.

We always have $W = 0$ if $n = 3$, while in dimensions $n \geq 4$ vanishing of W is, as shown by Schouten [105, p. 83], equivalent to conformal flatness of g (see [6, p. 48] and [46, p. 531]). Finally, we adopt the convention that $W = 0$ if $n \leq 2$, and, in any dimension, let $\mathring{W}$ stand for the bundle morphism $[T^*M]^{\odot 2} \to [T^*M]^{\odot 2}$ that corresponds to W just as $\mathring{R}$ defined in Remark 2.2 corresponds to R.

Lemma 4.2. *At every point of every Riemannian manifold* (M, g)*, we have*

$$|\mathrm{r}|^4 \;\geq\; \mathrm{s}\langle \mathrm{r}, \mathring{R}\mathrm{r}\rangle \;-\; \mathrm{s}\langle \mathrm{r}, \mathring{W}\mathrm{r}\rangle\,. \tag{4.1}$$

The inequality (4.1) *is strict except at points where* r *either is a multiple of* g*, or has exactly two eigenvalues, among them the eigenvalue* 0 *of multiplicity one.*

Proof. Let $n = \dim M$. Since $|\mathrm{r}|^4$ clearly coincides with $\mathrm{s}\langle \mathrm{r}, \mathring{R}\mathrm{r}\rangle$ at points where r is a multiple of g, our assertion follows when $n \leq 2$, and we will from now on assume that $n \geq 3$.

In terms of the bundle morphism $A : TM \to TM$ corresponding to $a = \mathrm{r}$ as in Remark 2.1, expression $|\mathrm{r}|^4 - \mathrm{s}\langle \mathrm{r}, \mathring{R}\mathrm{r}\rangle + \mathrm{s}\langle \mathrm{r}, \mathring{W}\mathrm{r}\rangle$ equals $[(n-1)(n-2)]^{-1}$ times $(n-1)(n-2)(\mathrm{tr}\,A^2)^2 + (\mathrm{tr}\,A)^4 - (2n-1)(\mathrm{tr}\,A)^2\,\mathrm{tr}\,A^2 + 2(n-1)(\mathrm{tr}\,A)\,\mathrm{tr}\,A^3 = n(n-2)\sigma^2\,\mathrm{tr}\,E^2 + 2n(n-1)\sigma\,\mathrm{tr}\,E^3 + (n-1)(n-2)(\mathrm{tr}\,E^2)^2$, where $\sigma = \mathrm{tr}\,A/n$ and $E = A - \sigma\,\mathrm{Id}$ is the traceless part of A.

As our claim is now obvious at points with $E = 0$, we may assume that $E \neq 0$. The quadratic polynomial in σ, appearing above, has the discriminant $4n(n-1)$ times $n(n-1)(\mathrm{tr}\,E^3)^2 - (n-2)^2(\mathrm{tr}\,E^2)^3$, which is nonpositive by Lemma 4.1 applied to the n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ of the eigenvalues of E. In view of the equality clause in Lemma 4.1, this completes the proof. □

Lemma 4.3. *Let* (M, g) *be a compact Ricci soliton of dimension* $n \geq 3$*, with* (0.1) *for some* w, λ*. If* $|\mathrm{r}|^4 \geq \mathrm{s}\langle \mathrm{r}, \mathring{R}\mathrm{r}\rangle$ *at every point of* M *and the scalar curvature* s *is positive everywhere in* M*, then* s *is constant.*

Proof. By (3.6), $\Delta\phi \geq d_v\phi$ for $\phi = |b|^2$ and $b = \mathrm{r}/\mathrm{s}$, as $\mathrm{s}|b|^4 \geq \langle b, \mathring{R}b\rangle$. Hence ϕ is constant in view of Corollary A.2 in Appendix A, which, combined with (3.6), gives $\nabla b = 0$. Now Lemma 3.1 yields our assertion. □

The following theorem, which provides an affirmative answer to the questions listed in (1.1), combines results of several authors. Specifically, the cases $n = 2$ and $n = 3$ were settled by Hamilton [63, §10] and, respectively, Ivey [71]. Bourguignon [9] showed that the scalar curvature s is not constant, and Ivey [71] proved positivity of s and λ. That $\lambda > 0$ and $\mathrm{s} \geq 0$ is also stated in Friedan's paper [52, Propositions 2.2.2 – 2.2.4 on p. 396].

Here is how the proof of Theorem 4.4 given below relates to existing presentations. The arguments excluding the cases $n \in \{2, 3\}$, although quite different from the original proofs of Hamilton [63, §10] and Ivey [71], are not new: the former is due to Chow [39, pp. 202–203] (see also Remark 4.6 below), and the latter, along with its immediate extension that excludes conformal flatness, is a variation on Hamilton's Lemma 10.5 in [61, p. 285]. The rest of the proof is essentially the same as the original arguments of Bourguignon [10, Proposition 3.11] and Ivey [71].

Theorem 4.4. *Let* (M, g) *be a compact Ricci soliton of dimension* n*, with* (0.1) *for some fixed* w *and* λ*. Let us also assume that* (M, g) *is not an Einstein manifold with* $\mathrm{r} = \lambda g$*. Then* $n \geq 4$*, the scalar curvature of* g *is nonconstant and positive,* $\lambda > 0$*, and the Weyl tensor* W *of* g *is nonzero somewhere in* M*.*

Proof. Let $\mathrm{r} \neq \lambda g$ (or, equivalently, $\pounds_w g \neq 0$) somewhere in M.

The scalar curvature s must be nonconstant, or else (3.4) would give $\pounds_w g = 0$. Next, at any fixed point x at which $\mathrm{s} = \mathrm{s}_{\min}$, the left-hand side of (3.3) is 0, while the first two terms on the right-hand side are nonnegative, and so the last term cannot be positive. However, $\mathrm{s}_{\min} < \mathrm{s}_{\mathrm{avg}}$ since s is not constant; thus, nonpositivity of the last term in (3.3) implies that $\mathrm{s}_{\min} \geq 0$.

Consequently, $\mathrm{s}_{\min} > 0$, that is, $\mathrm{s} > 0$ on M, as the case $\mathrm{s}_{\min} = 0$ is excluded by Theorem A.1 in Appendix A applied to $f = \mathrm{s}$, $\psi = 2(\mathrm{s}_{\mathrm{avg}} - \mathrm{s})/n$, $v = 2w$ and our g, with M replaced by the connected component U, containing our x, of the open set in M on which $\psi > 0$. (In fact, $\mathrm{s}_{\min} = 0$ would give $\mathrm{s} = 0$ on U, while $\mathrm{s} = \mathrm{s}_{\mathrm{avg}}$ on the boundary ∂U, due to how U was defined, and $U \neq M$, so that ∂U is nonempty; thus, ψ would be constant, and hence identically zero on U, contrary to the definition of U.)

Positivity of λ now follows (even without using Theorem A.1), since s is nonconstant and nonnegative: if we had $\lambda \leq 0$, (3.2.ii) would give $\mathrm{s}_{\min} < \mathrm{s}_{\mathrm{avg}} \leq 0$.

As g is not Einstein, $n \geq 2$. Thus, $n \geq 3$, for otherwise (3.4) and Theorem B.1 in Appendix B would yield $\pounds_w g = 0$. Finally, since $\mathrm{s} > 0$, the Weyl tensor W cannot vanish identically (and so, in particular, $n \neq 3$): if it did, (4.1) and Lemma 4.3 would contradict non-constancy of s. This completes the proof. □

Remark 4.5. Here is a different reason why $\mathrm{r} = \lambda g$ if s is constant in Theorem 7.3: by (3.3), constancy of s gives $2|\mathrm{e}|^2 = 0$, as the other three terms in (3.3) vanish. Hence g is an Einstein metric and (0.1) becomes $\pounds_w g = \mu g$ with the constant $\mu = \lambda - \mathrm{s}/n$, so that (2.7.iii) yields $4\delta w = n\mu$, and, by (2.15), $\mu = 0$, as required.

Remark 4.6. Chow's proof [39, pp. 202–203] of Hamiltons result [63, §10] stating that, under the hypotheses of Theorem 4.4, one has $n \neq 2$, relies on the uniformization theorem for metrics g on closed surfaces; namely, the argument of Bourguignon and Ezin [11], reproduced in Appendix B, requires choosing a constant-curvature metric conformal to g. However, Chen, Lu and Tian [35] prove that $n \neq 2$ in Theorem 4.4 without invoking the uniformization theorem.

5. Myers-type theorems: Questions (1.3)

Fernández-López and García-Río [50] showed that the answers to Questions (1.3) are all affirmative if, in the third question, one adds a boundedness hypothesis. They provided two separate proofs: one based on Ambrose's compactness criterion [1], the other using results of Lott [86]. Both arguments are presented below.

Before Fernández-López and García-Río's paper [50] appeared, Zhang [120] independently answered the first two questions in (1.3), using yet another argument. Zhang's proof, presented in Appendix C, reaches the conclusion directly, without relying on the results of either Ambrose [1] or Lott [86], although its basic idea is quite similar to Ambrose's.

In this section t always denotes a real variable ranging over a nontrivial closed interval $[a, b]$ or an upper half-line $[a, \infty)$, and $(\ )^\cdot = d/dt$.

We begin with two lemmas due, along with the proofs, to Myers [94] and, respectively, Ambrose [1]:

Lemma 5.1. *Let R and r be the curvature and Ricci tensors of an n-dimensional Riemannian manifold (M, g). Given a minimizing geodesic $[a, b] \ni t \mapsto x(t) \in M$,* *a* C^∞ *vector field* $t\mapsto u(t)\in[\dot x(t)]^\perp\subset T_{x(t)}M$ *normal to it with* $u(a)=u(b)=0$, *and any piecewise-*C^∞ *function* $\varphi:[a,b]\to\mathbf{R}$ *such that* $\varphi(a)=\varphi(b)=0$, *we have*

$$\text{(5.1)}\qquad \text{a) } (R(\dot x,u)\dot x,u)\le(\nabla_{\dot x}u,\nabla_{\dot x}u),\quad \text{b) } \int_a^b\varphi^2\mathrm{r}(\dot x,\dot x)\,dt\le(n-1)\int_a^b\dot\varphi^2\,dt,$$

$(\,,\,)$ *being the* L^2 *inner product of vector fields tangent to* M *along the geodesic.*

Proof. The formula $x(t,s)=\exp_{x(t)}su(t)$ defines a C^∞ mapping from a rectangle $[a,b]\times[c,-c]$ in the ts-plane, with some $c>0$, into M, such that $x(t,0)=x(t)$, $x(a,s)=x(a)$, $x(b,s)=x(b)$ and $x_s(t,0)=u(t)$ for all t,s, where $x_s=\partial x/\partial s$. Defining $\mathsf{L}(s)$ and $\mathsf{A}(s)$ for $s\in[c,-c]$ by $\mathsf{L}(s)=\int_a^b|x_t(t,s)|\,dt$ and $2\mathsf{A}(s)=\int_a^b|x_t(t,s)|^2\,dt$, with $x_t=\partial x/\partial t$, we obtain

$$2(b-a)\mathsf{A}(0)\;=\;[\mathsf{L}(0)]^2\;\le\;[\mathsf{L}(s)]^2\;\le\;2(b-a)\mathsf{A}(s)\quad\text{for all }\;s\in[c,-c].$$

In fact, the three relations, from left to right, follow from constancy of the function $t\mapsto|\dot x(t)|$, the minimizing property of the geodesic $t\mapsto x(t)=x(t,0)$ and, respectively, the Schwarz inequality. Thus, $\mathsf{A}(s)$ assumes its minimum value at $s=0$, and so $\mathsf{A}''(0)\ge0$, with $\mathsf{A}'=d\mathsf{A}/ds$. As $2\mathsf{A}(s)=(x_t,x_t)$, integrating by parts we see that $\mathsf{A}'(s)=(x_t,x_{ts})=(x_t,x_{st})=-(x_{tt},x_s)$, where the boundary term vanishes since $u(a)=u(b)=0$. The geodesic equation $x_{tt}(t,0)=0$ now gives $\mathsf{A}''(0)=-(x_{tts},x_s)$, at $s=0$. (A second or third subscript t or s stands for the covariant derivative along the curve $t\mapsto x(t,s)$ or $s\mapsto x(t,s)$.) Next, $x_{tts}=x_{stt}+R(x_t,x_s)x_t$ (cf. (2.1)). Since $\mathsf{A}''(0)\ge0$ and $x_s(t,0)=u(t)$, it follows that $0\ge(u,\nabla_{\dot x}\nabla_{\dot x}u+R(\dot x,u)\dot x)$, which proves (5.1.a).

Using (5.1.a) for $u=\varphi u$, for any C^∞ function $\varphi:[a,b]\to\mathbf{R}$ with $\varphi(a)=\varphi(b)=0$ and a parallel unit vector field $t\mapsto u(t)\in T_{x(t)}M$ normal to the geodesic $t\mapsto x(t)$, we get $(\varphi^2R(\dot x,u)\dot x,u)\le\int_a^b\dot\varphi^2\,dt$. Summing this over $n-1$ orthonormal fields u with the stated properties, we obtain (5.1.b). Finally, one easily generalizes (5.1.b) to the case where φ is only *piecewise* C^∞-differentiable by smoothing φ out in small neighborhoods of its nondifferentiability points. □

Lemma 5.2. *Let* $h:[a,\infty)\to\mathbf{R}$ *be a continuous function with* $\int_a^\infty h\,dt=\infty$, *and let* $p\in\mathbf{R}$. *Then there exist a real number* $b\in(a,\infty)$ *and a piecewise-*C^∞ *function* $\varphi:[a,b]\to\mathbf{R}$ *such that* $\varphi(a)=\varphi(b)=0$ *and* $\int_a^b\varphi^2h\,dt>p\int_a^b\dot\varphi^2\,dt$.

Proof. We fix $c\in(a,\infty)$ and a C^∞ function $\varphi:[a,c]\to\mathbf{R}$ with $\varphi(a)=0$ and $\varphi(c)=1$. Every $b\in(c,\infty)$ gives rise to an extension of φ to the interval $[a,b]$, still denoted by φ, and characterized by being linear on $[c,b]$ with $\varphi(c)=1$ and $\varphi(b)=0$. Given $b\in(c,\infty)$, using the corresponding $\varphi:[a,b]\to\mathbf{R}$, we set $\mathcal{I}_r^s=\int_r^s(\varphi^2h-p\dot\varphi^2)\,dt$, for $r,s\in[a,b]$. Thus, $\mathcal{I}_c^b+p/(b-c)=(b-c)^{-2}\int_c^b(b-t)^2h\,dt$. Choosing $\chi:[c,\infty)\to\mathbf{R}$ with $d^3\chi/dt^3=h$ and $\chi(c)=\dot\chi(c)=\ddot\chi(c)=0$, then integrating by parts, we get $\mathcal{I}_c^b+p/(b-c)=2(b-c)^{-2}\chi(b)$, which tends to ∞ as $b\to\infty$ due to l'Hospital's rule and the relation $\ddot\chi(b)=\int_c^b h\,dt\to\infty$. (All limits here are taken as $b\to\infty$, with fixed c.) Consequently, $\mathcal{I}_c^b\to\infty$, and so $\mathcal{I}_a^b=\mathcal{I}_a^c+\mathcal{I}_c^b\to\infty$, since $\mathcal{I}_a^c$ does not depend on b. Therefore, for some $b>c$, we have $\mathcal{I}_a^b>0$, that is, $\int_a^b\varphi^2h\,dt>p\int_a^b\dot\varphi^2\,dt$. This completes the proof. □

We now prove Ambrose's compactness criterion using his original argument [1]:

Theorem 5.3. *Let* r *denote the Ricci tensor of a complete Riemannian manifold* (M,g). *If* M *contains a point* y *such that* $\int_a^\infty\mathrm{r}(\dot x,\dot x)\,dt=\infty$ *for every unit-speed* *geodesic* $[a,\infty)\ni t\mapsto x(t)$ *with* $x(a)=y$*, then* M *is compact and the fundamental group* $\pi_1 M$ *is finite.*

Proof. Any unit-speed geodesic $t\mapsto x(t)$ emanating from y must eventually cease to be minimizing, for otherwise Lemma 5.2 for $p=\dim M-1$ and $h=\mathrm{r}(\dot x,\dot x)$ would contradict (5.1.b). Thus, we may associate with every unit vector $u\in T_yM$ the upper endpoint $\ell(u)\in(0,\infty)$ of the maximal interval $[0,\ell(u)]$ on which the geodesic $t\mapsto\exp_y tu$ is minimizing. Since the 'minimizing' property amounts to the distance equality $\mathrm{dist}(y,\exp_y tu)=t$ for every $t\in[0,\ell(u)]$, and the unit sphere $\varSigma\subset T_yM$ is compact, the function $\ell:\varSigma\to(0,\infty)$ just defined is bounded. In fact, if it were not, we could choose a sequence $u_k\in\varSigma$ converging to some $u\in\varSigma$ with $\ell(u_k)\to\infty$, so that $\mathrm{dist}(y,\exp_y tu_k)=t$ for any $t\in[0,\infty)$ and all large k (namely, k with $\ell(u_k)\ge t$). Taking the limit, we would then get $\mathrm{dist}(y,\exp_y tu)=t$ for every $t\in[0,\infty)$, contradicting finiteness of $\ell(u)$.

As $\ell:\varSigma\to(0,\infty)$ is bounded, (M,g) has a finite diameter by the Hopf-Rinow theorem, and so M is compact. The Riemannian universal covering space of (M,g) must be compact as well, as it satisfies the same hypotheses. Therefore, π_1M is finite, which completes the proof. □

The next four results are due to Fernández-López and García-Río [50], and so is the argument justifying Theorem 5.4. (The other three theorems follow.)

Theorem 5.4. *Let a complete Riemannian manifold* (M,g) *admit a* C^∞ *vector field* w *such that* $g(w,w)$ *is bounded on* M *and* $\pounds_w g+\mathrm{r}\ge\lambda g$ *for some constant* $\lambda>0$*, in the sense that* $\pounds_w g+\mathrm{r}-\lambda g$ *is positive semidefinite at every point. Then* M *is compact and has a finite fundamental group.*

In fact, Theorem 5.3 then applies to every point $y\in M$, as $\mathrm{r}(\dot x,\dot x)\ge\lambda-(\pounds_w g)(\dot x,\dot x)$, while, by (2.6.a), $(\pounds_w g)(\dot x,\dot x)=2g(\nabla_{\dot x}w,\dot x)=\dot\gamma$, where $\gamma=2g(w,\dot x)$. Thus, $\int_a^b(\pounds_w g)(\dot x,\dot x)\,dt=\gamma(b)-\gamma(a)$ is a bounded function of b.

Theorem 5.5. *Let* (M,g) *be a complete Ricci soliton such that, in* (0.1)*,* $\lambda>0$ *and* $g(w,w)$ *is bounded. Then* M *is compact and* π_1M *is finite.*

Theorem 5.6. *If a compact Riemannian manifold* (M,g) *admits a* C^∞ *vector field* w *such that the twice-covariant symmetric tensor* $\pounds_w g+\mathrm{r}$ *is positive definite at every point of* M*, then the fundamental group* π_1M *is finite.*

Theorem 5.7. *If* (M,g) *is a shrinking compact Ricci soliton, then* π_1M *is finite.*

Theorem 5.4 with $w=0$ amounts to a part of Myers's classical theorem [94], stating that *a complete Riemannian manifold* (M,g) *with* $\mathrm{r}\ge\lambda g$ *for a constant* $\lambda>0$ *is compact and has a finite fundamental group.* The remaining part of Myers's theorem is a diameter estimate: namely, the diameter of (M,g) then does not exceed the square root of $(n-1)\pi^2/\lambda$, where $n=\dim M$. (This is justified later, in the lines following Theorem 5.10.)

A similar diameter estimate also holds under the assumptions of Theorems 5.5 and 5.7. It is derived from a different argument, in the form of the following lemma and two theorems, all three of which are due to Lott [86]. Note that none of the results obtained earlier in this section are used below, except for Lemma 5.1.

We begin with a Poincaré-type estimate:

Lemma 5.8. *Let* $\varphi : [a,b] \to \mathbf{R}$ *and* $H : (a,b) \to \mathbf{R}$ *be* C^1 *functions such that* $\varphi(a) = \varphi(b) = 0$ *and there exist finite limits of* $(t-a)H(t)$ *as* $t \to a^+$ *and of* $(t-b)H(t)$ *as* $t \to b^-$. *Then* $\int_a^b (\dot H - H^2)\varphi^2\,dt \le \int_a^b \dot\varphi^2\,dt$, *with equality only if* $\varphi = Ce^{-F}$ *on* (a,b) *for some constant* C *and a fixed antiderivative* F *of* H.

In fact, $\int_a^b (\dot\varphi^2 - \dot H\varphi^2 + H^2\varphi^2)\,dt$ is nonnegative: it equals $\int_a^b (\dot\varphi + H\varphi)^2\,dt$, as one sees noting that $(\dot\varphi + H\varphi)^2 = \dot\varphi^2 + 2H\varphi\dot\varphi + H^2\varphi^2$, while the integrals of $2H\varphi\dot\varphi$ and $-\dot H\varphi^2$ coincide, since $2H\varphi\dot\varphi + \dot H\varphi^2 = (H\varphi^2)^{\dot{}}$.

Remark 5.9. The ordinary Poincaré inequality in dimension 1 is a special case of Lemma 5.8, obtained by setting $H(t) = -(b-a)^{-1}\pi\cot[\pi(t-a)/(b-a)]$. It states, in terms of the L^2 norm $\|\ \|$, that $\pi\|\varphi\| \le (b-a)\|\dot\varphi\|$ for any C^1 function $\varphi : [a,b] \to \mathbf{R}$ with $\varphi(a) = \varphi(b) = 0$, and the inequality is strict except when $\varphi(t)$ equals a constant times $\sin[\pi(t-a)/(b-a)]$.

The following theorem, due to Lott, is the main step in the proof [86, p. 868] of Theorem 1.1 of [86]. Lott assumes, in addition, that w is the gradient of a function, some power of which serves as the warping function of a warped product metric used in his proof. Lott's argument is rephrased here so as to avoid mentioning warped products, which makes assuming that w is a gradient unnecessary.

Theorem 5.10. *Let a complete* n*-dimensional Riemannian manifold* (M,g) *admit a* C^∞ *vector field* w *such that, for the corresponding* 1*-form* $\xi = \imath_w g$,

$$\text{(5.2)}\qquad \pounds_w g \;+\; \mathrm{r} \;-\; 4q^{-1}\xi\otimes\xi \;\ge\; \varepsilon g \text{ with some positive constants } \varepsilon \text{ and } q,$$

r *being the Ricci tensor, in the sense that* $\pounds_w g + \mathrm{r} - 4q^{-1}\xi\otimes\xi - \varepsilon g$ *is positive semidefinite at every point. Then* M *is compact, its fundamental group is finite, and the diameter of* (M,g) *does not exceed the square root of* $(q+n-1)\pi^2/\varepsilon$.

Proof. Let L be the length of any fixed minimizing geodesic $[a,b] \ni t \mapsto x(t) \in M$, and let $\varphi : [a,b] \to \mathbf{R}$ be any C^∞ function with $\varphi(a) = \varphi(b) = 0$. By (5.2), $\int_a^b \varphi^2(\pounds_w g + \mathrm{r} - 4q^{-1}\xi\otimes\xi)(\dot x,\dot x)\,dt \ge \varepsilon\int_a^b \varphi^2 g(\dot x,\dot x)\,dt$, while $(b-a)^2 g(\dot x,\dot x) = \mathsf{L}^2$, and, setting $H = 2q^{-1}g(w,\dot x)$, we get $(\pounds_w g - 4q^{-1}\xi\otimes\xi)(\dot x,\dot x) = q(\dot H - H^2)$ (cf. (2.6.a)). Lemma 5.8 and (5.1.b) now give $(b-a)^{-2}\varepsilon\mathsf{L}^2\int_a^b \varphi^2\,dt \le (q+n-1)\int_a^b \dot\varphi^2\,dt$.

So far the C^∞ function φ with $\varphi(a) = \varphi(b) = 0$ was arbitrary. The estimate $\varepsilon\mathsf{L}^2 \le (q+n-1)\pi^2$ follows from the last inequality with $\varphi(t) = \sin[\pi(t-a)/(b-a)]$ (cf. Remark 5.9). Thus, M must be compact, which, applied to the Riemannian universal covering space of (M,g), yields finiteness of $\pi_1 M$. □

Theorem 5.10 provides an alternative proof of Theorem 5.6, which is how Lott used it [86, Theorem 1.1 on p. 866] under the assumption that w is a gradient.

The diameter estimate in Myers's theorem, mentioned immediately after Theorem 5.6, is obvious from Theorem 5.10 with $w = 0$ and $q \to 0^+$. In addition, Theorem 5.10 implies a further diameter estimate:

Corollary 5.11. *Suppose that* (M,g) *is a compact Riemannian manifold admitting a* C^∞ *vector field* w *such that* $\pounds_w g + \mathrm{r} \ge \lambda g$ *for some* $\lambda \in (0,\infty)$. *For instance,* (M,g) *might be a shrinking compact Ricci soliton, with* w, λ *as in* (0.1).

Then $\mathrm{d}^2 \le 2\lambda^{-2}\pi^2\left[(n-1)\lambda + 8\max|w|^2\right]$, *where* $n = \dim M$ *and* d *denotes the diameter of* (M,g).

Proof. We then have (5.2) for $\varepsilon = \lambda/2$ and $q = 8\lambda^{-1}\max|w|^2$. □

Zhang's argument [120] yields yet another diameter estimate. See Appendix C.

Remark 5.12. On a related note, recent results of Perelman [102, Lemma 1.2] and Ni [96, Theorem 3] represent a case where completeness of a shrinking gradient Ricci soliton (M,g) implies compactness of M under a suitable positive-curvature assumption. Specifically, Perelman assumes that $\dim M = 3$, the sectional curvature is bounded, positive, and g is κ-noncollapsed for some $\kappa > 0$ (in the sense of [101, p. 20]), while Ni's theorem deals with Kähler-Ricci solitons having positive bisectional curvature. Note that neither result *assumes* the curvature to be bounded away from zero.

Remark 5.13. A stronger version of Theorem 5.7 is known to be true for compact shrinking *Kähler*-Ricci solitons (M,g). Namely, we then have $c_1(M) > 0$ (see Proposition 8.2), while compact complex manifolds M with $c_1(M) > 0$ are all simply connected [6, Theorem 11.26].

Remark 5.14. As shown by Ma [88], the answer to Question (1.4) is 'yes' if, in addition, $\|\mathrm{s}\|^2 \le 24\lambda^2\mathrm{V}$, where $\|\ \|$ is the L^2 norm, λ denotes the constant in (0.1), and V stands for the volume of (M,g). This follows from the well-known formula $96\pi^2[2\chi(M)+3\tau(M)] = 48\|W^+\|^2 - 12\|\mathrm{e}\|^2 + \|\mathrm{s}\|^2$ (see, for instance, [46, p. 598]), which gives $2\chi(M) \pm 3\tau(M) \ge 0$, for both signs $\pm$, if $\|\mathrm{s}\|^2 \ge 12\|\mathrm{e}\|^2$.

Specifically, under the assumption (0.1), one has $\|\mathrm{s}\|^2 \ge 12\|\mathrm{e}\|^2$ if and only if $\|\mathrm{s}\|^2 \le 24\lambda^2\mathrm{V}$. In fact, integrating (3.3) with $n = 4$, we get $6\int_M d_w\mathrm{s}\,\mathrm{d}g = 6\|\mathrm{e}\|^2 - 24\lambda^2\mathrm{V} + 3\|\mathrm{s}\|^2/2$, from the divergence theorem (2.15) and (3.2). On the other hand, (2.17) yields $6\int_M d_w\mathrm{s}\,\mathrm{d}g = -6\int_M \mathrm{s}\,\delta w\,\mathrm{d}g$, which, by (3.2.i) and (3.2.ii), equals $3\|\mathrm{s}\|^2 - 48\lambda^2\mathrm{V}$. Equating the two resulting expressions for $6\int_M d_w\mathrm{s}\,\mathrm{d}g$, we see that $\|\mathrm{s}\|^2 - 24\lambda^2\mathrm{V} = 6\|\mathrm{e}\|^2 - \|\mathrm{s}\|^2/2$.

6. Gradient Ricci solitons and Question (1.2)

An affirmative answer to Question (1.2) is provided by the next theorem, due to Perelman [101, Remark 3.2].

Theorem 6.1. *Let (M,g) be a compact Ricci soliton, with a vector field w satisfying* (0.1)*. Then w is the sum of a Killing field and a gradient.*

According to Theorem 6.1, which will be proved later in this section, every compact Ricci soliton (M,g) is a *gradient* Ricci soliton, in the sense that

(6.1) $\qquad \nabla df + \mathrm{r} = \lambda g$ for a constant λ and a C^∞ function $f : M \to \mathbf{R}$.

In terms of (0.1), relation (6.1) means that $2w - \nabla f$ is a Killing field (cf. (2.6.b)).

At least *ex post facto*, Perelman's proof of Theorem 6.1 presented below can be motivated as follows. If (6.1) holds, $2\varDelta f - |\nabla f|^2 + 2\lambda f + \mathrm{s}$ is constant (see Remark 6.5), and may be assumed equal to 0 by adding a constant to f. One may therefore try to exhibit a function f with (6.1) on a given compact Ricci soliton (M,g) by solving the equation $2\varDelta f - |\nabla f|^2 + 2\lambda f + \mathrm{s} = 0$. This equation, with $\lambda > 0$, is in fact solvable on every compact Riemannian manifold (M,g) (see Theorem 6.2), and, according to Theorem 6.3, it implies (6.1), provided that (M,g) is also a Ricci soliton.

We now present the details, beginning with a result of Rothaus [104]:

Theorem 6.2. *Given a Riemannian manifold (M,g) of dimension n, let $\mathcal{P}$ be the nonlinear operator acting on C^∞ functions $f : M \to \mathbf{R}$ by $\mathcal{P}f = \varDelta f - |\nabla f|^2/2$.*

If M is compact and $n \geq 3$, then, for any real constant $\lambda > 0$, the mapping $f \mapsto \mathcal{P}f + \lambda f$ sends the space of all C^∞ functions $M \to \mathbf{R}$ onto itself.

For a proof, see Appendix D. (When $n = 2$, Theorem 6.2 remains true, but requires a different argument [104].)

The next theorem constitutes the main step leading to Perelman's Theorem 6.1; our formulation is a minor variation on a result of Perelman [101, equality (3.4)], often called *Perelman's monotonicity formula* (cf. [108]). The proof given below amounts to a very detailed exposition of the argument outlined by Perelman in [101] and presented in much detail by Kleiner and Lott in [79].

Theorem 6.3. *Let* r *and* s *be the Ricci tensor and scalar curvature of a compact Riemannian manifold* (M, g)*. For any given* C^∞ *function* $f : M \to \mathbf{R}$*, any* C^∞ *vector field* w *on* M*, and any constant* $\lambda \in \mathbf{R}$*, let us define twice-covariant symmetric tensors* h, b *and a function* ψ *by* $h = \nabla df + \mathrm{r} - \lambda g$, $b = \pounds_w g + \mathrm{r} - \lambda g$, *and* $\psi = \varDelta e^{-f} + 2\delta[e^{-f}w]$, *cf.* (2.14.i) *and* (2.7.i)*. Then, with* $\mathcal{P}$ *as in* Theorem 6.2*,*

$$\int_M |h|^2 e^{-f} \mathrm{d}g \ + \ \int_M (\mathcal{P}f + \lambda f + \mathrm{s}/2)\psi \, \mathrm{d}g \ = \ \int_M \langle h, b\rangle e^{-f} \mathrm{d}g. \tag{6.2}$$

Proof. It suffices to prove vanishing of the four expressions

(a) $(\nabla df + \mathrm{r}, e^{-f}\nabla df) + (\mathcal{P}f + \mathrm{s}/2, \varDelta e^{-f})$,
(b) $(-\lambda g, e^{-f}\nabla df) + (\lambda f, \varDelta e^{-f})$,
(c) $(\nabla df + \mathrm{r}, e^{-f}[\mathrm{r} - \lambda g - b]) + (\mathcal{P}f + \mathrm{s}/2, 2\delta[e^{-f}w])$,
(d) $(-\lambda g, e^{-f}[\mathrm{r} - \lambda g - b]) + (\lambda f, 2\delta[e^{-f}w])$,

where $(\, , \,)$ stands for the L^2 inner product of functions or twice-covariant symmetric tensors on M.

To show that (a) – (d) are all zero, we set $v = \nabla f$, $a = \nabla df$, $Y = \varDelta f$, $Q = |v|^2$ and $u = \nabla e^{-f}$. Then, for any twice-covariant symmetric tensor c,

(e) $\int_M e^{-f}\langle c, a\rangle \, \mathrm{d}g \ = \ \int_M (\imath_u \delta c - \imath_u \imath_v c) \, \mathrm{d}g$,

by (2.15), since $e^{-f}\langle c, a\rangle - \imath_u\delta c + \imath_u\imath_v c = -\delta\imath_u c$, as one sees in local coordinates: $[c^{jk} f_{,k} e^{-f}]_{,j} = e^{-f} c^{jk} f_{,jk} - c^{jk}{}_{,j}(e^{-f})_{,k} + c^{jk} f_{,k}(e^{-f})_{,j}$.

Expression (a) is obtained by integrating the sum of five terms: $e^{-f}|\nabla df|^2$, $e^{-f}\langle \mathrm{r}, \nabla df\rangle$, $(\varDelta f)\varDelta e^{-f}$, $-|\nabla f|^2 \varDelta e^{-f}/2$, and $\mathrm{s}\varDelta e^{-f}/2$, which can also be rewritten as $e^{-f}|a|^2$, $e^{-f}\langle \mathrm{r}, a\rangle$, $Y\varDelta e^{-f}$, $-Q\varDelta e^{-f}/2$, and $\mathrm{s}\varDelta e^{-f}/2$. Since $\int_M \mathrm{s}\varDelta e^{-f}\mathrm{d}g$ equals $-\int_M g(\nabla \mathrm{s}, u)\, \mathrm{d}g = -2\int_M \imath_u \delta \mathrm{r} \, \mathrm{d}g$ (see (2.19) and (2.11.c)), the integrals of the second and fifth terms add up, by (e) for $c = \mathrm{r}$, to $-\int_M (\imath_u\imath_v \mathrm{r}) \, \mathrm{d}g$. Next, the sum of the integrals of the first, third and fourth terms is $\int_M (\imath_u\imath_v\mathrm{r}) \, \mathrm{d}g$ (and hence (a) equals 0). In fact, the integral of $-Q\varDelta e^{-f}/2$ coincides, by (2.19) and (2.10.ii), with that of $g(\nabla Q, u)/2 = \imath_u\imath_v a$, and hence $\int_M (e^{-f}|a|^2 - Q\varDelta e^{-f}/2) \, \mathrm{d}g = \int_M (\imath_u \delta a) \, \mathrm{d}g$, by (e) for $c = a$, while, from (2.19), $-\int_M Y\varDelta e^{-f}\mathrm{d}g = \int_M g(\nabla Y, u) \, \mathrm{d}g = \int_M (\imath_u dY) \, \mathrm{d}g$ and, by (2.13), $\int_M (e^{-f}|a|^2 + Y\varDelta e^{-f} - Q\varDelta e^{-f}/2) \, \mathrm{d}g = \int_M (\imath_u\imath_v\mathrm{r}) \, \mathrm{d}g$.

The sum in (b) vanishes by (2.19) for $\phi = e^{-f}$, since $\varDelta f = \langle \nabla df, g\rangle$.

Next, as $\mathrm{r} - \lambda g - b = -\pounds_w g$ (cf. the definition of b), with v, a, Y, Q, u as above, (c) equals $-\int_M e^{-f}\langle a + \mathrm{r}, \pounds_w g\rangle \, \mathrm{d}g + \int_M (2Y - Q + \mathrm{s})(\delta[e^{-f}w]) \, \mathrm{d}g$. The first of the two integrals is equal to $2\int_M [\imath_w \delta a - \delta\imath_w(a + \mathrm{r}) + d_w \mathrm{s}/2] e^{-f} \mathrm{d}g$, as one easily sees using (2.7.ii) and (2.11.e). Furthermore, (2.17) with f and the 1-form $\imath_w g$ replaced by e^{-f} and $\imath_w(a + \mathrm{r})$ gives $-\int_M [\delta\imath_w(a + \mathrm{r})] e^{-f}\mathrm{d}g = \int_M [\imath_u\imath_w(a + \mathrm{r})] \, \mathrm{d}g$. Also, by (2.13), $\imath_w\delta a = \imath_w\imath_v\mathrm{r} + \imath_w dY$, while $e^{-f}\imath_w\imath_v\mathrm{r} = -\imath_u\imath_w\mathrm{r}$ as $u = -e^{-f}v$, and, by (2.10.ii), $2\imath_u a = -2e^{-f}\imath_v a = -e^{-f}dQ$. Consequently, using (2.8.b) we get $2\int_M[\imath_w\delta a - \delta\imath_w(a+\mathrm{r}) + d_w\mathrm{s}/2]e^{-f}\mathrm{d}g = \int_M[d_w(2Y-Q+\mathrm{s})]e^{-f}\mathrm{d}g$, which, by (2.17), is the opposite of $\int_M(2Y-Q+\mathrm{s})(\delta[e^{-f}w])\,\mathrm{d}g$, thus showing that (c) is 0.

Finally, using (2.17) twice, we get $\int_M(\delta[e^{-f}w])f\,\mathrm{d}g = -\int_M g(\nabla f, e^{-f}w)\,\mathrm{d}g = \int_M g(\nabla e^{-f}, w)\,\mathrm{d}g = -\int_M e^{-f}(\delta w)\,\mathrm{d}g$. Therefore, (d) vanishes: by the definition of b and (2.7.iii), $\mathrm{r} - \lambda g - b = -\pounds_w g$, while $\langle g, \pounds_w g\rangle = 2\delta w$. □

Proof of Theorem 6.1. Let $n = \dim M$. If $n \le 2$ or $\lambda \le 0$ in (0.1), our assertion follows since, by Theorem 4.4, g is an Einstein metric with $\mathrm{r} = \lambda g$, and so $\pounds_w g = 0$. If $n \ge 3$ and $\lambda > 0$, choosing a C^∞ function $f : M \to \mathbf{R}$ with $\mathcal{P}f + \lambda f + \mathrm{s}/2 = 0$ (cf. Theorem 6.2), we get $\int_M |h|^2 e^{-f}\mathrm{d}g = 0$ from (6.2), since $b = 0$ and so the other two integrals vanish. Thus, $\nabla df + \mathrm{r} = \lambda g$, which completes the proof. □

Remark 6.4. Theorem 6.1 implies that *in every compact Ricci soliton,* (0.1) *is satisfied by a unique gradient vector field* w *and a unique constant* λ. In fact, λ is unique by (3.2.ii), while, applying $\langle g, \cdot\rangle$ to (6.1), we get $\Delta f = n\lambda - \mathrm{s}$, for $n = \dim M$. This determines f uniquely up to an additive constant (cf. (2.20.b)).

Remark 6.5. If (M, g) is a compact gradient Ricci soliton, with (6.1), we get $\int_M(\mathcal{P}f + \lambda f + \mathrm{s}/2)\psi\,\mathrm{d}g = 0$ from (6.2) for f satisfying (6.1) and $w = \nabla f/2$. (Note that $h = b = 0$.) This also follows directly (2.15), since ψ is, by definition, a divergence, while $\mathcal{P}f + \lambda f + \mathrm{s}/2$ is constant.

Constancy of $\mathcal{P}f + \lambda f + \mathrm{s}/2$ is, actually, a *local* consequence of (6.1). Namely, given an arbitary Riemannian manifold (M, g) with a C^∞ function f, and a constant λ, let us set $v = \nabla f$, $a = \nabla df$, $Q = g(v, v)$, $Y = \Delta f = \delta v$ and $h = a + \mathrm{r} - \lambda g$. Thus, $2\imath_v h = dQ + 2\imath_v\mathrm{r} - 2\lambda df$ (by (2.10)) and $2\delta h = 2\imath_u\mathrm{r} + 2dY + d\mathrm{s}$, from (2.13) and (2.11.c). Subtracting, we get $2\delta h - 2\imath_v h = d(2Y - Q + 2\lambda f + \mathrm{s})$, and so $2Y - Q + 2\lambda f + \mathrm{s}$ is constant if (6.1) holds, that is, if $h = 0$. Other interesting scalar equations also follow from (6.1): for instance [39, p. 201],

$$\Delta f - |\nabla f|^2 + 2\lambda f \quad \text{is constant,} \tag{6.3}$$

as one sees subtracting from $d(2Y - Q + 2\lambda f + \mathrm{s}) = 0$ the relation $d(Y + \mathrm{s}) = 0$ (immediate since $0 = \langle g, h\rangle = \langle g, a\rangle + \mathrm{s} - n\lambda = Y + \mathrm{s} - n\lambda$, where $n = \dim M$).

7. Ricci solitons and the curvature operator

This section presents four theorems, due to Böhm and Wilking [8], Hamilton [62], Ivey [74] and Tachibana [109]. The first of them answers Question 1.5 in the case where the curvature operator is positive.

The following result of Böhm and Wilking [8] was first proved by Hamilton [62] for $n = 4$. Hamilton also conjectured that the same conclusion remained valid for all dimensions n.

Theorem 7.1. *In any dimension* $n \ge 2$*, a compact Ricci soliton with positive curvature operator must have constant sectional curvature.*

The second result is due to Hamilton [62].

Theorem 7.2. *Every compact four-dimensional Ricci soliton with nonnegative curvature operator is a locally symmetric Einstein manifold.*

Neither of Theorems 7.1 and 7.2 is explicitly stated in [8] or [62]; they are, however, obvious consequences of much more general theorems about the Ricci flow with an initial metric that has positive or nonnegative curvature operator [8], [62]. The two papers never mention Ricci solitons at all.

In the next theorem, due to Ivey [74], the condition $W \leq \mathrm{s}/6$ in Theorem 7.3, known as *nonnegativity of the isotropic curvature* [93, p. 201], means that the difference $W - \mathrm{s}/6$ of the Weyl conformal tensor and multiplication by one-sixth of the scalar curvature, acting as a self-adjoint operator on exterior 2-forms, is nonpositive at every point.

Theorem 7.3. *Let (M, g) be a compact four-dimensional Ricci soliton, with* (0.1) *for some fixed w and λ, such that g is a Kähler metric and $W \leq \mathrm{s}/6$. Then g is an Einstein metric and $\mathrm{r} = \lambda g$.*

If, in addition, λ is positive, (M, g) must be isometric to $\mathbf{C}\mathrm{P}^2$ or $S^2 \times S^2$ with a multiple of the standard Kähler-Einstein metric.

Proofs of Theorems 7.1 – 7.3 are oulined at the end of this section.

The fourth result was proved by Tachibana [109] back in 1974. The metric g is assumed here to have *harmonic curvature* in the sense that $\delta R = 0$, which is true for all g with parallel Ricci tensor, including all Einstein metrics.

Theorem 7.4. *Let (M, g) be a compact Riemannian manifold with $\delta R = 0$.*

(i) *If the curvature operator of (M, g) is nonnegative, then (M, g) is locally symmetric.*
(ii) *If the curvature operator of (M, g) is positive, (M, g) must have constant sectional curvature.*

Proof. See Appendix E. □

Tachibana [109] proved Theorem 7.4 by a Bochner-type vanishing argument, based on a Weitzenböck formula due to Berger [5]. An attempt to adapt Tachibana's proof to the case of compact Ricci solitons with positive (or, nonnegative) curvature operator is hampered by the presence, in (0.1), of the term $\pounds_w g$ which (even when, using Theorem 6.1, one chooses w to be a gradient) cause the argument to break down. Applying the methods of Gursky and LeBrun [60], in dimension four, does not seem to produce immediate results either.

A more promising approach, inspired by Ivey's proof, in [71], that all compact three-dimensional Ricci solitons have constant sectional curvature (see Theorem 4.4), consists in using Hopf's maximum principle, which renders terms related to $\pounds_w g$ quite manageable.

In a Riemannian manifold (M, g) of any dimension $n \geq 3$, let us consider the following functions:

$$\begin{aligned} F &= |R|^2|\mathrm{r}|^2 - \mathrm{s}\,[\mathrm{tr}\,\hat{R}^3 + \mathrm{tr}\,R^3/2], \\ G &= |\mathrm{e}|^4 + 2(n-2)^{-1}\mathrm{s}\,\mathrm{tr}\,\mathrm{e}^3 + n^{-1}(n-1)^{-1}\mathrm{s}^2|\mathrm{e}|^2, \\ H &= |\mathrm{r}|^4 - \mathrm{s}\,\langle \mathrm{r}, R\mathrm{r}\rangle. \end{aligned} \tag{7.1}$$

For the meaning of $\mathrm{tr}\,R^3$, see the line following formula (E.2) in Appendix E.

further text in preparation

Proofs of Theorems 7.1 – 7.3.

in preparation

□

8. Kähler-Ricci solitons

By a *Kähler-Ricci soliton* we mean here a Ricci soliton which also happens to be a Kähler manifold. Most authors' definition of Kähler-Ricci solitons is different from ours, although equivalent to it in the compact case (see Remark 8.4 below).

In this section we use some definitions and facts about Kähler manifolds, a self-contained presentation of which is given in Appendices F through L. Note that a *holomorphic vector field* is always assumed to be real, the Kähler and Ricci forms of any given Kähler manifold (M, g) are denoted by Ω and ρ, while J stands for the complex structure tensor and $i\partial\overline{\partial}$ is the operator characterized by (G.2).

We begin with a lemma.

Lemma 8.1. *Let (M, g) be both a Kähler manifold and a gradient Ricci soliton, so that $\nabla df + \mathrm{r} = \lambda g$ for a constant λ and a C^∞ function $f : M \to \mathbf{R}$, where r is the Ricci tensor. Thus, we have* (0.1) *for $w = \nabla f/2$, cf.* (2.6.b). *Then*

(a) *the vector fields w and Jw are holomorphic,*
(b) *Jw is a Killing field,*
(c) *$i\partial\overline{\partial}f + \rho = \lambda\Omega$.*

Proof. Since r and g are Hermitian (Remark H.1(i)), so is $\nabla df = \lambda g - \mathrm{r}$. This has two consequences. First, $i\partial\overline{\partial}f = (\nabla df)J = (\lambda g - \mathrm{r})J = \lambda\Omega - \rho$, cf. (H.3), which yields (c). Secondly, (a) – (b) now follow from Remark L.1(d) with $\xi = df/2$. □

The next proposition provides an affirmative answer to the first question in (1.6). The proof given below uses Perelman's Theorem 6.1. However, to establish Theorem 6.1 itself one has to solve a nonlinear elliptic equation. For the reader's benefit we give in Appendix M a different, completely elementary proof of Proposition 8.2.

Proposition 8.2. *Let (M, g) be a compact Kähler-Ricci soliton, and let a vector field w satisfy* (0.1) *with a constant λ. Then w is holomorphic and $[\rho] = \lambda[\Omega]$ in $H^2(M, \mathbf{R})$, where Ω and ρ are the Kähler and Ricci forms.*

Proof. As stated immediately after Theorem 6.1, $\nabla df + \mathrm{r} = \lambda g$ for some constant λ and a function f such that $u = 2w - \nabla f$ is a Killing field. Our claim is now obvious from (a) and (c) in Lemma 8.1, since the Killing field u is holomorphic (Remark M.3 in Appendix M) and the 2-form $i\partial\overline{\partial}h$ is exact (cf. (G.2)). □

For later reference, we state the following immediate consequence of Lemma 8.1:

Corollary 8.3. *Given a compact Kähler-Ricci soliton (M, g), let w be the unique gradient vector field satisfying* (0.1), *cf.* Remark 6.4. *Then w is holomorphic and Jw is a Killing field.*

Remark 8.4. Most authors define a Kähler-Ricci soliton to be a Kähler manifold (M, g) whose Kähler and Ricci forms Ω and ρ satisfy the equality $\pounds_v\Omega + \rho = \lambda\Omega$ for some real constant λ and some holomorphic section v of the complexified tangent bundle $[TM]^{\mathbf{C}}$. (One often normalizes the metric so that λ is $1, 0$ or -1.) Since such v equals $w - iJw$, where w is a real holomorphic vector field, taking the real and imaginary parts we see that the above equality amounts to (0.1) with a real holomorphic field w such that Jw is a Killing field. Corollary 8.3 implies that, in the compact case, this definition is equivalent to ours.

9. Non-Einstein examples

The examples of non-Einstein compact Kähler-Ricci solitons found by Koiso [78] and, independently, Cao [17], are described in this section.

It is convenient to begin with a more general construction (which yields various non-soliton metrics as well), and then narrow it down to the Koiso-Cao case.

The construction uses three sets of data. First, let there be given

(9.1) an integer $k \geq 2$, the total space M of an S^k bundle with the structure group $\mathrm{SO}(k)$ over a manifold N, and the horizontal distribution $\mathcal{H}$ of a fixed $\mathrm{SO}(k)$ connection in the bundle M.

Here and below we refer to the standard action of $\mathrm{SO}(k)$ on S^k, which keeps two points fixed. Next, we have the fibre-related ingredients:

(9.2) a real number $c \neq 0$, a nontrivial closed interval $[t_{\min}, t_{\max}]$ of the variable t, and a C^∞ function $Q : [t_{\min}, t_{\max}] \to \mathbf{R}$ vanishing at $t_{\min}$ and $t_{\max}$, positive on the open interval $(t_{\min}, t_{\max})$, such that the values of its derivative $\dot{Q} = dQ/dt$ at the endpoints $t_{\min}, t_{\max}$ are $2c$ and $-2c$.

Finally, the data also include the geometry of the base manifold, namely,

(9.3) a C^∞ curve $[t_{\min}, t_{\max}] \ni t \mapsto p_t$ of Riemannian metrics on N.

The symbol t will simultaneously be used for a nonconstant function with the extrema $t_{\min}, t_{\max}$ on a manifold (such as S^k, $k \geq 2$, or the total space M of an S^k bundle). The dot $(\,)^{\cdot}$ always stands for the derivative d/dt.

Fixing the data (9.2) amounts to choosing a Riemannian metric b on S^k and a surjective C^∞ function $t : S^k \to [t_{\min}, t_{\max}]$, both $\mathrm{SO}(k)$-invariant, such that the b-Hessian of t equals some function times b. This is summarized as follows:

Lemma 9.1. *Given the data* (9.2)*, a solution* $r : (t_{\min}, t_{\max}) \to (0, \infty)$ *to the equation* $Q\dot{r} = cr$*, and a Euclidean space* V *of dimension* $k \geq 2$ *with the inner product* $\langle\,,\rangle$*, let* S^k *be the* k*-sphere obtained when two disjoint copies of* V *are glued together by the inversion diffeomorphism* $V \setminus \{0\} \ni v \mapsto v/\langle v, v\rangle \in V \setminus \{0\}$*. The first-copy embedding* $V \to S^k$ *allows us to identify* $V \setminus \{0\}$ *with an open subset of* S^k*. Denoting by* r *also the Euclidean norm function* $V \to \mathbf{R}$*, we use the inverse diffeomorphism* $r \mapsto t$ *of* $r : (t_{\min}, t_{\max}) \to (0, \infty)$ *to treat* t, Q *as functions of* r*, and, consequently, also as* C^∞ *functions* $V \setminus \{0\}$*. Then*

(a) S^k *admits a unique Riemannian metric* b *equal to* $(cr)^{-2}Q\langle\,,\rangle$ *on* $V \setminus \{0\}$*,*
(b) $t, Q, \dot{Q}$ *and* $\ddot{Q}$ *have unique extensions to* C^∞ *functions on* S^k*,*
(c) $2\mathrm{D}dt = \dot{Q}b$ *and* $b(\mathrm{D}t, \mathrm{D}t) = Q$ *on* S^k*, where* D *stands for both the Levi-Civita connection of* b *and the* b*-gradient,*
(d) *at points in* S^k *at which* $dt \neq 0$*, the gradient* $\mathrm{D}t$ *is an eigenvector of the Ricci tensor* r^b *of* b *for the eigenvalue* $(1-k)\ddot{Q}/2$*, while all nonzero vectors orthogonal to* $\mathrm{D}t$ *are eigenvectors of* r^b *corresponding to the eigenvalue* $(2-k)(\dot{Q}^2 - 4c^2)(4Q)^{-1} - \ddot{Q}/2$*.*

Proof. See Appendix S. □

Using the data (9.1) – (9.3) the metric b on S^k described in Lemma 9.1, we now define a Riemannian metric g on the total space M by

(9.4) i) $g = \gamma$ on $\mathcal{V}$, ii) $g = \pi^* p_t$ on $\mathcal{H}$, iii) $g(\mathcal{V}, \mathcal{H}) = 0$.

More precisely, (9.4.iii) means that the vertical distribution $\mathcal{V}$ on M is g-orthogonal to the horizontal distribution $\mathcal{H}$, while (9.4.ii) states that, at any point $x \in M$, the restriction of the metric g_x to the horizontal space H_x coincides with the restriction to H_x of the pullback $\pi^* p_{t(x)}$, where $\pi : M \to N$ is the bundle projection and $t(x)$ is the value at x of $t : M \to \mathbf{R}$.

the remainder of this section consists of
unfinished text, still in preparation

Although the metric b on S^k described in Lemma 9.1 depends on the choice of the positive function $t \mapsto r$ with $Q\dot{r} = cr$,

since the metric g constructed on M using b and the data the construction of

Given a C^∞ curve $[t_{\min}, t_{\max}] \ni t \mapsto p_t$ of Riemannian metrics on a manifold N, a real vector bundle $\mathcal{L}$ of fibre dimension $k \geq 2$ over N, a fibre metric $\langle\,,\rangle$ in $\mathcal{L}$, and a connection in $\mathcal{L}$ that makes $\langle\,,\rangle$ parallel, with the horizontal distribution $\mathcal{H}$ and the curvature tensor Z, let us consider the following five conditions. By $Z_{\phi\chi}$ and b_ϕ we mean here the 2-form and the symmetric 2-tensor at any point $y \in N$ with $Z_{\phi\chi} = \langle Z(\,\cdot\,,\cdot\,)\phi, \chi\rangle$ for fixed $\phi, \chi \in \mathcal{L}_y$, and $b_\phi = \sum_w \sum_\chi Z_{\phi\chi}(\,\cdot\,, w) Z_{\phi\chi}(\,\cdot\,, w)$, where $\phi \in \mathcal{L}_y$ is fixed, and w (or, χ) ranges over some/any basis of $T_y N$ (or, $\mathcal{L}_y$) orthonormal relative to p_t (or, $\langle\,,\rangle$).

Remark 9.2. $p_t(\Psi_t w, w') = \dot{p}_t(w, w')$,
$\mathrm{tr}^t \dot{p}_t = \mathrm{tr}\,\Psi_t$,
$\mathrm{tr}^t \ddot{p}_t = \mathrm{tr}\,\dot{\Psi}_t + \mathrm{tr}\,\Psi_t^2$,
$\mathrm{tr}\,\Psi_t^2 = p_t(\dot{p}_t, \dot{p}_t)$.

With $S = 2\dot{\phi} - \mathrm{tr}\,\Psi_t$, condition (6.1) holds for our (M, g) and ϕ if and only if

(a) $\delta^t \dot{p}_t = d\,\mathrm{tr}^t \dot{p}_t = 0$, where tr^t is the p_t-trace, and δ^t is the divergence operator corresponding to p_t,
(b) for any $t, s \in [t_{\min}, t_{\max}]$, the ratio $\mathrm{d}p_t/\mathrm{d}p_s$ of the volume elements of p_t and p_s is a constant, depending on t and s,
(c) $\delta^t Z = 0$, where δ^t may now depend on the connection in $\mathcal{L}$ as well,
(d) $p_t(Z_{\phi\chi}, Z_{\phi\chi}) = \Pi$ for some constant Π depending just on t, all $y \in N$, and all $\phi, \chi \in \mathcal{L}_y$ such that $\langle\phi,\phi\rangle = \langle\chi,\chi\rangle = 1$ and $\langle\phi,\chi\rangle = 0$.
(e) $b_\phi = \Sigma p_t$ at every $y \in N$, for all $\phi \in \mathcal{L}_y$ with $\langle\phi,\phi\rangle = 1$, and for some constant Σ depending only on t,
(f) and, finally,

$$(9.5)\qquad \begin{aligned} &4\mathrm{r}^t + SQ\dot{p}_t - 2Q\ddot{p}_t - k\dot{Q}\dot{p}_t - 2c^{-2}\Sigma Q p_t + 2Qp_t(\Psi_t\,\cdot\,, \Psi_t\,\cdot\,) = 4\lambda p_t\,,\\ &SQ\dot{Q} - 2Q\ddot{Q} + (2-k)(\dot{Q}^2 - 4c^2) + c^{-2}\Pi Q^2 = 4\lambda Q\,,\\ &S\dot{Q} + 2(1-k)\ddot{Q} + (2\dot{S} - \mathrm{tr}\,\Psi_t^2)Q = 4\lambda. \end{aligned}$$

10. Uniqueness of Kähler-Ricci solitons

An affirmative answer to the second question in (1.6) is provided by assertions (iii) – (iv) in following theorem, in which (ii) and (iii) are due to Calabi [16], while (iv), first proved in the Kähler-Einstein case by Bando and Mabuchi [3], was generalized to Kähler-Ricci solitons by Tian and Zhu [115].

Throughout this section, whenever (M,g) is a specific Kähler manifold, $\varOmega,\rho,J$ and $\mathfrak{h}(M)$ will denote its Kähler form, its Ricci form, the complex structure tensor of M, and the Lie algebra of all (real) holomorphic vector fields on M. For more on Kähler geometry, see Appendices F through L.

Theorem 10.1. *Given Kähler metrics g and $\hat{g}$ on a compact complex manifold M, let both (M,g) and $(M,\hat{g})$ be Ricci solitons, so that they satisfy* (0.1) *with some constants $\lambda,\hat{\lambda}$ and some vector fields $w,\hat{w}$.*

- (i) *The constants λ and $\hat{\lambda}$ then are both positive, both zero, or both negative.*
- (ii) *If $\lambda=0$ and the Kähler forms of g and $\hat{g}$ represent the same cohomology class in $H^2(M,\mathbf{R})$, then $\hat{g}=g$.*
- (iii) *If $\lambda<0$, then $\hat{\lambda}\hat{g}=\lambda g$.*
- (iv) *If $\lambda>0$, then $\hat{\lambda}\hat{g}=\lambda F^*g$ for some biholomorphism $F:M\to M$ which lies in the identity component of the complex automorphism group of M.*

Theorem 10.1 has no direct analogue for real Ricci solitons. Even in dimension 2, two hyperbolic metrics on a given compact surface of genus greater than 1 need not be isometric. In dimension 4, an example of this kind is provided by Page's Einstein metric [99] and the non-Einstein Koiso-Cao soliton [78], [17], which coexist on the complex surface obtained by blowing up a point in $\mathbf{C}\mathrm{P}^2$. Similarly, the two-point blow-up of $\mathbf{C}\mathrm{P}^2$ carries both the conformally-Kähler Einstein metric of Chen, LeBrun and Weber [34], and the toric Kähler-Ricci soliton of Wang and Zhu [116]. Also, various spheres S^n, $n\geq 5$, have been shown to admit nonstandard Einstein metrics [77], [12], [7].

We begin by proving assertions (i) – (iii) in Theorem 10.1. The crucial steps used here to establish (ii) and (iii) appear in Appendix H: for (iii), we follow Calabi's original argument [16], while the proof of (ii) is due to Yau [119].

Proof of (i) – (iii) *in* Theorem 10.1. As $\lambda[\varOmega]=[\rho]=2\pi c_1(M)\in H^2(M,\mathbf{R})$ according to Proposition 8.2 (and Remark K.1(ii)), the three possible values of $\operatorname{sgn}\lambda$ correspond to the cases where $c_1(M)$ is positive, zero, or negative, which are mutually exclusive (see the end of Appendix J). This proves (i). Assertion (ii) is in turn an immediate consequence of Theorem K.2(a) in Appendix K, since, according to Theorem 4.4, a compact Ricci soliton with $\lambda=0$ in (0.1) must be Ricci-flat.

Now let $\lambda<0$. By Theorem 4.4, g and $\hat{g}$ are Kähler-Einstein metrics with the negative Einstein constants λ and $\hat{\lambda}$. Hence, replacing g and $\hat{g}$ by λg and $\hat{\lambda}\hat{g}$, we obtain two Kähler metrics g and $\hat{g}$ with the Ricci tensors $\mathrm{r}=-g$ and $\hat{\mathrm{r}}=-\hat{g}$. Theorem K.2(b) in Appendix K now gives $g=\hat{g}$, completing the proof. □

Tian and Zhu derive Theorem 10.1(iv) from the following two results of theirs [114, Lemma A.2 and Theorem A in §7]. As usual, $\mathfrak{h}(M)$ is the complex Lie algebra of all (real) holomorphic vector fields on M. (See Remark L.2 in Appendix L.)

Theorem 10.2. *Given a compact Kähler-Ricci soliton (M,g), let w be the unique gradient vector field with* (0.1), *cf.* Remark 6.4. *The operator $\mathrm{Ad}\,w:\mathfrak{h}(M)\to\mathfrak{h}(M)$ sending v to $[w,v]$ then is self-adjoint and nonnegative relative to some Hermitian inner product in $\mathfrak{h}(M)$, while its kernel, as a real subspace of $\mathfrak{h}(M)$, is spanned by $\mathfrak{g}\cup J\mathfrak{g}$, where $\mathfrak{g}$ is the real Lie algebra of all Killing fields on (M,g).*

Theorem 10.3. *For any compact Kähler-Ricci soliton* (M,g)*, the identity component* $\mathrm{Isom}^{\mathrm{o}}(M,g)$ *of the isometry group of* (M,g) *is a maximal compact connected Lie subgroup of the biholomorphism group* $\mathrm{Aut}(M)$*.*

Proofs of Theorems 10.2, 10.3 and 10.1(iv) are given in Appendix P. They rely on some other results of Tian and Zhu [115], which we present next, beginning with some motivating remarks. First, if (M,g) is a compact Riemannian manifold and $\mathrm{s}_{\mathrm{avg}}$ denotes the average value of its scalar curvature s, (2.20.b) and (2.16) imply that

(10.1) $\quad \Delta f + \mathrm{s} = \mathrm{s}_{\mathrm{avg}}$ for some $f : M \to \mathbf{R}$, unique up to an additive constant.

Now, given a vector field v on M, we set

$$Lv = \delta v - d_v f \quad \text{for} \quad f \text{ as in (10.1)}, \tag{10.2}$$

where δ is the divergence, cf. (2.7.i). The resulting linear differential operator L sends vector fields on M to functions $M \to \mathbf{R}$, and gives rise to the functional

$$v \mapsto \int_M e^{Lv}\,\mathrm{d}g \tag{10.3}$$

on the space of all C^∞ vector fields on M. Furthermore,

$$Lu = 0 \quad \text{if} \quad u \text{ is a Killing field.} \tag{10.4}$$

In fact, by (10.1), the isometries forming the flow of u leave f invariant up to additive constants. Thus, $d_u f$ is constant and so, by (2.18), $d_u f = 0$. On the other hand, $\delta u = 0$ due to skew-adjointness of ∇u and (2.7.i).

The importance of (10.3) for our discussion lies in the fact that a critical point for (10.3) naturally arises in every compact gradient Ricci soliton (M,g) with $\mathrm{s}_{\mathrm{avg}} \neq 0$. Specifically, it is the vector field $u = \nabla f/(2\lambda)$, where f and λ are as in (6.1), so that, by (3.2.ii), $\lambda = \mathrm{s}_{\mathrm{avg}}/n$, for $n = \dim M$, and $\Delta f + \mathrm{s} = \mathrm{s}_{\mathrm{avg}}$. In fact, (2.14.i), (2.8.b) and (6.3) imply that Lu then differs from $-f$ by a constant. However, by (2.15), $\int_M (Lv)e^{-f}\mathrm{d}g = 0$ for *any* vector field v, since the integrand is the divergence of $e^{-f}v$ (cf. (2.8.a)).

That $u = \nabla f/(2\lambda)$ is a critical point for (10.3), whenever $\nabla df + \mathrm{r} = \lambda g$ with a constant $\lambda \neq 0$, will obviously remain true even if we modify (10.3), replacing the exponent Lv by $Pv = Lv + zL(Av)$ for a fixed complex number z and a fixed linear operator A acting on vector fields, as long as $L(Au) = 0$. An important case of such a modification of L arises when (M,g) is a compact, gradient, Kähler-Ricci soliton and $Av = Jv$, as one sees using (10.4) and noting that Jw, for $w = \nabla f/2$, is a Killing field (Lemma 8.1(b)). It is also natural to choose $z = -i$, since the corresponding P then is the unique complex-linear operator with the real part L.

Following Tian and Zhu [115, formula (2.3)], except for notation, we now set

$$\mathcal{F}(w) = \mu \int_M e^{Pw}\,\mathrm{d}g \quad \text{with} \quad Pw = Lw - iLJw \ \text{ and } \ \mu = (\mathrm{s}_{\mathrm{avg}})^m, \tag{10.5}$$

for any compact Kähler manifold (M,g) of complex dimension m, and for w which, rather than being an arbitrary C^∞ vector field on M, is now assumed to be holomorphic. In other words, (10.5) defines a function $\mathcal{F} : \mathfrak{h}(M) \to \mathbf{C}$, and P in (10.5) is obtained by restricting the modified version of (10.3) to the finite-dimensional Lie algebra $\mathfrak{h}(M)$ of all (real) holomorphic vector fields on M.

We use here the multiplicative notation, without parentheses, for differential operators (including bundle morphisms) acting on sections of vector bundles; thus, LJw in (10.5) stands for $L(Jw)$ (and ∇LJw will later denote its gradient). The factor $\mu = (\mathrm{s_{avg}})^m$ in (10.5) is introduced to make $\mathcal{F}$ scale-invariant, that is, unchanged when g is replaced by cg with $c \in (0, \infty)$. Furthermore,

(10.6) a) $d\mathcal{F}_w v = \mu \int_M (Pv) e^{Pw} \,\mathrm{d}g$, b) $(d_u d_v \mathcal{F})(w) = \mu \int_M (Pu)(Pv) e^{Pw} \,\mathrm{d}g$

for any $u, v, w \in \mathfrak{h}(M)$, where the real-linear function $d\mathcal{F}_w : \mathfrak{h}(M) \to \mathbf{C}$ is the differential of $\mathcal{F}$ at w, so that $d\mathcal{F}_w v$ equals $(d_v \mathcal{F})(w)$, the value at w of the directional derivative of $\mathcal{F}$ in the direction of the constant vector field v on $\mathfrak{h}(M)$, while $d_u d_v \mathcal{F} = d_u(d_v \mathcal{F})$ is a second-order directional derivative. In fact, since P is complex-linear, $d\mathcal{F}_w$ (or, $d(d_v\mathcal{F})_w$) is the composite of P with the differential of the function $\psi \mapsto \mu \int_M e^{\psi} \mathrm{d}g$ (or, $\psi \mapsto \mu \int_M (Pu) e^{\psi} \mathrm{d}g$), defined on a suitable finite-dimensional function space.

By (10.6.a), $d\mathcal{F}_w$ is complex-linear for any $w \in \mathfrak{h}(M)$, and so the function $\mathcal{F} : \mathfrak{h}(M) \to \mathbf{C}$ is holomorphic; note that $\mathfrak{h}(M)$ is a complex space, with $v \mapsto Jv$ serving as the multiplication by i (cf. Remark L.2). We will refer to $\mathcal{F}$ as the *Tian-Zhu invariant.*

If (M, g) is a compact Kähler manifold with $[\rho] = \lambda[\Omega] \in H^2(M, \mathbf{R})$ for some real number $\lambda \neq 0$, the restriction of the operator P in (10.5) to $\mathfrak{h}(M)$ is injective:

(10.7) $\mathfrak{h}(M) \cap \operatorname{Ker} P = \{0\}$ whenever $[\rho] = \lambda[\Omega] \neq 0$.

Namely, Lemma N.3(a) in Appendix N shows that $w = 0$ if $w \in \mathfrak{h}(M)$ satisfies the condition $Pw = 0$ (equivalent to $Lw = LJw = 0$).

The above discussion is summarized by the following result, due to Tian and Zhu [115, Proposition 3.1]:

Proposition 10.4. *Given a compact Kähler-Ricci soliton (M, g), let λ and w be the unique real constant and gradient vector field that satisfy* (0.1)*, cf.* Remark 6.4*. If $\lambda \neq 0$, then $\lambda^{-1} w$ is a critical point of the Tian-Zhu invariant $\mathcal{F} : \mathfrak{h}(M) \to \mathbf{C}$.*

As a next step, we define the *Futaki invariant* [53] of a compact Kähler manifold (M, g) to be the real-linear functional $\boldsymbol{F} : \mathfrak{h}(M) \to \mathbf{R}$ such that, if f is chosen as in (10.2) and $\mu = (\mathrm{s_{avg}})^m$ with $m = \dim_{\mathbf{C}} M$, then

(10.8) $\boldsymbol{F} v = \mu \int_M d_v f \,\mathrm{d}g$ for $v \in \mathfrak{h}(M)$, or, equivalently, $\boldsymbol{F} = -\operatorname{Re} d\mathcal{F}_0$.

That $\boldsymbol{F} = -\operatorname{Re} d\mathcal{F}_0$ is clear from (10.6.a) for $w = 0$, (10.5) and (2.15). The Futaki invariant constitutes a well-known obstruction [54] to the existence of Kähler-Einstein metrics on compact complex manifolds M with $c_1(M) > 0$. See Theorem N.1 in Appendix N.

Throughout this section, positivity of $c_1(M)$ is defined as at the beginning of Appendix N, and so it amounts to the existence, on the given compact complex manifold, of a Kähler metric whose Kähler form Ω and Ricci form ρ satisfy the relation $[\rho] = \lambda[\Omega] \in H^2(M, \mathbf{R})$ for some real number $\lambda > 0$.

The next two results are due to Tian and Zhu [115, p. 305], [115, Lemma 2.2]:

Theorem 10.5. *For any compact complex manifold M with $c_1(M) > 0$, the Tian-Zhu invariant $\mathcal{F} : \mathfrak{h}(M) \to \mathbf{C}$, defined with the aid of a Kähler metric g satisfying the condition $[\rho] = \lambda[\Omega] \in H^2(M, \mathbf{R})$ for some $\lambda \in \mathbf{R}$, depends only on the complex structure of M, and not on the choice of such a metric g.*

Proof. See Appendix N. □

Lemma 10.6. *Suppose that M is a compact complex manifold with $c_1(M) > 0$, while $\mathfrak{p} \subset \mathfrak{h}(M)$ is the image, under the transformation $v \mapsto Jv$, of the real Lie* *subalgebra of* $\mathfrak{h}(M)$ *corresponding to any fixed maximal compact connected Lie subgroup* K *of the biholomorphism group of* M*. The restriction to* $\mathfrak{p}$ *of the Tian-Zhu invariant* $\mathcal{F} : \mathfrak{h}(M) \to \mathbf{C}$ *then is real-valued, and* $\mathcal{F} : \mathfrak{p} \to \mathbf{R}$ *has exactly one critical point.*

Only the uniqueness part of Lemma 10.6 is needed for proving Theorem 10.1. This is why the proof of the existence assertion is postponed until §11.

Proof of uniqueness of the critical point. According to Theorem 10.5, we can evaluate $\mathcal{F}$ using any Kähler metric g on M with the Kähler form Ω such that $[\Omega] = 2\pi c_1(M)$. Let us choose g with this property which is also K-invariant, for instance, one obtained by starting from any such metric, then averaging it over K.

For any $w \in \mathfrak{p}$, the function Pw in (10.5) is real-valued (by (10.4), since Jw is a Killing field). Thus, (10.7) and real-valuedness of Pw, Pv in (10.6.b) give $d_v d_v \mathcal{F} > 0$ on $\mathfrak{p}$ whenever $v \in \mathfrak{p}$ and $v \neq 0$.

If $\mathcal{F} : \mathfrak{p} \to \mathbf{R}$ now had two different critical points w and $w + v$, setting $\chi(t) = \mathcal{F}(w + tv)$ and $(\ )^\cdot = d/dt$ for $t \in \mathbf{R}$ we would get $\ddot{\chi} > 0$, contrary to the equalities $\dot{\chi}(0) = \dot{\chi}(1) = 0$. □

11. Existence of Kähler-Ricci solitons

in preparation

We will need the remaining part of Lemma 10.6:

The existence of a critical point. First, the operator $\mathfrak{h}(M) \ni w \mapsto Pw$ given by (10.5) is injective: if $Pw = 0$, that is, $Lw = LJw = 0$, Lemma N.3(a) in Appendix N gives $w = 0$. Thus, $\mathfrak{p} \ni w \mapsto Pw \in \mathcal{X}$ is a linear isomorphism.

To prove that $\mathcal{F} : \mathfrak{p} \to \mathbf{R}$ has a critical point (actually, a minimum), we now consider the space $\mathcal{X}$ of real-valued functions on M, obtained as the image of $\mathfrak{p}$ under the linear operator $w \mapsto Pw$. For the function $\boldsymbol{F} : \mathcal{X} \to \mathbf{R}$ given by $\boldsymbol{F}(\psi) = \int_M e^\psi \mathrm{d}g$, we have $\boldsymbol{F}(\psi) \to \infty$ as $|\psi| \to \infty$, where $|\ |$ is some (or any) norm in $\mathcal{X}$. In fact, given a sequence $\psi[j] \in \mathcal{X}$, $j = 1, 2, 3, \ldots$, with $|\psi[j]| \to \infty$ as $j \to \infty$, we may assume, passing to a subsequence, that $\psi[j]/|\psi[j]| \to \psi$ as $j \to \infty$, for some $\psi \in \mathcal{X}$. Since $|\psi| = 1$ and $\int_M \psi e^{-f} \mathrm{d}g = 0$ in (10.5), we may fix $\varepsilon \in (0, \infty)$ and a nonempty open set $U \subset M$ with $\psi > 2\varepsilon$ on U. Choosing $|\ |$ to be the supremum norm, we now get $\psi[j] \geq \varepsilon|\psi[j]|$ on U for large j, and so $\boldsymbol{F}(\psi[j]) \geq \int_U e^{\psi[j]}\,\mathrm{d}g \geq \int_U e^{\varepsilon|\psi[j]|}\mathrm{d}g \to \infty$ as $j \to \infty$, which shows that $\boldsymbol{F}(\psi) \to \infty$ as $|\psi| \to \infty$. Next, since $w \mapsto Pw$ is a linear isomorphism $\mathfrak{p} \to \mathcal{X}$ and $\mathcal{F}(w) = \boldsymbol{F}(Pw)$, we also have $\mathcal{F}(w) \to \infty$ as $w \to \infty$ in $\mathfrak{p}$. Thus, $\mathcal{F}$ has a minimum value in $\mathfrak{p}$, which completes the proof. □

We say that a compact complex manifold M with $c_1(M) > 0$ is *toric* if its biholomorphism group contains a Lie subgroup isomorphic to the real torus T^m, where $m = \dim_{\mathbf{C}} M$. If this is the case, we usually fix such a subgroup, and refer to its action on N simply as *the torus action.*

The following theorem is due to Wang and Zhu [116].

Theorem 11.1. *Let* M *be a compact complex manifold with* $c_1(M) > 0$*. If* M *is toric, then there exists a Kähler metric on* M *which is at the same time a Ricci soliton invariant under the torus action.*

Proof.

in preparation

□

Appendix A. Hopf's maximum principle

For the reader's convenience, we give here a standard proof of Hopf's maximum principle [69] for manifolds without boundary.

Theorem A.1. *If f is a C^2 function on a Riemannian manifold (M,g) and $\Delta f \le \psi f + d_v f$ for some nonnegative function ψ and a vector field v, such that both ψ and $|v|$ are locally bounded, then f cannot have a nonpositive minimum in M unless it is constant on M.*

Proof. Suppose that, on the contrary, $f_{\min} = c \le 0$ and the set $f^{-1}(c)$ has a boundary point y' (that is, $\emptyset \ne f^{-1}(c) \ne M$). Let us now fix a diffeomorphic identification of a neighborhood U' of y' with a Euclidean ball $\{x : |x - y'| < 1\}$ and then choose $z' \in U'$ with $|z' - y'| < 1/3$ and $f(z') > c$. A point y in $U' \cap f^{-1}(c)$, nearest z', thus lies on the boundary sphere $\partial U \subset U'$ of an open ball U of some radius $r' < 1/2$, centered at z', and $f > c$ on U. Let us now fix a point z with $y \ne z \ne z'$ on the segment joining z' to y. The formula $h_a(x) = e^{-a|y-z|^2} - e^{-a|x-z|^2}$, with $a > 0$, defines a function $h_a : U \to \mathbf{R}$. For the operator $\mathcal{P}$ with $\mathcal{P}f = \Delta f - \psi f - d_v f$ and the closed ball $Y = \{x : |x - y| \le r\}$ of any fixed radius $r < |y - z|$, the expression $a^{-2} e^{a|x-z|^2}(\mathcal{P}h_a + e^{-a|y-z|^2}\psi)$ converges to some negative function, uniformly on Y, as $a \to \infty$, which one easily verifies noting that $Y \subset U'$ and $\mathcal{P}f = g^{jk}\partial_j\partial_k f + w^j\partial_j f - \psi f$ for some coefficient functions g^{jk}, w^j. Since $\psi \ge 0$, we may thus choose a such that $\mathcal{P}h_a < 0$ on Y. Setting $f_\varepsilon = f + \varepsilon h_a$ for any $\varepsilon > 0$ we now get $\mathcal{P}f_\varepsilon < 0$ on Y, due to our assumption that $\mathcal{P}f \le 0$ on M. For ε close to 0, we also have $f_\varepsilon > c$ on the sphere ∂Y. Namely, as $f \ge c$ on U', this is obvious, for any ε, at those points of ∂Y at which $h_a > 0$. On the other hand, points $x \in \partial Y$ with $h_a(x) \le 0$ (that is, $|x - z| \le |y - z|$) form a set Z *contained in* U, due to the relation $|x - z'| \le |x - z| + |z - z'| \le |y - z| + |z - z'| = |y - z'| = r'$. (Note that the resulting inequality $|x - z'| \le r'$ must be strict: otherwise we would have $|x - z'| = |x - z| + |z - z'|$ and so x, y would both lie on the ray emanating from z' through z, at the same distance r' from z', which would give $x = y$, even though $|x - y| = r > 0$ as $x \in \partial Y$.) Since $Z \subset U$, we have $f > c$ on Z, so that compactness of Z gives $f_\varepsilon > c$ on Z (and hence on ∂Y) for sufficiently small $\varepsilon > 0$. However, $f_\varepsilon(y) = c \le 0$. The minimum value of f_ε on Y thus must be nonpositive, and is assumed at an interior point of Y. At such a point, relation $\mathcal{P}f_\varepsilon < 0$ gives $0 \le \Delta f_\varepsilon < \psi f_\varepsilon \le 0$. This contradiction completes the proof. □

Corollary A.2. *A nonconstant C^∞ function ϕ on a Riemannian manifold (M,g), such that $\Delta\phi \ge d_v\phi$ for some C^∞ vector field v, cannot assume a maximum value anywhere in M.*

In fact, that would contradict Theorem A.1 with $\psi = 0$ and $f = \phi_{\max} - \phi$.

Appendix B. The Bourguignon-Ezin theorem

The following result of Bourguignon and Ezin [11] is used to prove Theorem 4.4.

Theorem B.1. *Let* w *be a conformal vector field on a Riemannian manifold* (M,g)*, in the sense that* $n\pounds_w g = 2(\delta w)g$*, where* $n = \dim M$*. If* M *is compact, then* $\int_M d_w\mathrm{s}\,\mathrm{d}g = 0$*, with* s *denoting the scalar curvature of* g*.*

Proof. We have $2\mathrm{s}\delta w = \langle \mathrm{r}, \pounds_w g\rangle$, since w is conformal and $\mathrm{s} = \langle g, \mathrm{r}\rangle$. Now (2.11.e) and (2.8.a) for $f = \mathrm{s}$ give $(n-2)d_w\mathrm{s} = 2n\delta\imath_w\mathrm{e}$ (cf. (2.3.iii)), and (2.15) yields our assertion for $n > 2$.

Now let $n = 2$. As g is conformal to a metric of constant curvature, it suffices to show that $\int_M d_w\mathrm{s}\,\mathrm{d}g$ is a conformal invariant. This in turn follows from (2.15), since, for $\tilde{\mathrm{s}}, \mathrm{d}\tilde{g}$ depending on the metric $\tilde{g} = e^\phi g$, conformal to g, just as $\mathrm{s}, \mathrm{d}g$ depend on g, we have $(d_w\tilde{\mathrm{s}})\mathrm{d}\tilde{g} = d_w\mathrm{s}\,\mathrm{d}g + \sigma\,\mathrm{d}g$ with a function σ such that $\sigma = \delta v$ for some vector field v on M.

Namely, $\sigma = (d_w\phi)\Delta\phi - (d_w\Delta\phi + \mathrm{s}d_w\phi)$, since $\tilde{\mathrm{s}} = e^{-\phi}(\mathrm{s}-\Delta\phi)$ and $\mathrm{d}\tilde{g} = e^\phi\,\mathrm{d}g$. We now show that, if $n = 2$ and w is conformal, σ defined by this formula with any given function ϕ is, explicitly, a divergence. First, $d_w\Delta\phi + \mathrm{s}d_w\phi = \delta[2\nabla_w u - (\Delta\phi)w]$, for $u = \nabla\phi$, as one sees from the local-coordinate calculation $2(\phi_{,jk}w^k)^{,j} - (\phi_{,j}{}^j w^k)_{,k} = 2\phi_{,j}{}^j{}_k w^k + 2\phi_{,jk}w^{k,j} - \phi_{,j}{}^j{}_k w^k - \phi_{,j}{}^j w^k{}_{,k} = \phi_{,j}{}^j{}_k w^k + \mathrm{s}w^k\phi_{,k} = d_w\Delta\phi + \mathrm{s}d_w\phi$. Here we reduced four terms to two by replacing $2\phi_{,j}{}^j{}_k$ (and $2\phi_{,jk}w^{k,j}$) with $2\phi_{,j}{}^{jk} + \mathrm{s}\phi^k$ (and, respectively, $\phi_{,jk}(w^{j,k} + w^{k,j}) = \phi_{,j}{}^j w^k{}_{,k}$), using (2.11.b) with $\mathrm{r} = \mathrm{s}g/2$ for $u = \nabla\phi$ instead of w (or, respectively, using relation $w^{j,k} + w^{k,j} = w^l{}_{,l}g^{jk}$, which states that w is conformal). Finally, for any function ϕ and vector field w in dimension $n = 2$, setting $a = \nabla d\phi - d\phi\otimes d\phi$ and $u = \nabla\phi$ we have $8(d_w\phi)\Delta\phi = 2\langle\pounds_w g - (\delta w)g, a\rangle + \delta v'$, where $v' = 4(\Delta\phi)\phi w - 4\phi\nabla_w u + 4(d_w\phi)u + \mathrm{s}\phi^2\delta w + \phi^2\nabla\mu - 2\mu\phi u$ and $\mu = \delta w$. □

Appendix C. Zhang's argument

The result presented here and its proof are due to Zhang [120]. The meaning of inequalities between tensors is the same as in Theorem 5.4.

Theorem C.1. *Let the Ricci tensor* r *of a complete Riemannian manifold* (M,g) *of dimension* $n \geq 2$ *and a* C^∞ *vector field* w *on* M *satisfy the inequalities* $\lambda g - \pounds_w g \leq \mathrm{r} \leq \kappa g$ *for some positive constants* κ *and* λ*. Then*

(a) *the fundamental group of* M *is finite*;

(b) M *is compact if, in addition,* $g(w,w)$ *is bounded on* M*.*

Proof. To prove (a) (or, respectively, (b)), we fix a point $y \in M$, set $y' = y$, and use the constant C by $C = |w(y)|$ (or, respectively, consider two arbitrary points $y, y' \in M$ and denote by C the supremum, over M, of the norm $|w| = [g(w,w)]^{1/2}$). Let $[0,\mathsf{L}] \ni t \mapsto x(t) \in M$ be a unit-speed geodesic with $y = x(0)$ and $y' = x(\mathsf{L})$, having the minimum length: in case (a), among piecewise-C^∞ loops in its fixed-end homotopy class, or, in case (b), among all piecewise-C^∞ curves joining y to y'. It follows that

$$\text{(C.1)} \qquad \lambda\mathsf{L} \leq 4C + \int_0^{\mathsf{L}}[(n-1)\dot\varphi^2 + \kappa(1-\varphi^2)]\,dt$$

for every piecewise-C^∞ function $\varphi : [0,\mathsf{L}] \to [-1,1]$ with $\varphi(0) = \varphi(\mathsf{L}) = 0$. In fact, we have (5.1.b), since our proof of Lemma 5.1 uses only the assumption that the geodesic minimizes the length among all nearby curves in its fixed-end homotopy class. As $\int_0^{\mathsf{L}} \varphi^2 \mathrm{r}(\dot{x},\dot{x})\,dt = \int_0^{\mathsf{L}} \mathrm{r}(\dot{x},\dot{x})\,dt - \int_0^{\mathsf{L}}(1-\varphi^2)\mathrm{r}(\dot{x},\dot{x})\,dt$, (C.1) is now immediate from (5.1.b): the inequality $\mathrm{r} \ge \lambda g - \pounds_w g$ (or, $\mathrm{r} \le \kappa g$ with $\varphi^2 \le 1$) gives $\int_0^{\mathsf{L}} \mathrm{r}(\dot{x},\dot{x})\,dt \ge \lambda\mathsf{L} - 4C$ (or, respectively, $\int_0^{\mathsf{L}}(1-\varphi^2)\mathrm{r}(\dot{x},\dot{x})\,dt \le \kappa\int_0^{\mathsf{L}}(1-\varphi^2)\,dt$). Note that $\int_0^{\mathsf{L}}(\pounds_w g)(\dot{x},\dot{x})\,dt = \gamma(\lambda)-\gamma(0)$, where $\gamma = 2g(w,\dot{x})$, cf. the proof of Theorem 5.4.

For any $\varepsilon \in (0,\mathsf{L}/2)$, we have $\lambda\mathsf{L} \le F(\varepsilon)$, with $F:(0,\infty)\to(0,\infty)$ given by $F(\varepsilon) = 4C + 2(n-1)\varepsilon^{-1} + 4\kappa\varepsilon/3$. This is clear from (C.1) applied to the function φ vanishing at 0 and L, equal to 1 on $[\varepsilon,\mathsf{L}-\varepsilon]$, and linear on $[0,\varepsilon]$ and $[\mathsf{L}-\varepsilon,\mathsf{L}]$. Let $\mu > 0$ be the minimum value of F in $(0,\infty)$, assumed at a unique point $\varepsilon_0 \in (0,\infty)$. The lengths L now have an upper bound depending just on n,κ,λ and C. Namely, $\mathsf{L} \le \max(2\varepsilon_0,\mu/\lambda)$. In fact, if $\mathsf{L} > 2\varepsilon_0$, setting $\varepsilon = \varepsilon_0$ we get $\varepsilon \in (0,\mathsf{L}/2)$, and so $\lambda\mathsf{L} \le F(\varepsilon) = F(\varepsilon_0) = \mu$, that is, $\mathsf{L} \le \mu/\lambda$.

As M is compact whenever (M,g) is bounded, assertion (b) follows. To obtain (a), note that boundedness of the set of lengths of all geodesic loops at a fixed point y minimizing the length in their homotopy classes implies finiteness of the fundamental group. This is clear since an infinite sequence of such geodesic loops at y with uniformly bounded lengths L_j, $j=1,2,3,\ldots$, cannot represent infinitely many distinct homotopy classes, as one sees choosing a convergent subsequence of the sequence (u_j,L_j), where u_j is the initial unit tangent vector of the jth geodesic, and using continuity of the geodesic exponential mapping $\exp_y : T_yM \to M$. □

Theorem C.1 provides proofs of Theorems 5.6 and 5.7, more direct than those given in §5 (which use results of Ambrose [1] or Lott [86]). In addition, the inequality $\mathsf{L} \le \max(2\varepsilon_0,\mu/\lambda)$ in the above proof amounts to a diameter estimate similar to the one in Corollary 5.11 (and valid under the same assumptions), but differing from it by depending, via κ, on the maximum of the Ricci curvature.

Appendix D. Proof of Rothaus's Theorem 6.2

The argument presented here, due to Rothaus [104], follows the standard variational approach of realizing the equation $\mathcal{P}f + \lambda f = \Psi$, with a fixed C^∞ function $\Psi : M \to \mathbf{R}$ and an unknown C^∞ function f, as the Euler-Lagrange equation for some functional, and then proving the existence of a *minimizer*, that is, a C^∞ function f for which the functional assumes its minimum value. Since the functional (D.1) most obviously associated with our equation is unbounded, both from above and below, an additional (integral) constraint is needed as well.

It suffices to find f such that $\mathcal{P}f + \lambda f - \Psi$ is constant, since adding a suitable constant to f we then get $\mathcal{P}f + \lambda f - \Psi = 0$. In terms of the positive C^∞ function $\varphi = e^{-f/2} : M \to \mathbf{R}$, constancy of $\mathcal{P}f + \lambda f - \Psi$ means that $2\Delta\varphi + 2\lambda\varphi\log\varphi + \Psi\varphi$ is a constant multiple of φ, which in turn amounts to its being L^2-orthogonal to every L^2 function L^2-orthogonal to φ. Thus, the positive C^∞ function φ is required to be a critical point of the functional

$$\text{(D.1)} \qquad \varphi \mapsto \int_M (\varepsilon|\nabla\varphi|^2 - \varphi^2\log\varphi + H\varphi^2)\,\mathrm{d}g,$$

subject to the constraint $\|\varphi\| = 1$. Here $\|\ \|$ is the L^2 norm of functions on (M,g), while $\mathrm{d}g$ denotes, as usual, the Riemannian volume element, $\varepsilon = 1/\lambda$, and $H = -\varepsilon\Psi/2$. (The discussion would be exactly the same if we fixed the value of $\|\varphi\|$ to be any given positive real number, rather than 1.)

Recall that our compact Riemannian manifold (M,g) is assumed to be of dimension $n \geq 3$. Now, for any real constant $\varepsilon > 0$,

$$\inf \textstyle\int_M (\varepsilon|\nabla\varphi|^2 - \varphi^2 \log\varphi)\,\mathrm{d}g \; > \; -\infty\,, \tag{D.2}$$

the infimum being taken over all positive C^∞ functions $\varphi : M \to \mathbf{R}$ with $\|\varphi\| = 1$.

To obtain (D.2), we first establish *Jensen's inequality*, valid for any smooth positive probability-measure density ω on a compact manifold M and any continuous function $F : M \to \mathbf{R}$, which reads

$$\exp \int_M F\omega \; \leq \; \int_M e^F \omega\,, \tag{D.3}$$

This is immediate from the corresponding inequality for finite Riemann sums:

$$\mu_1^{c_1} \ldots\, \mu_k^{c_k} \; \leq \; c_1\mu_1 + \ldots + c_k\mu_k\,, \tag{D.4}$$

for $c_j \in [0,\infty)$ with $c_1 + \ldots + c_k = 1$ and $\mu_j \in (0,\infty)$, $j = 1,\ldots,k$, which is in turn easily verified by applying $d/d\mu_1$ to find the maximum of $\mu_1^{c_1} \ldots \mu_k^{c_k} - c_1\mu_1 - \ldots - c_k\mu_k$, where $\mu_2,\ldots,\mu_k$ and all c_j are kept fixed.

Next, $a\int_M \varphi^2 \log\varphi\,\mathrm{d}g = \int_M \varphi^2 \log\varphi^a \mathrm{d}g$ for φ as in (D.2) and any real $a > 0$, and $\int_M \varphi^2 \log\varphi^a \mathrm{d}g \; \leq \; (2+a)\log\|\varphi\|_{2+a}$, in view of (D.3) for $\omega = \varphi^2\mathrm{d}g$ and $F = \log\varphi^a$. Choosing $a = 4/(n-2)$, we now use the $p = 2$ case of the Sobolev inequality $\|\varphi\|_r \leq C\|\varphi\|_{p,1}$ with $r = np/(n-p)$, which holds, with a constant C depending only on M, g and p, whenever $p \in \mathbf{R}$ and $1 < p < n$. We thus get $2\int_M \varphi^2 \log\varphi\,\mathrm{d}g \; \leq \; n\log C\|\varphi\|_{2,1}$. As $\|\varphi\|_{2,1}^2 = \int_M (|\nabla\varphi|^2 + \varphi^2)\,\mathrm{d}g$, we now have $\int_M (\varepsilon|\nabla\varphi|^2 - \varphi^2\log\varphi)\,\mathrm{d}g \geq \Phi(\xi)$, where $\Phi(\xi) = \varepsilon(\xi^2 - 1) - (1 + 2/a)\log C\xi$ for $\xi = \|\varphi\|_{2,1} \geq \|\varphi\| = 1$. Now (D.2) follows since $\inf\{\Phi(\xi) : \xi \in [1,\infty)\} > -\infty$.

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further text in preparation

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Appendix E. Proof of Tachibana's Theorem 7.4

Our presentation, although phrased differently, follows Tachibana's argument in [109]. We begin with a simple fact from linear algebra.

Lemma E.1. *For nonnegative self-adjoint operators F, F' in a Euclidean space,*

(a) $\operatorname{tr} FF' \geq 0$, *while*

(b) $\operatorname{tr} FF' > 0$ *if, in addition,* $F > 0$ *and* $F' \neq 0$.

Proof. Evaluate $\operatorname{tr} FF'$ in an orthonormal basis diagonalizing F'. □

Let (M,g) be a Riemannian manifold, and let $\mathcal{E}$ be the real vector bundle over M whose sections are the four-times covariant tensor fields S on M with $S_{kjlm} = -S_{jklm} = S_{jkml} = S_{mljk}$ (that is, skew-symmetric both in the first and in the last pair of arguments, as well as symmetric under the switch of the two pairs); an example of such S is provided by the curvature tensor R of g. With every section S of $\mathcal{E}$ we can associate a bundle morphism $\hat{S} : [T^*M]^{\wedge 2} \to [T^*M]^{\wedge 2}$ corresponding to S just as $\hat{R}$ in Remark 2.2 corresponds to R. Symmetries of S clearly imply that $\hat{S}$ is self-adjoint, at each point, relative to the fibre metric $\langle\,,\rangle$ described in

Remark 2.1, and the assignment $S \mapsto \hat{S}$ is an isomorphic identification between sections of $\mathcal{E}$ and (pointwise) self-adjoint bundle morphisms $[T^*M]^{\wedge 2} \to [T^*M]^{\wedge 2}$. Setting $\langle S, S'\rangle = \operatorname{tr}\hat{S}\hat{S}'$, we obtain a fibre metric in $\mathcal{E}$, also denoted by $\langle\,,\rangle$. The curvature tensor R gives rise to a bundle morphism

$$\text{(E.1)}\qquad \underline{R} : [T^*M]^{\wedge 2} \to \mathcal{E}\,,\ \text{ with } \hat{S} = [\operatorname{Ad}\omega, \hat{R}] \text{ for } S = \underline{R}\omega \text{ and any 2-form } \omega.$$

More precisely, the self-adjoint bundle morphism $\hat{S} : [T^*M]^{\wedge 2} \to [T^*M]^{\wedge 2}$, for $S = \underline{R}\omega$, is the commutator of $\operatorname{Ad}\omega$ and $\hat{R}$, while $\operatorname{Ad}\omega$ denotes the skew-adjoint morphism $[T^*M]^{\wedge 2} \to [T^*M]^{\wedge 2}$ which, under the identification between 2-forms and skew-adjoint morphisms $TM \to TM$ provided by Remark 2.1, acts as the commutator $[\omega, \cdot]$. In local coordinates, for the adjoint $\underline{R}^* : \mathcal{E} \to [T^*M]^{\wedge 2}$ of $\underline{R}$ relative to the two fibre metrics, and any 2-form ω,

(i) $(\underline{R}\omega)_{jklm} \,=\, R_{jkl}{}^p\omega_{pm} - R_{jkm}{}^p\omega_{pl} + R_{lmj}{}^p\omega_{pk} - R_{lmk}{}^p\omega_{pj}$,
(ii) $2(\underline{R}^*S)_{jk} \,=\, R_j{}^{mpq}S_{kmpq} - R_k{}^{mpq}S_{jmpq}$ whenever S is a section of $\mathcal{E}$.

In fact, for $\underline{R}\omega$ and $\underline{R}^*S$ *defined* by (i) – (ii), $\langle S, \underline{R}\omega\rangle \,=\, S^{jklm}R_{jkl}{}^p\omega_{pm} = \langle\omega, \underline{R}^*S\rangle$, the contributions of the four (or, two) terms on the right-hand side of (i) (or, (ii)) being all equal due to their (skew)symmetry properties.

A section S of $\mathcal{E}$ also gives rise to a bundle morphism $S : [T^*M]^{\otimes 2} \to [T^*M]^{\otimes 2}$, denoted here simply by S, which acts on arbitrary twice-covariant tensor fields a via $(Sa)_{jl} = a^{km}S_{jklm}$. (Equivalently, $[S(\xi\odot\eta)](w,w') = g(S(u,w)v,w')$ for any vector fields u,v,w,w' and the 1-forms $\xi = \imath_u g$, $\eta = \imath_v g$.) On every Riemannian manifold (M,g) we thus have the function $\phi : M \to \mathbf{R}$ given by

$$\text{(E.2)}\qquad \phi \,=\, 2\langle R\bullet R, \mathrm{r}\rangle - 2\operatorname{tr}\hat{R}^3 - \operatorname{tr}R^3, \quad \text{where } S = g\wedge\mathrm{r}\,.$$

Here "products" represent composites, r stands for the Ricci tensor, $\operatorname{tr}R^3$ denotes the trace of the third iteration of $R : [T^*M]^{\otimes 2} \to [T^*M]^{\otimes 2}$, and $R\bullet R$ is the twice-covariant symmetric tensor a with the components given by $4a_{jk} = R^{lmp}{}_jR_{lmpk}$. Note that $2\langle R\bullet R, \mathrm{r}\rangle = \operatorname{tr}\hat{S}\hat{R}^2$, where $S = g\wedge\mathrm{r}$ is the section of $\mathcal{E}$ with $S_{jklm} = g_{jl}R_{km} + g_{km}R_{jl} - g_{kl}R_{jm} - g_{jm}R_{kl}$ (cf. §4). Hence

$$\text{(E.3)}\qquad 2\phi \,=\, R^{jklm}(R_{smj}{}^pR_{pkl}{}^s + R_{smk}{}^pR_{jpl}{}^s + R_{sml}{}^pR_{jkp}{}^s + R_{mp}R_{jkl}{}^p)\,.$$

In fact, of the four terms appearing (by distributivity) on the right-hand side of (E.3), the first and second are both clearly equal to $\operatorname{tr}R^3$. In the third term, skew-symmetry of R^{jklm} in l,m allowes us to replace $R_{sml}{}^p$ with $(R_{sml}{}^p - R_{slm}{}^p)/2$, which, by the first Bianchi identity, is the same as $-R^p{}_{slm}$. Thus, the third terms equals $-R^{lmjk}R_{jk}{}^{ps}R_{pslm}/2 \,=\, -4\operatorname{tr}\hat{R}^3$, while the fourth one is precisely $4\langle R\bullet R, \mathrm{r}\rangle$.

Given a Riemannian manifold (M,g), the *Laplacian* $\varDelta R$ of the curvature tensor R, its *divergence* δR, and the *exterior derivative* $d\delta R$ of δR, are the covariant tensor fields with the components $R_{jklm,s}{}^s$, $R_{jkl}{}^s{}_{,s}$ and $R_{jkm}{}^s{}_{,sl} - R_{jkl}{}^s{}_{,sm}$.

In the following lemma, assertion (b) and the identity $\phi = \operatorname{tr}\underline{R}\hat{R}\underline{R}^*$ are due to Tachibana [109], while the equality $\langle R, \varDelta R + d\delta R\rangle = \phi$ is a result of Berger [5]; $\langle R, \varDelta R + d\delta R\rangle : M \to \mathbf{R}$ is well defined since $R, \varDelta R$ and $d\delta R$ are sections of $\mathcal{E}$.

Lemma E.2. *In a Riemannian manifold (M,g) of any dimension n,*

(a) $\langle R, \varDelta R + d\delta R\rangle \,=\, \phi \,=\, \operatorname{tr}\underline{R}\hat{R}\underline{R}^*$ *for $\underline{R}, \phi$ given by* (E.1) – (E.2)*,*
(b) $\underline{R} = 0$ *identically if and only if g has constant sectional curvature or $n \le 2$.*

Proof. The second Bianchi identity gives $0 = R^{jklm}(R_{jklm,s}{}^s + R_{jkms,l}{}^s + R_{jksl,m}{}^s) = R^{jklm}(R_{jklm,s}{}^s - 2R_{jkls,m}{}^s)$. Thus, $4\langle R, \Delta R\rangle = R^{jklm}R_{jklm,s}{}^s = 2R^{jklm}R_{jkl}{}^s{}_{,ms}$, and so $2\langle R, \Delta R + 2d\delta R\rangle = R^{jklm}(R_{jkl}{}^s{}_{,ms} - R_{jkl}{}^s{}_{,sm})$, which, by the Ricci identity (2.12.a) combined with (E.3), equals 2ϕ, proving the first equality in (a).

Next, $\operatorname{tr} \underline{R}\hat{R}\underline{R}^* = (R_{jkl}{}^pR_{pmq}{}^m - R_{jkm}{}^pR_{plq}{}^m + R_{lmj}{}^pR_{pkq}{}^m - R_{lmk}{}^pR_{pjq}{}^m)R^{jkql}$, with the four terms corresponding precisely to those in (i).

The 'if' part in (b) is obvious since constancy of the sectional curvature of g means that $\hat{R}$ is, at each point, a multiple of the identity operator. Conversely, suppose that $\underline{R} = 0$ at a point $x \in M$. For the basis $\xi_1, \dots, \xi_n$ of T_x^*M, dual to any given orthonormal basis $e_1, \dots, e_n$ of T_xM, let $A = A_{jk}$ and $B = B_{jk}$ be the operators $T_xM \to T_xM$ corresponding as in Remark 2.1 to $a = \omega_{jk}$ and $b = \hat{R}\omega_{jk}$, where we have set $\omega_{jk} = \xi_j \wedge \xi_k$. By (2.4), $A_{jk}e_j = e_k$, $A_{jk}e_k = -e_j$, and $A_{jk}e_l = 0$ whenever $j \neq k \neq l \neq j$, which yields the commutation relations $[A_{jk}, A_{jl}] = A_{kl}$ if $j \neq k \neq l \neq j$ and $[A_{jk}, A_{lm}] = 0$ if $\{j,k\} \cap \{l,m\} = \emptyset$. At the same time, $\hat{R}\omega_{jk} = \sum_{l,m} R_{jklm}\omega_{lm}$, with $R_{jklm} = R(e_j, e_k, e_l, e_m)$. As the assumption $\underline{R} = 0$ gives $[A_{jk}, B_{lm}] = 0$ for all j,k,l,m, we thus have $R_{jkjl} = 0$ if $j \neq k \neq l \neq j$ and $R_{jklm} = 0$ if $\{j,k\} \cap \{l,m\} = \emptyset$. Consequently, every nonzero 2-form ω at x which is decomposable, i.e., equal to $\xi \wedge \eta$ for some 1-forms ξ, η, is an eigenvector of $\hat{R}$ at x. (In fact, we just verified this when $\omega = \omega_{jk}$ and $j \neq k$, for an *arbitrary* orthonormal basis $e_1, \dots, e_n$.) Denoting by $\mu_{jk} = \mu_{kj}$, for $j \neq k$, the eigenvalue μ with $\hat{R}\omega_{jk} = \mu\omega_{jk}$, we have $\mu_{jk} = \mu_{jl}$ whenever $j \neq k \neq l \neq j$. Namely, all nonzero 2-forms in the plane spanned by ω_{jk} and ω_{jl} are decomposable; thus, they are eigenvectors of $\hat{R}$, and must all correspond to the same eigenvalue. As a result, all the eigenvalues μ_{jk} are the same, and $\hat{R}$ is a multiple of the identity operator, which proves (b). □

Proof of Theorem 7.4. Let (M, g) be compact, with $\delta R = 0$ and $\hat{R} \geq 0$.

The function $\langle R, \Delta R\rangle = \operatorname{tr} \hat{R}\underline{R}^*\underline{R}$ (cf. Lemma E.2(a)) vanishes identically, and (M, g) is locally symmetric: $\int_M \langle R, \Delta R\rangle \, \mathrm{d}g = -\int_M |\nabla R|^2 \, \mathrm{d}g \leq 0$ by (2.15), since $4\langle R, \Delta R\rangle + 4|\nabla R|^2 = R^{jklm}R_{jklm,s}{}^s + R^{jklm,s}R_{jklm,s} = (R^{jklm}R_{jklm,s})^{,s} = 2\delta d(|R|^2)$, while $\operatorname{tr} \hat{R}\underline{R}^*\underline{R} \geq 0$ from Lemma E.1(a) applied to $F = \hat{R}$ and $F' = \underline{R}^*\underline{R}$ in the tangent space of (M, g) at any point.

Finally, if $\hat{R} > 0$, positivity of $F = \hat{R}$ at each point implies, by Lemma E.1(b), vanishing of $F' = \underline{R}^*\underline{R}$, and hence of $\underline{R}$ (as $\operatorname{tr} FF' = 0$), so that g has constant curvature in view of Lemma E.2(b). This completes the proof. □

Appendix F. The first Chern class

Given a manifold M and an integer r, let $\Omega^r M$ be the vector space of all differential r-forms on M (that is, C^∞ sections of $[T^*M]^{\wedge r}$). Thus, $\Omega^r M$ is infinite-dimensional if $\dim M = n \geq 1$ and $0 \leq r \leq n$, while, by definition, $\Omega^r M = \{0\}$ if $r < 0$ or $r > \dim M$. The spaces $Z^r M$ and $B^r M$ of *closed* or, respectively, *exact* r-forms are defined to be, respectively, the kernel of the exterior derivative $d : \Omega^r M \to \Omega^{r+1} M$ and the image of $d : \Omega^{r-1} M \to \Omega^r M$. Consequently, $B^r M \subset Z^r M \subset \Omega^r M$, as $dd = 0$. The quotient space $H^r(M, \mathbf{R}) = Z^r M / B^r M$ is known as the rth *de Rham cohomology space* of M. We denote by $[\zeta] \in H^r(M, \mathbf{R})$ the cohomology class of $\zeta \in Z^r M$ (that is, its equivalence class in $Z^r M / B^r M$).

As an example, the (real) *first Chern class* $c_1(\mathcal{L}) \in H^2(M, \mathbf{R})$ of a complex line bundle $\mathcal{L}$ over a manifold M is given by $2\pi c_1(\mathcal{L}) = [\operatorname{Im}\zeta]$, where $\operatorname{Im}\zeta$ is the imaginary part of the curvature form ζ of any given connection ∇ in $\mathcal{L}$. More precisely, the curvature tensor of ∇ is defined as in (2.1), except that the vector field w has to be replaced by a section ψ of $\mathcal{L}$. Since the fibre dimension is 1, for vector fields u, v on M and sections ψ of $\mathcal{L}$, the section $R(u,v)\psi$ equals the product of ψ and a function $\zeta(u,v) : M \to \mathbf{C}$, which gives rise to the (complex-valued) curvature form ζ. A fixed section ψ of $\mathcal{L}$ without zeros, defined on an open set $U \subset M$, leads to the complex-valued *connection form* Γ of ∇ (relative to ψ), with $\nabla_v\psi = \Gamma(v)\psi$ for all vector fields v on U. Now, by (2.1) and (2.24.a), $\zeta = -d\Gamma$, and so ζ is closed (although not necessarily exact, as Γ is defined only locally). Thus, $\operatorname{Im}\zeta$ is closed as well. Finally, $c_1(\mathcal{L})$ does not depend on the choice of the connection ∇. In fact, for another connection ∇', with the corresponding ζ' and Γ', we clearly have $\zeta' - \zeta = d\Gamma - d\Gamma' = d\xi$, for the complex-valued 1-form ξ on M such that $\nabla' = \nabla - \xi$.

One also defines the first Chern class $c_1(\mathcal{E})$ of a complex vector bundle $\mathcal{E}$ of any fibre dimension $m \geq 1$ over a manifold M by setting $c_1(\mathcal{E}) = c_1(\mathcal{L})$ for the line bundle $\mathcal{L} = \mathcal{E}^{\wedge m}$, that is, the highest complex exterior power of $\mathcal{E}$.

The exterior multiplication $\wedge$ of differential forms preserves closedness, and descends to a multiplication $\cup$ of cohomology classes, known as the *cup product*; explicitly, $[\zeta] \cup [\eta] = [\zeta \wedge \eta]$. This is clear from the Leibniz rule for $\wedge$ and d.

Appendix G. Almost complex manifolds

An *almost complex manifold* is a manifold M carrying a fixed *almost complex structure* (a C^∞ vector-bundle morphism $J : TM \to TM$ with $J^2 = -\operatorname{Id}$). In other words, TM then is the underlying real bundle of a complex vector bundle, in which J is the multiplication by i. This allows us to define the first Chern class $c_1(M) \in H^2(M, \mathbf{R})$ by $c_1(M) = c_1(TM)$. (See Appendix F.)

We always use the symbol J for the almost complex structure under consideration, while the almost complex manifold in question is simply denoted by M (rather than, for instance, (M, J)). The *complex dimension* of M is then defined to be $\dim_{\mathbf{C}} M = n/2$, where n stands for the ordinary (real) dimension of M.

The automorphism group $\mathrm{GL}(V) \approx \mathrm{GL}(m, \mathbf{C})$ of any complex vector space V with $1 \leq \dim V = m < \infty$ is connected, since every automorphism of V is represented in some basis by a triangular matrix, and that matrix can be joined to Id by an obvious curve of nonsingular triangular matrices. The underlying real space of V thus becomes naturally oriented, as it has a distinguished connected set of real bases, namely, $e_1, ie_1, \ldots, e_m, ie_m$, where $e_1, \ldots, e_m$ runs through the set of all complex bases of V (and the latter set is connected, being an orbit of the connected group $\mathrm{GL}(V)$). This has the following obvious consequence:

(G.1) Every almost complex manifold is canonically oriented.

Given an almost complex manifold M, we denote by $i\partial\overline{\partial}$ the operator sending every C^∞ function $h : M \to \mathbf{R}$ to the exact 2-form $i\partial\overline{\partial}h$ such that

$$2i\partial\overline{\partial}h = -d[(dh)J]. \tag{G.2}$$

Here $(dh)J$ is the 1-form equal, at any point $x \in M$, to the composite in which $J_x : T_xM \to T_xM$ is followed by $dh_x : T_xM \to \mathbf{R}$. For our purposes, $i\partial\overline{\partial}$ may be treated as a single symbol, even though the notation reflects an actual factorization.

Remark G.1. A twice-covariant tensor field a on an almost complex manifold M gives rise to two more such tensor fields, $b = aJ$ (or, $b = Ja$), characterized by $b(u,v) = a(Ju,v)$ (or, respectively, $b(u,v) = -a(u,Jv)$) for any vector fields u,v on M. The tensor field a is said to be *Hermitian* (or, *skew-Hermitian*) if it is symmetric (or, skew-symmetric) at every point and $aJ = Ja$, that is, if $a(Ju,Jv) = a(u,v)$ for all vector fields u,v on M. Clearly, a is Hermitian if and only if aJ is skew-Hermitian, while $(aJ)J = J(Ja) = -a$.

Note that a twice-covariant skew-symmetric tensor field is nothing else than a differential 2-form.

Remark G.2. By a *Hermitian metric* on a given almost complex manifold M we mean a Riemannian metric g on M which is a Hermitian tensor, that is, $gJ = Jg$. This amounts to g-skew-adjointness of J at every point; equivalently, J is required to act in every tangent space as a linear isometry.

If g is Hermitian, the operation $a \mapsto b = Ja$ (or, $a \mapsto b = aJ$), defined in Remark G.1 for twice-covariant tensor fields a, coincides with the ordinary composition $B = JA$ (or, $B = AJ$) of bundle morphisms $TM \to TM$, provided that one identifies a,b with A,B as in Remark 2.1. In the case where a is also symmetric (or, skew-symmetric) at every point, its being Hermitian (or, skew-Hermitian) is obviously equivalent to complex-linearity of the corresponding bundle morphism $A : TM \to TM$, which in turn means that A commutes with J.

Let M be an almost complex manifold. If a Riemannian metric g on M is Hermitian, the formula $\varOmega = gJ$ clearly defines a skew-symmetric twice-covariant tensor field (that is, a differential 2-form), which is also skew-Hermitian. Moreover,

$$\varOmega^{\wedge m} = m!\,\mathrm{d}g, \qquad \text{where } m = \dim_{\mathbf{C}} M. \tag{G.3}$$

(Since M is oriented according to (G.1), the volume element $\mathrm{d}g$ may be treated as a positive differential $2m$-form.) In fact, let $x \in M$ and let a complex basis $e_1, \ldots, e_m$ of T_xM be orthonormal relative to the Hermitian inner product with real part g_x. Now $\varOmega_x = \xi_1 \wedge \xi_2 + \ldots + \xi_{2m-1} \wedge \xi_{2m}$ for the real basis $\xi_1, \ldots, \xi_{2m}$ of T_x^*M, dual to the g_x-orthonormal real basis $e_1, Je_1, \ldots, e_m, Je_m$ of T_xM, which one easily sees using (2.4) to evaluate both sides on any pair of vectors from the basis $e_1, Je_1, \ldots, e_m, Je_m$. Thus, $\varOmega_x^{\wedge m} = m!\,\xi_1 \wedge \ldots \wedge \xi_{2m}$, as required.

Appendix H. Kähler metrics

By a *Kähler manifold* we mean a Riemannian manifold (M,g) which is simultaneously an almost complex manifold, such that g is Hermitian (Remark G.2) and $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

The simplest example of a Kähler manifold (M,g) arises when a finite-dimensional complex vector space V with a Hermitian inner product $\langle\,,\rangle$ is given: we then set $M = V$, let J operate in each tangent space T_xM via the ordinary multiplication by i (with the standard identification $T_xV = V$), and choose g to be the constant (translation-invariant) metric $\mathrm{Re}\,\langle\,,\rangle$. Another example is provided by any oriented 2-dimensional Riemannian manifold (M,g), with J that acts in each tangent plane T_xM as the positive rotation by the angle $\pi/2$. Further examples are provided by locally symmetric Kähler manifolds, described below in Appendix L.

Speaking of a Kähler manifold (M,g), we usually skip the word ‘almost’ and call J the (underlying) *complex structure* of (M,g), while g is referred to as *a Kähler metric on the complex manifold* M. See also the end of Appendix L.

By the *Ricci form* of a Kähler manifold (M,g) one means the twice-covariant tensor field $\rho = \mathrm{r}J$ (cf. Remark G.1), where r the Ricci tensor of g. We have

$$\text{(H.1)}\quad \text{a) } \mathrm{tr}_{\mathbf{R}}\, J[R(v,w)] = -2\rho(v,w),\ \text{ b) } \delta[J(\nabla w)^*] = \imath_w\rho,\ \text{ c) } R(Jv,Jw) = R(v,w),$$

for δ as in (2.9) and any vector fields v,w on M. In coordinates, (a) – (c) read $R_{klp}{}^qJ_q^p = -2\rho_{kl}$, $J_q^pw_{k,}{}^q{}_p = \rho_{lk}w^l$ and $J_k^rJ_l^sR_{rsp}{}^q = R_{klp}{}^q$.

In fact, as $\nabla J = 0$, the Levi-Civita connection ∇ is a connection in the *complex* vector bundle TM, and so, for any vector fields u,v on M, the vector-bundle morphism $R(u,v) : TM \to TM$ in (2.2) is complex-linear (commutes with J). At every point, the commuting morphisms $R(u,v)$ and J are skew-adjoint, and so their composite is self-adjoint. Hence $R_{qlsp}J_k^p = R_{qlkp}J_s^p$, which, contracted against J_r^k or g^{qs}, gives (H.1.c) or, respectively, $\rho_{kl} = R_{pkl}{}^qJ_q^p$. However, due to the well-known symmetries of R and skew-adjointness of J, the expression $R_{pkl}{}^qJ_q^p$ is skew-symmetric in k,l, so that, from the first Bianchi identity, $0 = (R_{kpl}{}^q + R_{kl}{}^q{}_p + R_k{}^q{}_{pl})J_q^p = -2R_{pkl}{}^qJ_q^p - R_{klp}{}^qJ_q^p$, and (H.1.a) follows. Finally, since J is skew-adjoint, $2J_q^pw_{k,}{}^q{}_p = J_q^p(w_{k,}{}^q{}_p - w_{k,p}{}^q) = J_q^pR^q{}_{plk}w^l = J_q^pR_{klp}{}^qw^l$, by the Ricci identity (2.12.a). Now (H.1.a) yields (H.1.b).

For any vector field v on a Kähler manifold (M,g), we have, with δ as in (2.7.i),

$$\text{(H.2)}\quad \begin{array}{ll} \text{i)} & \mathrm{tr}\, JAJA = (\mathrm{tr}\, JA)^2 - \mathrm{r}(v,v) + \delta[JAJv - (\mathrm{tr}\, JA)Jv] \quad \text{and} \\ \text{ii)} & \mathrm{tr}\, JAJA^* = \delta(JA^*Jv) - \mathrm{r}(v,v), \quad \text{where } A = \nabla v : TM \to TM, \end{array}$$

A^* being the (pointwise) adjoint of A. Namely, in local coordinates, $\mathrm{tr}\, JAJA = J_q^pv^q{}_{,k}J_l^kv^l{}_{,p} = (J_q^pv^q{}_{,k}J_l^kv^l)_{,p} - J_q^pv^q{}_{,kp}J_l^kv^l$. Next, $(J_q^pv^q{}_{,k}J_l^kv^l)_{,p} = \delta(JAJv)$ and, by the Ricci identity (2.12.a), $-J_q^pv^q{}_{,kp}J_l^kv^l = -J_q^pv^q{}_{,pk}J_l^kv^l + J_q^pJ_l^kR_{pks}{}^qv^sv^l$, while $-J_q^pv^q{}_{,pk}J_l^kv^l = -(J_q^pv^q{}_{,p}J_l^kv^l)_{,k} + J_q^pv^q{}_{,p}J_l^kv^l{}_{,k} = -\delta[(\mathrm{tr}\, JA)Jv] + (\mathrm{tr}\, JA)^2$ (as $J_q^pv^q{}_{,p} = \mathrm{tr}\, JA$). Also, by (H.1.c), $J_q^pJ_l^kR_{pks}{}^qv^sv^l$ equals $R_{qls}{}^qv^sv^l$, that is, $-\mathrm{r}(v,v)$. This proves (H.2.i). Finally, $\mathrm{tr}\, JAJA^* = J_q^pv^q{}_{,k}J_l^kv_{p,}{}^l = (J_q^pv^qJ_l^kv_{p,}{}^l)_{,k} - J_q^pv^qJ_l^kv_{p,}{}^l{}_k$, while $(J_q^pv^qJ_l^kv_{p,}{}^l)_{,k} = \delta(JA^*Jv)$ and, by (H.1.b), $-J_q^pv^qJ_l^kv_{p,}{}^l{}_k = -J_q^pv^q\rho_{kp}v^k = \rho(Jv,v) = -\mathrm{r}(v,v)$, which gives (H.2.ii).

Remark H.1. If (M,g) is a Kähler manifold,

(i) the Ricci tensor r of (M,g) is Hermitian;
(ii) its Ricci form $\rho = \mathrm{r}J$ is a *closed* differential 2-form;
(iii) as g is Hermitian, $\Omega = gJ$ is a skew-Hermitian 2-form on M, called the *Kähler form* of (M,g). Being parallel, Ω is closed as well.

In fact, (i) amounts to skew-symmetry of ρ (obvious from (H.1.a)), while the relation $d\rho = 0$, that is, $\rho_{sk,l} + \rho_{kl,s} + \rho_{ls,k} = 0$ (cf. (2.24.c)), is immediate from the coordinate version of (H.1.a) and the second Bianchi identity (since $\nabla J = 0$).

For any function $f : M \to \mathbf{R}$ on a Kähler manifold (M,g), we have

$$\text{(H.3)}\qquad \text{i) } 2i\partial\overline{\partial}f = (\nabla df)J + J(\nabla df), \quad \text{ii) } \mathrm{tr}_g[(i\partial\overline{\partial}f)J] = -\Delta f$$

(notation of (2.14.i), (G.2) and Remark G.1 for $a = \nabla df$). Namely, (G.2) and (2.24.b) give (i), which in turn implies (ii). Thus, by (2.20.b),

$$\text{(H.4)}\qquad \text{a function } f : M \to \mathbf{R} \text{ is constant if } M \text{ is compact and } i\partial\overline{\partial}f = 0.$$

Lemma H.2. *Let an exact differential* 2*-form* ζ *on a compact Kähler manifold* (M,g) *be skew-Hermitian in the sense that* $J\zeta = \zeta J$*, cf.* Remark G.1.

(a) *There exists a* C^∞ *function* $\theta : M \to \mathbf{R}$ *with* $\zeta = i\partial\overline{\partial}\theta$.

(b) *The function* θ *in* (a) *is unique up to an additive constant.*
(c) *Denoting by* $\|\ \|$ *the* L^2 *norm, both for functions and tensor fields on* M*, we have* $\sqrt{2}\,\|\zeta\| = \|\mathrm{tr}_g\,\zeta J\|$.
(d) *If* $\mathrm{tr}_g\,\zeta J = 0$*, then* $\zeta = 0$.

Proof. We first prove (c). Let v be a vector field with $\zeta = d\xi$ for the 1-form $\xi = \imath_v g$, and let $A = \nabla v$, so that, by (2.24.b), $A - A^*$ is the vector-bundle morphism $TM \to TM$ corresponding to ζ as in Remark 2.1. We clearly have $\mathrm{tr}\,(A-A^*)A^* = -\,\mathrm{tr}\,(A-A^*)A$. Thus, $\|\zeta\|^2 = -\int_M \mathrm{tr}\,(A-A^*)^2\,\mathrm{d}g = -2\int_M \mathrm{tr}\,(A-A^*)A\,\mathrm{d}g$. Since ζ is skew-Hermitian, $[J, A-A^*] = 0$, that is, $A - A^* = JA^*J - JAJ$. Thus, $\|\zeta\|^2 = 2\int_M \mathrm{tr}\,JAJ(A-A^*)\,\mathrm{d}g$, and so (c) follows from (H.2) and (2.15), as $2\,\mathrm{tr}\,JA = \mathrm{tr}\,J(A-A^*) = \mathrm{tr}_g\,\zeta J$ due to skew-adjointness of J.

Next, (d) is obvious from (c). To prove (a), let us choose $\theta : M \to \mathbf{R}$ with $\varDelta\theta = -\,\mathrm{tr}_g\,\zeta J$. (Such θ exists by (2.16), since, as we just saw, $\mathrm{tr}_g\,\zeta J = 2\,\mathrm{tr}\,JA$, so that $\mathrm{tr}_g\,\zeta J = 2J_q^p v^q{}_{,p} = 2\delta(Jv)$, and $\int_M \mathrm{tr}_g\,\zeta J\,\mathrm{d}g = 0$.) Applying (d) to $\zeta - i\partial\overline{\partial}\theta$ rather than ζ, and noting that the premise of (d) is then satisfied in view of (H.3.ii), we now see that $\zeta = i\partial\overline{\partial}\theta$. Finally, (b) is immediate from (H.4). ☐

Appendix J. Almost-Kähler manifolds

An *almost-Kähler metric* on an almost complex manifold M is any Hermitian metric g on M (cf. Remark G.2) for which the skew-Hermitian 2-form $\varOmega = gJ$ is closed. Such pairs (M, g) are referred to as *almost-Kähler manifolds*; obvious examples are provided by Kähler manifolds (cf. Remark H.1(iii)).

Remark J.1. If g is just a Hermitian metric, the differential 2-form $\varOmega = gJ$ is skew-Hermitian, but need not, in general, be parallel relative to the Levi-Civita connection ∇, or even closed. The condition $\nabla\varOmega = 0$ is necessary and sufficient for a given Hermitian metric g to be a Kähler metric: it means the same as $\nabla J = 0$, since $\varOmega = gJ$ and $\nabla g = 0$.

One easily finds examples of non-Kähler, almost-Kähler metrics, also on compact manifolds. On the other hand, as we will see below (Theorem J.3), for an almost complex manifold M on which a Kähler metric exists, all almost-Kähler metrics on M are Kähler metrics. By Lemma J.2, the same conclusion holds even if one replaces the existence of a Kähler metric with the weaker requirement that J be parallel relative to some torsionfree connection on M. A conjecture of Goldberg [55], stating that a compact almost-Kähler Einstein manifold is necessarily a Kähler manifold, is still open [98].

Lemma J.2. *Let* ∇ *be the Levi-Civita connection of a Hermitian metric* g *on an almost complex manifold* M*, and let* $\hat{\nabla}J = 0$ *for some torsionfree connection* $\hat{\nabla}$ *on* M*. Then, for the skew-Hermitian* 2*-form* $\varOmega = gJ$ *and any vector field* w *on* M*, we have* $2\nabla_w\varOmega = \imath_w d\varOmega + J(\imath_w d\varOmega)J$*, in the notation of* Remark G.1.

Proof. Let v, w always stand for arbitrary vector fields on M. Denoting by B the section of $\mathrm{Hom}([TM]^{\odot 2}, TM)$ with $\hat{\nabla} = \nabla - B$, we have $\hat{\nabla}_w = \nabla_w - B_w$, and B sends v, w to a vector field $B_v w = B_w v$, its symmetry being due to the fact that $\hat{\nabla}, \nabla$ are both torsionfree. As J is $\hat{\nabla}$-parallel, $\nabla_w J = [B_w, J]$, where $[\ ,\ ]$ denotes the commutator of bundle morphisms $TM \to TM$. In coordinates, $B_w, \nabla_w J, \nabla_w\varOmega, \hat{\nabla}_w g$ and $\hat{\nabla}_w\varOmega$ have the components $(B_w)_k^l = w^s B_{sk}^l, (\nabla_w J)_k^l =$

$w^s C^l_{sk}$, $(\nabla_w \Omega)_{kl} = w^s C_{skl}$, $(\hat\nabla_w g)_{kl} = w^s D_{skl}$ and $(\hat\nabla_w \Omega)_{kl} = w^s E_{skl}$, for some functions B, C, D, E (with subscripts and superscripts) such that

$$\begin{aligned}&\text{(a) } C^l_{pk} = J^s_k B^l_{ps} - J^l_s B^s_{pk}, \ \text{(b) } B^r_{kl} = B^r_{lk}, \ \text{(c) } C_{pkl} = C^s_{pk} g_{sl}, \ \text{(d) } E_{pkl} = J^s_k D_{psl},\\ &\text{(e) } D_{pkl} = B^s_{pk} g_{sl} + B^s_{pl} g_{ks}, \ \text{(f) } C_{lpk} = -C_{lkp}, \ \text{(g) } J^s_k E_{pls} = -J^s_k E_{psl} = D_{pkl},\\ &\text{(h) } (d\Omega)_{pkl} = C_{pkl} + C_{klp} + C_{lpk}, \ \text{(i) } (d\Omega)_{pkl} = E_{pkl} + E_{klp} + E_{lpk}.\end{aligned}$$

In fact, (a) is the coordinate version of $\nabla_w J = [B_w, J]$, (b) expresses symmetry of B, the relation $\Omega = gJ$ along with $\nabla g = 0$ (or, $\hat\nabla J = 0$) yields (c) (or, respectively, (d)), while (e) follows since $\nabla g = 0$ and $\hat\nabla = \nabla + B$, (f) is due to skew-symmetry of Ω and $\nabla_w \Omega$, and (d) implies (g) as $J^2 = -\mathrm{Id}$. Finally, (h) (or, (i)) amounts to (2.24.c) for $\zeta = \Omega$ and the torsionfree connection ∇ (or, $\hat\nabla$).

We need to prove the equality $2\nabla_w \Omega - \imath_w d\Omega = J(\imath_w d\Omega)J$, equivalent, in view of (h) and (i), to $C_{pkl} - C_{klp} - C_{lpk} = -J^r_k J^s_l (E_{prs} + E_{rsp} + E_{spr})$. (Note that $2C_{pkl} - (C_{pkl} + C_{klp} + C_{lpk}) = C_{pkl} - C_{klp} - C_{lpk}$.) First, (f) and (c) give $C_{pkl} - C_{klp} - C_{lpk} = C_{lkp} - C_{klp} + C_{pkl} = (C^s_{lk} - C^s_{kl}) g_{sp} + C^s_{pk} g_{sl}$. In view of (a), this equals $J^r_k B^s_{lr} g_{sp} - J^r_l B^s_{kr} g_{sp} + J^r_k B^s_{pr} g_{sl} - J^s_r B^r_{pk} g_{sl}$ (two other terms cancel each other by (b)). Using (e) and (b) we can rewrite the last expression as $J^r_k D_{rlp} - J^r_l B^s_{kr} g_{sp} - J^s_r B^r_{pk} g_{sl}$, which equals $J^r_k D_{rlp} - J^r_l B^s_{kr} g_{sp} + J^r_l B^s_{pk} g_{rs}$ (where $J^s_r g_{sl} = -J^s_l g_{sr}$ as $J^s_r g_{sl} = \Omega_{rl}$, and the indices r, s have been switched). Applying (e) and (b) again, we see that this coincides with $J^r_k D_{rlp} + J^r_l (D_{pkr} - D_{rkp})$.

On the other hand, by (g), $-J^r_k J^s_l (E_{prs} + E_{rsp} + E_{spr})$ is equal to $J^r_k D_{rlp} + J^r_l (D_{pkr} - D_{rkp})$ as well, which completes the proof. □

Suppose that (M, g) is an almost-Kähler manifold. As in the Kähler case, we call $\Omega = gJ$ the *Kähler form* of (M, g). Being closed, Ω gives rise to a cohomology class $[\Omega] \in H^2(M, \mathbf{R})$ (see Appendix F) known as the *Kähler cohomology class* of (M, g), or, briefly, its *Kähler class*.

Theorem J.3. *Let $\mathcal{A}$ be the set of all almost-Kähler metrics on a given almost complex manifold M.*

(i) *$\mathcal{A}$ is a convex subset of the vector space of all Hermitian twice-covariant C^∞ tensor fields a on M such that the differential 2-form aJ is closed.*

(ii) *The set of all Kähler metrics on M is either empty, or coincides with $\mathcal{A}$.*

Proof. Assertion (i) is obvious since $\mathcal{A}$ is defined by imposing on a metric g the linear equations $gJ = Jg$ and $d(gJ) = 0$. To prove (ii), let us suppose that M admits a Kähler metric. For an arbitrary almost-Kähler metric g on M, denoting by ∇ and Ω the Levi-Civita connection and Kähler form of g, we have $\nabla\Omega = 0$ by Lemma J.2, and so g is a Kähler metric (Remark J.1), as required. □

For an almost-Kähler metric g on a compact almost complex manifold M,

(J.1) its volume $\mathrm{V} = \int_M \mathrm{d}g$ depends only on its Kähler class $[\Omega] \in H^2(M, \mathbf{R})$.

In fact, the oriented integral $\int_M \sigma$ of a differential $2m$-form σ, for $m = \dim_{\mathbf{C}} M$, depends only on the cohomology class $[\sigma]$ (as $\int_M \sigma = 0$ when σ is exact, by Stokes's formula (2.23)). That $\mathrm{V} = \int_M \mathrm{d}g$ depends on g only through $[\Omega]$ is clear from (G.3), since $[\Omega^{\wedge m}] = [\Omega]^{\cup m}$, where $\cup$ is the cup product (Appendix G).

Also, $[\Omega] \neq 0$ in $H^2(M, \mathbf{R})$, for the Kähler form Ω of any compact almost-Kähler manifold (M, g). Namely, if we had $\Omega = d\xi$ for some 1-form ξ, it would follow that $\Omega^{\wedge m} = d[\xi \wedge \Omega^{\wedge(m-1)}]$, and so $\mathrm{V} = 0$ by (G.3) and (2.23).

Given a compact almost complex manifold M, one calls an element of $H^2(M,\mathbf{R})$ *positive* (or *negative*) if it equals $[\Omega]$ (or, $-[\Omega]$) for the Kähler form Ω of some almost-Kähler metric on M. A cohomology class in $H^2(M,\mathbf{R})$ cannot be simultaneously positive and zero, or zero and negative, or positive and negative: if it were, a suitable difference would be both positive and zero, giving $[\Omega]=0$ for the Kähler form Ω of some almost-Kähler metric, contrary to the last paragraph.

Appendix K. Comparing Kähler metrics

For any C^1 curve $t\mapsto F=F(t)\in\mathrm{GL}(V)$ of linear automorphisms of a finite-dimensional real/complex vector space V, setting $(\;)^{\cdot}=d/dt$ we have

$$(\det F)^{\cdot}\,=\,(\det F)\,\mathrm{tr}\,(F^{-1}\dot F). \tag{K.1}$$

In fact, shifting the variable, we see that it suffices to establish (K.1) at $t=0$. When $F(0)=\mathrm{Id}$, (K.1) at $t=0$ means that tr the differential of the homomorphism $\det$ at $\mathrm{Id}\in\mathrm{GL}(V)$, and so (K.1) follows since $1+(\mathrm{tr}\,A)\,t$ is the first-order part of $\det(\mathrm{Id}+tA)$ treated as a polynomial in t. The general case is reduced to the above by replacing the curve $t\mapsto F(t)$ with $t\mapsto[F(0)]^{-1}F(t)$.

Suppose that g and $\hat g$ are Riemannian metrics on a manifold M of any (real) dimension n and $\gamma:M\to(0,\infty)$ is the ratio of their volume elements, in the sense that $\mathrm{d}\hat g=\gamma\,\mathrm{d}g$. Then, with tr_g denoting the g-trace, as in Remark 2.1,

$$\text{a)}\ \ \det{}_g\,\hat g\,=\,\gamma^2,\qquad\qquad \text{b)}\ \ \mathrm{tr}_g\,\hat g\,\ge\,n\gamma^{2/n}. \tag{K.2}$$

Here $\det_g\hat g:M\to\mathbf{R}$ assigns to each $x\in M$ the determinant, at x, of the vector-bundle morphism $A:TM\to TM$ corresponding to $\hat g$ (via the fixed metric g) as in Remark 2.1. Namely, (K.2.a) follows since, in local coordinates, $\det A=(\det g)^{-1}\det\hat g$, while the component function of $\mathrm{d}g$ is $(\det g)^{1/2}$, and similarly for $\hat g$. (By $\det g$ we mean the coordinate-dependent function $\det[g_{jk}]$.) Next, for A as above, $\mathrm{tr}_g\,\hat g=\mathrm{tr}\,A$. As the eigenvalues of A at any given point $x\in M$ are positive, (K.2.b) is obvious from (K.2.a) and the inequality between the arithmetic and geometric means, that is, (D.4) with $k=n$ and $c_1=\ldots=c_n=1/n$.

Remark K.1. Let ρ be the Ricci form of a Kähler manifold (M,g).

- (i) The curvature form ζ (see Appendix F) of the connection ∇ which the Levi-Civita connection of g, also denoted by ∇, induces in the complex exterior power $[TM]^{\wedge m}$, for $m=\dim_{\mathbf{C}}M$, is given by $\zeta=i\rho$.
- (ii) In cohomology, $[\rho]\,=\,2\pi c_1(M)\in H^2(M,\mathbf{R})$, cf. Appendix F.
- (iii) The Ricci form $\hat\rho$ of any other Kähler metric $\hat g$ on the same underlying complex manifold M is related to ρ by $\hat\rho=\rho-i\partial\overline{\partial}\log\gamma$, where $\mathrm{d}\hat g=\gamma\,\mathrm{d}g$, that is, $\gamma:M\to(0,\infty)$ is the ratio of the volume elements.

In fact, let the vector fields e_a, $a=1,\ldots,m$, trivialize the *complex* vector bundle TM over an open set $U\subset M$, and let Γ_a^b be the corresponding (complex-valued) *connection forms* on U, with $\nabla_v e_a=\Gamma_a^c(v)\,e_c$. (Here and below repeated indices are summed over, and v,w are arbitrary vector fields on U.) Thus, by (2.1), $R(v,w)e_a=R_a^c(v,w)e_c$, where $R_a^b=-d\Gamma_a^b+\Gamma_a^c\wedge\Gamma_c^b$, with d and $\wedge$ as in (2.24.a) and (2.4). On the other hand, $i\rho(v,w)$ equals the complex trace of the complex-linear bundle morphism $R(v,w):TM\to TM$ defined as in (2.2). To see this, note that, at each point, $R(v,w)$ is skew-adjoint relative to g, as a real operator, and hence also relative to the Hermitian fibre metric $g^{\mathbf{C}}$ in TM with $\mathrm{Re}\,g^{\mathbf{C}}=g$. Consequently, $i\,\mathrm{tr}_{\mathbf{C}}[R(v,w)]$ is real and coincides with the complex trace of the self-adjoint composite morphism $J[R(v,w)] = [R(v,w)]J$, which equals $1/2$ of its real trace, and so $i\,\mathrm{tr}_{\mathbf{C}}[R(v,w)] = -\rho(v,w)$ by (H.1.a).

In other words, $\rho = i\,d\varGamma_a^a$ on U, as $i\rho = R_a^a = -d\varGamma_a^a$, with $\varGamma_a^c \wedge \varGamma_c^a = 0$ due to obvious pairwise cancellations. Now (i) and (ii) are immediate from the discussion in the second paragraph of Appendix F applied to $\mathcal{L} = [TM]^{\wedge m}$ and $\psi = e_1 \wedge \ldots \wedge e_m$ (with the connection form $\varGamma = \varGamma_a^a$).

The formulae $\mathfrak{G} = [g^{\mathbf{C}}(e_a, e_b)]$ and $\mathfrak{D} = \det_{\mathbf{C}}\mathfrak{G}$ define functions on U valued in $m \times m$ Hermitian matrices and, respectively, in positive real numbers. For any vector field w on U we have $d_w \log \mathfrak{D} = \mathrm{tr}_{\mathbf{C}}(\mathfrak{G}^{-1} d_w \mathfrak{G})$, in view of (K.1) for $F = \mathfrak{G}$ treated as a function of the parameter t of any given integral curve of w. As $\nabla g^{\mathbf{C}} = 0$, the Leibniz rule gives $d_w h_{ab} = \varGamma_a^c(w) h_{cb} + \overline{\varGamma_b^c(w) h_{ca}}$ for the entries $h_{ab} = g^{\mathbf{C}}(e_a, e_b)$ of $\mathfrak{G}$, that is, $d_w \mathfrak{G} = \mathfrak{T}\mathfrak{G} + (\mathfrak{T}\mathfrak{G})^*$, where * stands for the conjugate transpose, and $\mathfrak{T}$ is the matrix-valued function with the entries $\varGamma_a^b(w)$. (In both $\varGamma_a^b(w)$ and h_{ab}, the index a is the row number and b the column number.) This gives $d_w \log \mathfrak{D} = \mathrm{tr}_{\mathbf{C}}(\mathfrak{G}^{-1} d_w \mathfrak{G}) = \mathrm{tr}_{\mathbf{C}}[\mathfrak{G}^{-1}\mathfrak{T}\mathfrak{G} + (\mathfrak{G}^{-1}\mathfrak{T}\mathfrak{G})^*] = 2\,\mathrm{Re}\,\mathrm{tr}_{\mathbf{C}}\mathfrak{T} = 2\,\mathrm{Re}\,\varGamma_a^a(w)$. Hence $d\log\mathfrak{D} = 2\,\mathrm{Re}\,\varGamma$, where $\varGamma = \varGamma_a^a$ denotes, as in the previous paragraph, the connection form in $[TM]^{\wedge m}$ with $\rho = i\,d\varGamma$.

Let $\hat{g}^{\mathbf{C}}, \hat{h}_{ab}, \hat{\mathfrak{D}}, \hat{\nabla}, \hat{\varGamma}_a^b$ and $\hat{\varGamma}$ be the analogous objects for another Kähler metric $\hat{g}$ on M (with the same vector fields e_a on U), and let $H : TM \to TM$ be the complex-linear bundle morphism such that $\hat{g}^{\mathbf{C}}(v,w) = g^{\mathbf{C}}(Hv, w)$. For the matrix H_a^b of functions $U \to \mathbf{C}$ given by $He_a = H_a^c e_c$ we thus have $\hat{h}_{ab} = H_a^c h_{cb}$, and so $\hat{\mathfrak{D}}/\mathfrak{D} = \det_{\mathbf{C}} H$. Moreover, since H is, at each point, a self-adjoint positive operator, $\det_{\mathbf{C}} H$ is real-valued and equals $[\det_{\mathbf{R}} H]^{1/2}$. Finally, taking the real part of the equality $\hat{g}^{\mathbf{C}}(v,w) = g^{\mathbf{C}}(Hv,w)$ we obtain $\hat{g}(v,w) = g(Hv,w)$, and so, by (K.2.a), $[\det_{\mathbf{R}} H]^{1/2} = [\det_g \hat{g}]^{1/2} = \gamma$. Consequently, $\hat{\mathfrak{D}}/\mathfrak{D} = \gamma$.

The equalities $\rho = i\,d\varGamma$, $d\log\mathfrak{D} = 2\,\mathrm{Re}\,\varGamma$ and their analogues for $\hat{g}$ now give $d\log\gamma = 2\,\mathrm{Re}\,(\hat{\varGamma} - \varGamma)$ and $\rho - \hat{\rho} = i\,d(\varGamma - \hat{\varGamma}) = d[i(\varGamma - \hat{\varGamma})]$. However, $\varGamma - \hat{\varGamma}$ is, at every point x, complex-linear as a mapping $T_xM \to \mathbf{C}$. (In fact, so is $\varGamma_a^b - \hat{\varGamma}_a^b$ for each pair of indices a, b, since $\hat{\nabla}_v w - \nabla_v w$ depends on v, w symmetrically and complex-bilinearly: symmetry follows as both connections are torsionfree, while $\mathbf{C}$-linearity in v is immediate from symmetry and $\mathbf{C}$-linearity in w, the latter being due to the relations $\nabla J = \hat{\nabla} J = 0$.) Therefore, $\rho - \hat{\rho} = d[(\varGamma - \hat{\varGamma})J]$. Since ρ and $\hat{\rho}$ are real-valued, this equals $d\,\mathrm{Re}\,[(\varGamma - \hat{\varGamma})J] = -d[(d\log\gamma)J]/2 = i\partial\overline{\partial}\log\gamma$ (see (G.2)), which proves (iii).

The next result is due to Calabi [16, pp. 86–87]. The proof of assertion (a) given here comes from Yau [119, p. 375]. See also Bérard Bergery's exposition [4].

Theorem K.2. *Let $g, \hat{g}$ be two Kähler metrics on a compact complex manifold M, with the Ricci tensors r and $\hat{\mathrm{r}}$, and the Kähler classes $[\varOmega], [\hat{\varOmega}] \in H^2(M, \mathbf{R})$.*

(a) *If $\mathrm{r} = \hat{\mathrm{r}}$ and $[\varOmega] = [\hat{\varOmega}]$, then $g = \hat{g}$.*
(b) *If $\mathrm{r} = -g$ and $\hat{\mathrm{r}} = -\hat{g}$, then $g = \hat{g}$.*

Proof. Let $\gamma : M \to (0, \infty)$ be the ratio of the volume elements, with $\mathrm{d}\hat{g} = \gamma\,\mathrm{d}g$.

The assumption $\mathrm{r} = \hat{\mathrm{r}}$ made in (a) gives $\rho = \hat{\rho}$ for the Ricci forms. Hence γ is constant in view of Remark K.1(iii) and (H.4). The other assumption, $[\varOmega] = [\hat{\varOmega}]$, now has two consequences. First, by (J.1), the constant γ must be equal to 1. Secondly, the 2-form $\varOmega - \hat{\varOmega}$ is exact, so that, in view of Lemma H.2(a), $\varOmega = \hat{\varOmega} - i\partial\overline{\partial}\alpha$ for some C^∞ function $\alpha : M \to \mathbf{R}$. Taking the g-trace of both sides of the corresponding equality $\hat{g} = g - (i\partial\overline{\partial}\alpha)J$ involving the metrics $g = -\Omega J$ and $\hat{g} = -\hat{\Omega}J$, we see, using (H.3.ii), (K.2.b) with $\gamma = 1$ and $\mathrm{tr}_g g = n$, for $n = \dim_{\mathbf{R}} M$, that $n = n\gamma^{2/n} \leq \mathrm{tr}_g\,\hat{g} = \mathrm{tr}_g[g - (i\partial\overline{\partial}\alpha)J] = n + \Delta\alpha$. Hence $\Delta\alpha \geq 0$. Thus, by (2.20.b), α is constant, and so $g = \hat{g}$, which proves (a).

Under the hypotheses of (b), $\rho = -\Omega$ and $\hat{\rho} = -\hat{\Omega}$, so that, for $\alpha = \log\gamma$, Remark K.1(iii) yields $\Omega = \hat{\Omega} - i\partial\overline{\partial}\alpha$. As in the preceding paragraph, this gives $\hat{g} = g - (i\partial\overline{\partial}\alpha)J$. By (H.3.i), $-2[(i\partial\overline{\partial}\alpha)J](u,v) = (\nabla d\alpha)(u,v) + (\nabla d\alpha)(Ju,Jv)$ for any point $x \in M$ and any vectors $u, v \in T_xM$. Hence, as $\alpha = \log\gamma$, we have $\hat{g} \leq g$ (or, $\hat{g} \geq g$) at points where $\gamma = \gamma_{\max}$ (or, respectively, $\gamma = \gamma_{\min}$). The inequalities between tensors have here the usual meaning: for instance, $\hat{g} \leq g$ states that $\hat{g} - g$ is negative semidefinite, or, equivalently, that if $\hat{g}$ is treated, with the aid of g, as a bundle morphism $A : TM \to TM$ (see Remark 2.1), then its eigenvalues do not exceed 1 at the point in question. Since the eigenvalues of $\hat{g}$ are all positive, we now have, from (K.2.a), $\gamma^2 = \det_g\hat{g} \leq 1$ wherever $\gamma = \gamma_{\max}$ and, similarly, $\gamma^2 \geq 1$ wherever $\gamma = \gamma_{\min}$. Consequently, $\gamma_{\max} \leq 1 \leq \gamma_{\min}$ and so $\gamma = 1$ everywhere in M, that is, $\alpha = 0$ and $g = \hat{g}$. □

Appendix L. Holomorphic vector fields

We say that a C^∞ mapping $F : M \to N$ between almost complex manifolds M and N is *holomorphic* if, at every $x \in M$, the differential $dF_x : T_xM \to T_{F(x)}N$ is complex-linear. A diffeomorphism $F : M \to N$ which is holomorphic is referred to as a *biholomorphism*, and, if such F exists, M and N are called *biholomorphic*. By a (real) *holomorphic* vector field on an almost complex manifold M we mean any C^∞ vector field w on M for which $\pounds_w J = 0$, that is, the flow of w consists of (local) biholomorphisms. For more on terminology, see the end of this section.

Remark L.1. Let w be a vector field on a Kähler manifold (M, g). We treat the covariant derivative ∇w of M, as well as the complex structure J, as bundle morphisms $TM \to TM$, while $[\,,\,]$ denotes the commutator of such morphisms.

(a) For $u = Jw$, we have $\nabla u = J\nabla w$.
(b) The Lie derivative $\pounds_w J$ equals $[J, \nabla w]$. Thus, w is holomorphic if and only if $[J, \nabla w] = 0$.
(c) If w is holomorphic, so is Jw.
(d) The following three conditions are equivalent:
  i) w is holomorphic and is, locally, the gradient of a function;
  ii) Jw is a holomorphic Killing field;
  iii) the tensor field $\nabla\xi$, where $\xi = \imath_w g$, is symmetric and Hermitian.

In fact, as $\nabla J = 0$, we get (a) and $\pounds_w J = [J, \nabla w]$, which yields (b). (The relation $\pounds_w u = [w, u] = \nabla_w u - \nabla_u w$, for any vector field u, gives $(\pounds_w J)u = \pounds_w(Ju) - J(\pounds_w u) = [J, \nabla w]u$.) Now (c) is obvious from (a) and (b). Next, in (d), let $u = Jw$. Condition (i) states that $[J, \nabla w] = 0$ (cf. (b)) and $(\nabla w)^* = \nabla w$, and so $[J, \nabla u] = 0$ and $(\nabla u)^* = -\nabla u$ (as $\nabla u = J\nabla w$ by (a)); hence (ii) follows. Assuming (ii) we similarly get $[J, \nabla u] = 0$ and $(\nabla u)^* = -\nabla u$, while $\nabla w = -J\nabla u$, which yields $[J, \nabla w] = 0$ and $(\nabla w)^* = \nabla w$, that is, (i). Finally, as (i) amounts to $[J, \nabla w] = 0$ and $(\nabla w)^* = \nabla w$, it is equivalent to (iii) (cf. Remark G.2), since $a = \nabla\xi$ in (iii) corresponds to $A = \nabla w$ as in Remark 2.1.

Remark L.2. The real vector space $\mathfrak{h}(M)$ of all holomorphic vector fields on a Kähler manifold (M, g) is a *complex* Lie algebra: in addition to being closed under the Lie bracket, it has the structure of a complex space, with $v \mapsto Jv$ serving as the multiplication by i (cf. Remark L.1(c)).

By a *locally symmetric Kähler manifold* we mean any Riemannian manifold (M,g) which is simultaneously an almost complex manifold, such that the metric g is Hermitian (Remark G.2) and, for every $x \in M$, there exists a holomorphic g-isometry $\varPhi_x : U_x \to U_x$ of some neighborhood U_x of x in M with the differential at x equal to $-\mathrm{Id} : T_xM \to T_xM$. The terminology makes sense since such (M,g) is automatically a Kähler manifold (and, in addition, its curvature tensor is parallel). In fact, any k-times covariant tensor field T on M, for odd k, which is invariant under $\varPhi_x$ for every x, must vanish identically (as the differential of $\varPhi_x$ at x sends T_x to T_x and, at the same time, to $-T_x$). Applying this to $T = \nabla\varOmega$ and $T = \nabla R$, for $\varOmega = gJ$ and the four-times covariant curvature tensor R, we get $\nabla\varOmega = 0$ and $\nabla R = 0$, as required.

In any complex dimension m, one prominent example of a locally symmetric Kähler manifold is the standard $\mathbf{C}^m$. Another is the complex projective space $\mathbf{C}\mathrm{P}^m$, formed by all complex lines through 0 in $\mathbf{C}^{m+1}$, and hence equal to the quotient S^{2m+1}/S^1 of the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$ under the action, by multiplication, of the unit circle $S^1 \subset \mathbf{C}$. Since S^1 acts on the ambient space $\mathbf{C}^{m+1}$ by holomorphic isometries, a Riemannian metric and an almost complex structure on $\mathbf{C}\mathrm{P}^m$ can be uniquely defined by projecting them, via the isomorphism $d\pi_y$, from the orthogonal complement of $\mathrm{Ker}\, d\pi_y$ in T_yS^{2m+1} onto $T_x\mathbf{C}\mathrm{P}^m$, where $\pi : S^{2m+1} \to \mathbf{C}\mathrm{P}^m$ is the quotient projection, while $y \in S^{2m+1}$ and $x = \pi(y)$. The holomorphic isometry $\varPhi_x$ required in the last paragraph is provided by the unitary reflection about the line $\mathbf{C}y$ in $\mathbf{C}^{m+1}$, which obviously descends to $\mathbf{C}\mathrm{P}^m$.

The *Fubini-Study* metric g on $\mathbf{C}\mathrm{P}^m$, described above, is also an Einstein metric. In fact, the unitary automorphisms of $\mathbf{C}^{m+1}$ keeping a given unit vector y fixed descend to isometries $\mathbf{C}\mathrm{P}^m \to \mathbf{C}\mathrm{P}^m$ which fix the point $x = \pi(y)$. The differentials of these isometries at x form a group acting on $T_x\mathbf{C}\mathrm{P}^m$ in a manner equivalent to how $\mathrm{U}(m)$ acts on $\mathbf{C}^m$ (as one sees identifying $y^\perp \approx \mathbf{C}^m$ with $T_x\mathbf{C}\mathrm{P}^m$ via the isomorphism $d\pi_y$). The Ricci tensor of g at x now must be a multiple of g_x, or else its eigenspaces would correspond to nontrivial proper $\mathrm{U}(m)$-invariant real subspaces of $\mathbf{C}^m$ (which do not exist, since $\mathrm{U}(m)$ acts transitively on the unit sphere $S^{2m-1} \subset \mathbf{C}^m$).

Here is the reason why we are speaking of Kähler metrics on *complex* manifolds (without the word 'almost'). One normally defines a *complex manifold* to be any almost complex manifold M whose almost complex structure J is *integrable* in the sense that every point of M has a connected neighborhood biholomorphic to an open set in $\mathbf{C}^m$, $m = \dim_{\mathbf{C}} M$. In other words, M is required to be covered by a collection of $\mathbf{C}^m$-valued charts, the transition mappings between which are all holomorphic. The term 'holomorphic' that we used for F or w at the beginning of this section is usually reserved for objects on complex manifolds; in the general almost-complex case, such F and w are called *pseudoholomorphic*. However, in a Kähler manifold, J is always integrable (which is a well-known fact, not used here). Our terminology thus agrees, in the end, with the standard usage.

Appendix M. An elementary proof of Proposition 8.2

Although the argument deriving Proposition 8.2 from Theorem 6.1 is trivial (and well known), proving the latter requires solving a nonlinear elliptic equation; see §6. This is why we give below a completely elementary proof of Proposition 8.2.

The symbols $\varOmega$ and ρ stand for the Kähler and Ricci forms of any given Kähler manifold (M,g), so that $\varOmega(u,v)=g(Ju,v)$ and $\rho(u,v)=\mathrm{r}(Ju,v)$ for all vector fields u,v. Both $\varOmega$ and ρ are closed differential 2-forms, and the real cohomology class $[\rho]$ equals $2\pi c_1(M)$. (See Remarks H.1 and K.1(ii) in Appendices H and K.)

Lemma M.1. *Given a Kähler manifold (M,g), let $A:TM\to TM$ be a bundle morphism anticommuting with J and skew-adjoint at every point, and let a be the twice-covariant tensor corresponding to A as in* Remark 2.1. *If a is closed as a differential 2-form, then $\delta A=0$, where δ is the divergence operator with* (2.9).

Proof. Let $\xi=\delta A$. In local coordinates, $J_k^p\xi_p=J_k^pA_{p,l}^l$. As $AJ=-JA$, this equals $-J_p^lA_{k,l}^p=\varOmega^{lp}a_{kp,l}$, that is, $-1/2$ times $\varOmega^{lp}a_{pl,k}=J_l^pA_{p,k}^l=(J_l^pA_p^l)_{,k}$. (Since $da=0$, (2.24.c) gives $0=\varOmega^{lp}(a_{kp,l}+a_{lk,p}+a_{pl,k})=2\varOmega^{lp}a_{kp,l}+\varOmega^{lp}a_{pl,k}$.) However, $J_l^pA_p^l=\operatorname{tr}JA=0$, as $JA=-AJ$. Thus, $J_k^p\xi_p=0$ and $\delta A=\xi=0$. □

In any Kähler manifold (M,g), given a bundle morphism $B:TM\to TM$, the commutator $A=[J,B]:TM\to TM$ satisfies the obvious relations

(M.1) a) $A+A^*=[J,B+B^*]$, b) $JA+AJ=0$, c) $A=JB-(JB)^*-(B+B^*)J$.

If, in addition, $B+B^*$ commutes with J, then

$$\operatorname{tr}A^2\;=\;2\operatorname{tr}AJB. \tag{M.2}$$

In fact, (M.1.a) gives $A^*=-A$, and so $\operatorname{tr}A(B+B^*)J=\operatorname{tr}ABJ+\operatorname{tr}(AB^*J)^*=\operatorname{tr}(ABJ+JBA)=\operatorname{tr}(JAB+AJB)=\operatorname{tr}(JA+AJ)B$, which equals 0 by (M.1.b). Now (M.1.c) yields $\operatorname{tr}A^2=\operatorname{tr}AJB-\operatorname{tr}A(JB)^*$, while $\operatorname{tr}A(JB)^*=\operatorname{tr}[A(JB)^*]^*=-\operatorname{tr}JBA=-\operatorname{tr}AJB$, which implies (M.2).

In the next lemma aJ, for $a=\pounds_wg$, is defined as in Remark G.1.

Lemma M.2. *Let w be a vector field on a Kähler manifold (M,g). If the symmetric tensor $\pounds_wg$ is Hermitian and the 2-form $(\pounds_wg)J$ is closed, then w is holomorphic and $(\pounds_wg)J$ is exact.*

Proof. Let $B=\nabla w$, $A=[J,B]$, $u=Jw$ and $\xi=\imath_ug$. Hermitian symmetry of $\pounds_wg$ means, by (2.6.a) and Remark G.2, that $[J,B+B^*]=0$, and hence $A^*=-A$ (see (M.1.a)). Moreover, (M.1.c), combined with Remark G.2, states that the twice-covariant tensor a, corresponding to A as in Remark 2.1, is given by

$$a\;=\;d\xi\;-\;(\pounds_wg)J. \tag{M.3}$$

(In fact, $JB=\nabla u$, cf. Remark L.1(a), so that, in view of (2.24.b) and (2.6.a), the morphisms $TM\to TM$ associated with $d\xi$, $\pounds_wg$ and $(\pounds_wg)J$ via Remark 2.1 are $JB-(JB)^*$, $B+B^*$ and, respectively, $(B+B^*)J$.) Closedness of $(\pounds_wg)J$ and (M.3) now give $da=0$, and so $\delta A=0$ by (M.1.b) and Lemma M.1. Next, from (M.2), $\operatorname{tr}A^2=2\operatorname{tr}AJB=2\operatorname{tr}A(\nabla u)$, while $\operatorname{tr}A(\nabla u)=\delta(Au)$, since $\delta A=0$. Integrating and using (2.15), we obtain $\int_M\operatorname{tr}A^2\mathrm{d}g=0$. Hence $A=0$. (In fact, skew-adjointness of A yields $\operatorname{tr}A^2<0$ at points where $A\neq 0$). Thus, w is holomorphic (Remark L.1(b)). Also, (M.3) with $a=0$ implies exactness of $(\pounds_wg)J$. □

Remark M.3. One obvious consequence of Lemma M.2 is the well-known fact that, on a compact Kähler manifold (M,g), every Killing vector field w is holomorphic.

Proof of Proposition 8.2. Equation (0.1) implies that $\pounds_w g$ is Hermitian (since so are r and g) and $(\pounds_w g)J = (\lambda g - \mathrm{r})J = \lambda\varOmega - \rho$ is closed (since so are ρ and $\varOmega$); see Remark H.1. Our assertion now follows from Lemma M.2. □

Appendix N. The Futaki and Tian-Zhu invariants

By a *compact complex manifold with* $c_1(M) > 0$, or $c_1(M) < 0$, we mean any compact almost complex manifold M that admits a Kähler metric with the Kähler cohomology class $c_1(M)$ or, respectively, $-c_1(M)$. (Cf. the text preceding Theorem J.3.) This is equivalent to the requirement that M be a compact almost complex manifold admitting a Kähler metric and, at the same time, having a positive (or, respectively, negative) first Chern class in the sense defined at the end of Appendix J. Namely, an almost-Kähler metric with the Kähler form $\varOmega$ such that $c_1(M) = \pm[\varOmega]$ must then be a Kähler metric by Theorem J.3(ii).

The *Futaki invariant* [53] of a compact Kähler manifold (M,g) is the real-linear functional $\boldsymbol{F} : \mathfrak{h}(M) \to \mathbf{R}$ on the Lie algebra $\mathfrak{h}(M)$ (see Remark L.2), defined as follows. With $\varOmega$ and ρ denoting, as usual, the Kähler and Ricci forms, and with $\mathrm{s}_{\mathrm{avg}}$ standing for the average value of the scalar curvature s, let $f : M \to \mathbf{R}$ be a function such that $\varDelta f + \mathrm{s} = \mathrm{s}_{\mathrm{avg}}$. We set

$$\text{(N.1)} \qquad \boldsymbol{F}v = \mu\textstyle\int_M d_v f \, \mathrm{d}g \;\text{ for }\; v \in \mathfrak{h}(M), \;\text{ where }\; \mu = (\mathrm{s}_{\mathrm{avg}})^m \text{ and } m = \dim_{\mathbf{C}} M.$$

The Futaki invariant $\boldsymbol{F}$ is particularly interesting for compact complex manifolds M with $c_1(M) > 0$, since M then admits a Kähler metric g with $[\rho] = \lambda[\varOmega]$ for some $\lambda \in (0,\infty)$ (e.g., $\lambda = 1$), and $\boldsymbol{F}$ turns out to be the same for all such metrics g. In other words, $\boldsymbol{F}$ then is an invariant of the complex structure of M. As such, it constitutes a well-known obstruction [54] to the existence of Kähler-Einstein metrics on compact complex manifolds M with $c_1(M) > 0$. All of this is summarized by the following result of Futaki [53]:

Theorem N.1. *Given a compact complex manifold* (M,g) *with* $c_1(M) > 0$*, the Futaki invariant* $\boldsymbol{F} : \mathfrak{h}(M) \to \mathbf{C}$*, defined with the aid of a Kähler metric* g *such that* $[\rho] = \lambda[\varOmega]$ *for a constant* λ*, does not depends on the choice of such* g*. Furthermore,* $\boldsymbol{F} = 0$ *if* M *admits a Kähler-Einstein metric.*

The final clause of Theorem N.1 is immediate from its first part: using a Kähler-Einstein metric g to evaluate $\boldsymbol{F}$, we get $\boldsymbol{F} = 0$, since f in (N.1) is constant.

Theorem N.1 can be derived from Tian and Zhu's Theorem 10.5 (see Remark N.7 below). However, we establish the two theorems separately, since a direct proof of Theorem N.1 is much shorter than one needed for Theorem 10.5.

We begin with two lemmas, in which $\varOmega, \rho$ and $\mathrm{s}_{\mathrm{avg}}$ denote, as before, the Kähler form, Ricci form, and the average value of the scalar curvature s.

Lemma N.2. *If* (M,g) *is a compact Kähler manifold,* $\lambda \in \mathbf{R}$*, and* $[\rho] = \lambda[\varOmega]$ *in* $H^2(M,\mathbf{R})$*, then* $\lambda = \mathrm{s}_{\mathrm{avg}}/n$*, where* $n = \dim_{\mathbf{R}} M$*, and*

$$\text{(N.2)} \qquad i\partial\overline{\partial} f \;+\; \rho \;=\; \lambda\varOmega \quad \text{ for } \; f : M \to \mathbf{R} \;\text{ such that }\; \varDelta f + \mathrm{s} \,=\, \mathrm{s}_{\mathrm{avg}}.$$

In fact, $i\partial\overline{\partial} f + \rho = \lambda\varOmega$ for *some* function f, as $[\rho] = \lambda[\varOmega]$ (see Lemma H.2(a)). Now (H.3.ii) gives $\varDelta f + \mathrm{s} = n\lambda$, and so $\lambda = \mathrm{s}_{\mathrm{avg}}/n$.

In the next lemma, only parts (a) and (b) are needed for a proof of Theorem N.1. The symbol L denotes the operator given by (10.2), i.e., $Lv = \delta v - d_v f$ for vector fields v, with f as in (N.2), while P in (e) is defined as in (10.5).

Lemma N.3. *Let (M,g) be a compact Kähler manifold such that $[\rho] = \lambda[\Omega]$ in $H^2(M,\mathbf{R})$ for some $\lambda \in \mathbf{R}$. Then, for any holomorphic vector field v on M,*

(a) $\nabla Lv - J\nabla LJv = -2\lambda v$, (b) $\Delta Lv = -2\lambda\delta v$,
(c) $|\nabla Lv|^2 + 2\lambda d_v Lv = |\nabla LJv|^2 + 2\lambda d_{Jv}LJv$,
(d) $g(\nabla Lv, \nabla LJv) + \lambda(d_{Jv}Lv + d_v LJv) = 0$.
(e) $\lambda\psi - d_w\psi + \Delta\psi/2 = i\,d_{Jw}\psi$, *where* $\psi = Pv$ *and* $w = \nabla f/2$.

Proof. The cotangent-vector version of (a): $d(Lv) + \{d[L(Jv)]\}J = -2\lambda\imath_v g$, reads

$$v^k{}_{,kl} - (v^k f_{,k})_{,l} + J_l^q J_k^p v^k{}_{,pq} - J_l^q J_k^p (v^k f_{,p})_{,q} = -2\lambda v_l \tag{N.3}$$

in local coordinates. On the other hand, $-v^k f_{,kl} - J_l^q J_k^p v^k f_{,pq}$ is the lth component of $-\imath_v\nabla df - (\imath_u\nabla df)J = 2\imath_u(i\partial\overline{\partial}f)$ for $u = Jv$ (by (H.3.i)), which equals $2\imath_u(\lambda\Omega - \rho) = -2\lambda\imath_v g + 2\imath_v\mathrm{r}$ by (N.2). Next, as $[J,\nabla v] = 0$ (see Remark L.1(b)), $-J_l^q J_k^p v^k{}_{,q} f_{,p} = -J_l^q J_q^k v^p{}_{,k} f_{,p} = v^p{}_{,l} f_{,p}$, while $J_l^q J_k^p v^k{}_{,pq} = J_l^q J_k^p v^k{}_{,qp} + J_l^q J_k^p R_{pqs}{}^k v^s$ by the Ricci identity (2.12.a). The relation $[J,\nabla v] = 0$ also gives $J_l^q J_k^p v^k{}_{,qp} = J_l^q J_q^k v^p{}_{,kp} = -v^p{}_{,lp}$. Moreover, by (H.1.c), $J_l^q J_k^p R_{pqs}{}^k v^s = R_{kls}{}^k v^s = -R_{ls}v^s$. Combining these equalities and using (2.12.b), we get (N.3), that is, (a). Now (b) follows if we apply the divergence operator δ to (a), where $\delta(J\nabla\phi) = 0$ for any function ϕ, as $\delta(J\nabla\phi) = J_l^k \phi_,{}^l{}_k$, while J is skew-adjoint and $\nabla d\phi$ is symmetric. Next, $|\nabla Lv|^2 + 2\lambda d_v Lv = g(\nabla Lv, \nabla Lv + 2\lambda v) = g(\nabla Lv, J\nabla LJv)$ by (a). The same equality for Jv rather than v, cf. Remark L.1(c), reads $|\nabla LJv|^2 + 2\lambda d_{Jv}LJv = -g(\nabla LJv, J\nabla Lv)$, and, as J is skew-adjoint, the two equalities together prove (c). The left-hand side in (d) is 1/2 times $g(\nabla LJv, \nabla Lv + 2\lambda v) + g(\nabla Lv, \nabla LJv + 2\lambda Jv) = g(\nabla LJv, J\nabla LJv) - g(\nabla Lv, J\nabla Lv)$ (by (a)); now (d) follows due to skew-adjointness of J. Finally, (b) applied to both v and Jv (see Remark L.1(c)) gives $\Delta\psi = -2\lambda[\delta v - i\delta(Jv)]$ for $\psi = Pv$, since $Pv = Lv - iLJv$. Now, by (10.2), $\Delta\psi = -2\lambda(Lv + d_v f) + 2i\lambda(LJv + d_{Jv}f) = -2\lambda(\psi + d_v f - i\,d_{Jv}f)$, and so (a) implies (e). □

Let us now suppose that (M,g) is a compact Riemannian manifold, $f : M \to \mathbf{R}$, and u, v are vector fields on M. With $Lw = \delta w - d_w f$, we get, for any vector field w and any $\phi : M \to \mathbf{R}$,

$$\text{(N.4)}\quad \begin{array}{l} \text{a) } \delta\nabla(e^{-f}u) = \delta(e^{-f}\nabla u) - e^{-f}[\nabla_u df + (Lu)df], \ \text{ b) } e^{-f}Lu = \delta(e^{-f}u), \\ \text{c) } -d\delta w = \imath_w\mathrm{r} - \delta\nabla w, \ \text{ d) } \int_M \phi\,\delta v\,\mathrm{d}g = -\int_M \imath_v d\phi\,\mathrm{d}g. \end{array}$$

In fact, (b) – (d) are trivial special cases of (2.8.a), (2.11.b) and (2.17), while (a) follows since $\nabla(e^{-f}u) = e^{-f}\nabla u - e^{-f}df \otimes u$, and $\delta(e^{-f}df \otimes u) = e^{-f}\nabla_u df + e^{-f}(Lu)df$ due to the definition of L. Let us denote by $(\ ,\ )_f$ the weighted L^2 inner product with $(\phi,\phi)_f = \int_M \phi^2 e^{-f}\,\mathrm{d}g$, by $\|\ \|_f$ the corresponding norm, both for functions and vector fields, and by $(\ ,\ )$ the ordinary L^2 inner product. Using, respectively, (N.4.b), (N.4.d) (for $\phi = \delta(e^{-f}u)$), and (N.4.c) (for $w = e^{-f}u$), we see that $(Lu, \delta v)_f = (\delta(e^{-f}u), \delta v) = -\int_M \imath_v d\delta(e^{-f}u)\,\mathrm{d}g = (\mathrm{Ric}\,u, v)_f - \int_M \imath_v\delta\nabla(e^{-f}u)\,\mathrm{d}g$, where Ric is the bundle morphism $A : TM \to TM$ corresponding as in Remark 2.1 to the Ricci tensor $a = \mathrm{r}$ of (M,g). Thus, by (N.4.a), $(Lu, \delta v)_f =$

Given a compact Riemannian manifold (M,g) and a function $f : M \to \mathbf{R}$, let us denote by Ric_f the bundle morphism $A : TM \to TM$ corresponding as in Remark 2.1 to $a = \nabla df + \mathrm{r}$, where r is the Ricci tensor of (M,g), by δ_f the operator sending a vector field w to the function $\delta_f w = e^f \delta(e^{-f} w)$ (so that, when f is the zero function, δ_f becomes the ordinary divergence δ, cf. (2.7.i)), and by $(\,,\,)_f$ the weighted L^2 inner product of tensor fields on M with $(A,B)_f = \int_M \langle A,B\rangle e^{-f}\,\mathrm{d}g$. The symbol $\langle\,,\rangle$ in the integrand represents the inner product induced by g, including the ordinary product (when A, B are functions) and g (when they are vector fields). The divergence theorem (2.15) now implies that

$$\text{(N.5)}\qquad \text{a) } (\delta_f w, 1)_f \;=\; 0, \;\; \text{b) } (\nabla\chi, w)_f = -(\chi, \delta_f w)_f \;\text{ whenever } \chi : M \to \mathbf{R}.$$

Also, for any vector fields u, v on M, and any $f : M \to \mathbf{R}$,

$$\text{(N.6)}\qquad (\mathrm{Ric}_f u, v)_f \;=\; (\delta_f u, \delta_f v)_f \;-\; (\nabla u, (\nabla v)^*)_f\,,$$

$(\nabla v)^*$ being the (pointwise) adjoint of $\nabla v : TM \to TM$. If $f = 0$, (N.6) is nothing else than Bochner's integral formula (2.21).

To verify (N.6), note that $\delta_f[\nabla_v u - (\delta_f u)v] = \mathrm{tr}\,(\nabla u)(\nabla v) + (\mathrm{r} + \nabla df)(u,v) - (\delta_f u)\delta_f v$ (as one easily sees in local coordinates, using (2.12.b) and the Leibniz rule); then apply (N.5.a).

It is obvious from (N.6) and (N.5.b), for $\chi = \delta_f u$ and $w = v$, that

$$\text{(N.7)}\qquad -(\nabla\delta_f u, v)_f \;=\; (\mathrm{Ric}_f u, v)_f \;+\; (\nabla u, (\nabla v)^*)_f\,.$$

Lemma N.4. *Suppose that u, v are vector fields on a Kähler manifold (M,g) and $f : M \to \mathbf{R}$. Then, for $A = \nabla u$ and $B = \nabla v$,*

i) $\pounds_{\nabla f} J = [J, \nabla\nabla f]$,
ii) the bundle morphism $(\pounds_{\nabla f} J)J : TM \to TM$ is self-adjoint at every point,
iii) $\langle Ju, \nabla_{Jv}\nabla f\rangle = (\nabla df)(u,v) + \langle u, (\pounds_{\nabla f} J)Jv\rangle$,
iv) $\delta_f[(d_{Ju} f)Jv] = (\nabla df)(u,v) + (d_{Ju} f)(\mathrm{tr}\, JB - d_{Jv} f) + \langle\nabla f, JAJv\rangle + \langle u, (\pounds_{\nabla f} J)Jv\rangle$.

Proof. Assertion (i) is obvious from Remark L.1(b). By (i), $\pounds_{\nabla f} J$ anticommutes with J. As $J^* = -J$ and $(\nabla\nabla f)^* = \nabla\nabla f$, (i) also implies that $\pounds_{\nabla f} J$ is self-adjoint at every point, and (ii) follows.

Next, $(\nabla df)(u,v) = \langle u, (\nabla\nabla f)v\rangle = \langle Ju, J(\nabla\nabla f)v\rangle$, which is nothing else than $\langle Ju, (\nabla\nabla f)Jv\rangle - \langle Ju, [J, \nabla\nabla f]v\rangle$, so that (i) yields (iii).

Finally, in local coordinates, $\delta_f[(d_{Ju} f)Jv] = e^f[e^{-f}(Ju)^l f_{,l}(Jv)^k]_{,k}$ equals

$$-f_{,k}(Ju)^l f_{,l}(Jv)^k + J_s^l u^s{}_{,k} f_{,l}(Jv)^k + (Ju)_s f_{,}{}^s{}_k (Jv)^k + (Ju)^l f_{,l} J_s^k v^s{}_{,k}\,.$$

These four terms are, respectively, $-(d_{Ju} f) d_{Jv} f$, $\langle\nabla f, JAJv\rangle$, $\langle Ju, \nabla_{Jv}\nabla f\rangle$ and $(d_{Ju} f)\,\mathrm{tr}\, JB$. Now (iv) is immediate from (iii). □

The expression $(\mathrm{Ric}_f u, v)_f$ also appears in another integral identity, requiring additional hypotheses. Specifically, we have the following lemma.

Lemma N.5. *Let* $f : M \to \mathbf{R}$ *be a function on a compact Kähler manifold* (M,g). *If vector fields* u,v *on* M *are local gradients, that is, the* 1*-forms* $\imath_u g, \imath_v g$ *are closed, then, with* J *denoting the complex-structure tensor of* (M,g),

$$(\text{N.8})\quad (\pounds_u J, \pounds_v J)_f/2 = (\nabla u, \nabla v)_f - (\mathrm{Ric}_f u, v)_f + (d_{Ju} f, d_{Jv} f)_f - ((\pounds_{\nabla f} J)u, Jv)_f,$$

where $\pounds$ *stands for the Lie derivative. Furthermore,*

$$(\text{N.9})\quad -(\nabla\delta_f u, v)_f = 2(\mathrm{Ric}_f u, v)_f - (d_{Ju} f, d_{Jv} f)_f + ((\pounds_{\nabla f} J)u, Jv)_f + (\pounds_u J, \pounds_v J)_f/2.$$

In the remainder of Appendix N, all tensor fields, such as a Riemannian metric g, and operators (including connections), depend C^∞-differentiably on a *time parameter* t varying in a fixed interval, in the sense that their components in a local coordinate system are C^∞ functions of the coordinates and t. Their dependence on t will, however, be suppressed in our notation. The same will apply to the volume element $\mathrm{d}g$, divergence operator δ, and the g-inner product $\langle\,,\rangle$ of twice-covariant symmetric tensors. Rather than speaking of curves of metrics, connections, etc., we will refer to such objects as *time-dependent* (and call them *time-independent* when appropriate). Writing $(\;)^{\cdot}$ for d/dt, we have

$$(\text{N.10})\quad \text{a) } \dot\delta = d\varphi \text{ and b) } \langle g, \dot g\rangle = 2\varphi \text{ for } \varphi : M \to \mathbf{R} \text{ such that: c) } (\mathrm{d}g)^{\cdot} = \varphi\,\mathrm{d}g,$$

(a) meaning that $(\delta w)^{\cdot} = d_w\varphi$ for any time-independent vector field w on M. In fact, contracting the Christoffel symbol formula $2\Gamma_{jk}^l = g^{ls}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})$ we get $\Gamma_{jk}^j = g^{jl}\partial_k g_{jl}$, that is, by (K.1), $\Gamma_{jk}^j = \partial_k \log\det[g_{jl}]$. Also, $\mathrm{d}g$ has the component function $(\det[g_{jl}])^{1/2}$, and hence (K.1) gives $\langle g, \dot g\rangle = g^{jl}\dot g_{jl} = 2\varphi$. Finally, applying d/dt to $\delta w = \partial_j w^j + \Gamma_{jk}^j w^k = \partial_j w^j + w^k \partial_k \log\det[g_{jl}]$ and switching d/dt with ∂_k, we obtain (N.10.a).

Lemma N.6. *Suppose that* $\dot\Omega = 2i\partial\overline{\partial}\chi$ *for some time-dependent function* χ *and the Kähler form* Ω *of a time-dependent Kähler metric* g *on a given complex manifold* M *with a time-independent complex structure* $J : TM \to TM$, *where* $(\;)^{\cdot} = d/dt$. *Then, for* ρ, L *as above and* φ *given by* (N.10.c),

(i) $\varphi = \Delta\chi$, (ii) $\dot\rho = -i\partial\overline{\partial}\Delta\chi$, (iii) $\dot L = -2\lambda d\chi$,

(iv) f *with* (N.2) *may be chosen so that* $\dot f = \Delta\chi + 2\lambda\chi$,

(iii) *meaning that* $(Lw)^{\cdot} = -2\lambda d_w\chi$ *for all time-independent vector fields* w.

Proof. As $\dot g J = \dot\Omega = 2i\partial\overline{\partial}\chi$, we have $\dot g = -2(i\partial\overline{\partial}\chi)J$. Hence, by (H.3.ii), $\langle g, \dot g\rangle = 2\Delta\chi$, and (N.10.b) yields (i). By (i), Remark K.1(iii) and (N.10.c), $\dot\rho = -i\partial\overline{\partial}\varphi = -i\partial\overline{\partial}\Delta\chi$, and (ii) follows. Next, choosing $f : M \to \mathbf{R}$ so that $\Delta f + \mathrm{s} = \mathrm{s}_{\mathrm{avg}}$ for *some* t and $\dot f = \Delta\chi + 2\lambda\chi$ for all t, and then applying d/dt to $i\partial\overline{\partial}f + \rho - \lambda\Omega$, we see that, by (ii), $i\partial\overline{\partial}f + \rho = \lambda\Omega$ for all t, which proves (iv). Using (10.2) and (N.10.a) with $\varphi = \Delta\chi$ we now obtain (iii). □

We now proceed to prove Theorems N.1 and 10.5. Rescaling two given Kähler metrics with the stated property, we may assume that they have the same value of λ, which will also be the case for all intermediate metrics in a line segment of Kähler metrics joining them (Theorem J.3). We thus have a C^∞ curve $t \mapsto g = g(t)$ of Kähler metrics on the complex manifold M, with Kähler forms Ω such that $\dot\Omega = 2i\partial\overline{\partial}\chi$ for some function $\chi : M \to \mathbf{R}$. (We use the shorthand conventions of the last paragraph.) We will from now on ignore the fact that the curve is a line segment, although we do make use of its consequence in the form of differentiability of the assignment $t \mapsto \chi$ (which is in fact constant).

Proof of Theorem N.1. Applying d/dt to $-\mu^{-1}\boldsymbol{F}v = \int_M Lv\,\mathrm{d}g$ (cf. (N.1) and (2.15)), we obtain the integral of $(\varDelta\chi)Lv - 2\lambda d_v\chi$. Integration by parts shows that this equals the L^2 inner product of χ and the function $\varDelta Lv + 2\lambda\delta v$, which vanishes by Lemma N.3(b). □

Proof of Theorem 10.5. The relation $\dot{L} = -2\lambda d\chi$ gives $(Pw)^{\cdot} = 2i\lambda d_{Jw}\chi - 2\lambda d_w\chi$, and, since $(\mathrm{d}g)^{\cdot} = (\varDelta\chi)\,\mathrm{d}g$, we get $\mu^{-1}\dot{\mathcal{F}}(w) = \int_M(2i\lambda d_{Jw}\chi - 2\lambda d_w\chi + \varDelta\chi)e^{Pw}\,\mathrm{d}g$ from (10.5). Integrating by parts we see that this is equal to the integral of χ times

$$\varDelta e^{Pw} + 2\lambda(d_w e^{Pw} - i d_{Jw}e^{Pw}) + 2\lambda[\delta w - i\delta(Jw)]e^{Pw}. \tag{N.11}$$

To prove that (N.11) vanishes for every holomorphic vector field w, we use the identity $\varDelta e^{\psi} = e^{\psi}[\varDelta\psi + g(\nabla\psi,\nabla\psi)]$, immediate when the function ψ is real-valued, but also easily verified to complex-valued functions ψ, with g extended complex-bilinearly to complex vector fields (sections of the complexified tangent bundle). Thus, $g(\nabla\psi,\nabla\psi) = |\nabla\,\mathrm{Re}\,\psi|^2 - |\nabla\,\mathrm{Im}\,\psi|^2 + 2ig(\nabla\,\mathrm{Re}\,\psi,\nabla\,\mathrm{Im}\,\psi)$. For $\psi = Pw$, we have $\mathrm{Re}\,\psi = Lw$, $\mathrm{Im}\,\psi = -LJw$, and (N.11) equals e^{Pw} times

$$\begin{aligned}&\varDelta Lw + 2\lambda\delta w - i\,[\varDelta LJw + 2\lambda\delta(Jw)]\\ &+ |\nabla Lw|^2 + 2\lambda d_w Lw - [|\nabla LJw|^2 + 2\lambda d_{Jw}LJw]\\ &- 2i\,[g(\nabla Lw,\nabla LJw) + \lambda(d_{Jw}Lw + d_w LJw)].\end{aligned}$$

Each of the three lines is separately equal to zero, due to a part of Lemma N.3: the first, by (a); the second, by (c); and the third, in view of (d). □

Remark N.7. Theorem N.1 is also a direct consequence of Theorem 10.5 combined with (10.8).

Appendix P. Proofs of Theorems 10.2, 10.3 and 10.1(iv)

We begin by proving Theorem 10.2. To this end, let us fix a compact Kähler-Ricci soliton (M,g). By (6.1), $\nabla df + \mathrm{r} = \lambda g$ for some constant λ and some function f. If $\lambda \le 0$, our assertion follows: g is an Einstein metric with $\mathrm{r} = \lambda g$ (see Theorem 4.4); thus, formula (Q.5) and Lemma Q.2 in Appendix Q show that all holomorphic vector fields on M are parallel. Consequently, they are commuting Killing fields, and $\mathfrak{h}(M) = \mathfrak{g} = J\mathfrak{g} = \mathrm{Ker}\,(\mathrm{Ad}\,w)$. (If $\lambda < 0$, (Q.5) also shows that $\mathfrak{h}(M) = \{0\}$.) From now on we assume that $\lambda > 0$.

According to Lemma N.3(e) in Appendix N, the complex-linear operator P defined in (10.5), with L given by (10.2), sends the complex Lie algebra $\mathfrak{h}(M)$ into the space $\mathcal{Y}$ of all C^∞ functions $\psi : M \to \mathbf{C}$ with $\lambda\psi - d_w\psi + \varDelta\psi/2 = id_{Jw}\psi$, where $w = \nabla f/2$.

Furthermore, $P : \mathfrak{h}(M) \to \mathcal{Y}$ is a *complex-linear isomorphism*. In fact, its injectivity is immediate from (10.7). Lemma N.3(a) also states that the operator H sending any function $\psi : M \to \mathbf{C}$ to the vector field $-(2\lambda)^{-1}[\nabla\,\mathrm{Re}\psi + J\nabla\,\mathrm{Im}\psi]$, restricted to the image $P(\mathfrak{h}(M)) \subset \mathcal{Y}$, is the inverse of $P : \mathfrak{h}(M) \to P(\mathfrak{h}(M))$. It is also a trivial exercise to see that $PH\psi = \psi$ for every $\psi \in \mathcal{Y}$. Isomorphicity of $P : \mathfrak{h}(M) \to \mathcal{Y}$ thus will follow once we verify that $H\psi$ is holomorphic whenever $\psi \in \mathcal{Y}$, which is achieved via the following integration by parts:

in preparation

Next, for H defined as above (with a constant $\lambda \neq 0$), in *any* Kähler manifold (M, g), and for any holomorphic Killing field u on (M, g), one easily verifies that $[u, H\psi] = H d_u \psi$ for all functions $\psi : M \to \mathbf{C}$. Applied to the Killing field $u = Jw$ this shows that, under the isomorphic identification $\mathfrak{h}(M) = \mathcal{Y}$ provided by P (or H), our Z corresponds to the operator $-i d_{Jw} : \mathcal{Y} \to \mathcal{Y}$. (That $d_{Jw}(\mathcal{Y}) \subset \mathcal{Y}$ can be seen directly: as Jw is a Killing field, d_{Jw} commutes with Δ, and hence with the operator whose kernel is $\mathcal{Y}$.) We now easily obtain self-adjointness of $-i d_{Jw}$ relative to the ordinary L^2 inner product of functions in $\mathcal{Y}$.

further text in preparation

This completes the proof of Theorem 10.2.

Note that Theorem 10.2 establishes a *Cartan decomposition* of $\mathfrak{h}(M)$. Namely, $\mathfrak{h}(M)$ then is the direct sum of the eigenspaces of the operator $\mathrm{Ad}\, w : \mathfrak{h}(M) \to \mathfrak{h}(M)$, and, setting $\mathfrak{q}_c = \mathrm{Ker}\, [(\mathrm{Ad}\, w)Z - c] \subset \mathfrak{h}$ for $c \in \mathbf{R}$, we have $[\mathfrak{q}_c, \mathfrak{q}_d] \subset \mathfrak{q}_{c+d}$ whenever $c, d \in \mathbf{R}$.

We now proceed to prove Theorem 10.3. To this end, let us fix a compact Kähler-Ricci soliton (M, g). By (6.1), $\nabla df + \mathrm{r} = \lambda g$ for some constant λ and some function f. If $\lambda \leq 0$, our assertion follows: by Theorem 4.4, g is an Einstein metric with $\mathrm{r} = \lambda g$, and so formula (Q.5) in Appendix Q shows that all holomorphic vector fields on M are (parallel) Killing fields; hence $\mathrm{Isom}^{\mathrm{o}}(M, g) = \mathrm{Aut}^{\mathrm{o}}(M)$ (and, if $\lambda < 0$, both groups are trivial). From now on we will therefore assume that $\lambda > 0$.

further text in preparation

Proof of Theorem 10.1. First, let us replace g and $\hat{g}$ with λg and $\hat{\lambda}\hat{g}$, that is, rescale both metrics so as to have $\lambda = \hat{\lambda} = 1$ in (0.1). Next, let w and $\hat{w}$ be the unique vector fields with the property stated in Proposition 10.4, for g and $\hat{g}$. Now By Corollary 8.3, Jw and $J\hat{w}$ are Killing fields. Thus, their flows, consisting of isometries of (M, g) and, respectively, $(M, \hat{g})$, are both contained in some maximal compact connected Lie subgroups K and $\hat{K}$ of the identity component $\mathrm{Aut}^{\mathrm{o}}(M)$ of the biholomorphism group of M. By the Malcev-Iwasawa theorem (Appendix T), K and $\hat{K}$ are conjugate to each other in $\mathrm{Aut}^{\mathrm{o}}(M)$, and so, replacing $\hat{g}$ with its pullback under a suitable element of $\mathrm{Aut}^{\mathrm{o}}(M)$, we may assume that, in addition, both w and $\hat{w}$ lie in the same subspace $\mathfrak{p} \subset \mathfrak{h}(M)$ with the properties listed in Lemma 10.6. Proposition 10.4 and Lemma 10.6 now imply that $\hat{w} = w$.

further text in preparation

□

Appendix Q. Kähler-Einstein metrics

On an arbitrary Riemannian manifold (M, g), we denote by D the operator sending any vector field w on M to the vector field Dw characterized by

$$\text{(Q.1)} \qquad \imath_{Dw} g = -\Delta \imath_w g - \imath_w \mathrm{r}, \quad \text{that is,} \quad (Dw)_j = -w_{j,k}{}^k - R_{jk} w^k.$$

Replacing $R_{jk}w^k$ by $w^k{}_{,jk} - w^k{}_{,kj}$ (cf. (2.12.b)), we get $(Dw)_j = -(w_{j,k} + w_{k,j})^k + w^k{}_{,kj}$. Rewritten with the aid of (2.6.a), this equality gives

$$\imath_{Dw} g = -\delta \pounds_w g + d\delta w, \tag{Q.2}$$

while, applied to $w = \nabla\psi$ for a function $\psi : M \to \mathbf{R}$, it yields, again by (2.12.b), $(Dw)_j = -2\psi_{,kj}{}^k + \psi_{,k}{}^k{}_j = -2R_{jk}w^k - \psi_{,k}{}^k{}_j$, that is,

$$\imath_{Dw} g = -d\Delta\psi - 2\imath_w \mathrm{r} \quad \text{if} \quad w = \nabla\psi. \tag{Q.3}$$

Also, for any vector field w on a Riemannian manifold,

$$\Delta\delta w - -\delta D w - 2\delta\imath_w \mathrm{r}, \tag{Q.4}$$

since, in local coordinates, (2.12.b) gives $w_{j,}{}^{jk}{}_k = w_{j,}{}^{kj}{}_k + (R_{jk}w^j)^{,k}$, while formula (2.11.f) (or, more precisely, its coordinate form, cf. the lines following (2.12)) yields $w_{j,}{}^{kj}{}_k = w_{j,}{}^k{}_{kj} = \delta\Delta w$ (and so (Q.1) implies (Q.4)).

Note that D is a second-order elliptic differential operator; it is also self-adjoint, in view of symmetry of r and the relation $-g(\Delta w, v) = \langle \nabla w, \nabla v\rangle - \delta[(\nabla w)^* v]$ (which has the local-coordinate form $-v^j w_{j,k}{}^k = v^{j,k}w_{j,k} - (v^j w_{j,k})^{,k}$). Applied to $v = w$, this last relation shows that, on a compact Riemannian manifold (M, g),

$$(Dw, w) = \|\nabla w\|^2 - \int_M \mathrm{r}(w, w)\, \mathrm{d}g \tag{Q.5}$$

for any vector field w on M. Here and below $(\ ,\ )$ stands for the L^2 inner product of functions and vector or tensor fields, while $\|\ \|$ is the corresponding L^2 norm.

Similarly, any function ϕ and vector field w on a compact Riemannian manifold satisfy the L^2 inner-product relations

$$2(\nabla w, \nabla d\phi) = (Dw - \nabla\delta w, \nabla\phi). \tag{Q.6}$$

In fact, $2\phi^{,jk}w_{j,k} = \phi^{,jk}w_{j,k} + \phi^{,jk}w_{k,j}$ differs by a divergence from $-\phi^{,j}w_{j,k}{}^k + \phi^{,j}w_{k,j}{}^k$ which, in view of (Q.1) and (2.12.b), equals $\phi^{,j}(Dw)_j - \phi^{,j}w^k, kj$.

Remark Q.1. Our discussion of the operator D, defined by (Q.1) on a Riemannian manifold (M, g), deals mainly with the case where M is compact. In many cases, however, one has $Dw = 0$ for purely local reasons:

(i) $Dw = 0$ if w is a Killing field;
(ii) $Dw = 0$ if w satisfies the soliton equation (0.1);
(iii) $Dw = 0$ if (M, g) is a Kähler manifold and w is holomorphic;
(iv) $Dw = -\nabla(\Delta\psi + 2\lambda\psi)$ whenever (M, g) is an Einstein manifold with the Einstein constant λ and $w = \nabla\psi$ for a function $\psi : M \to \mathbf{R}$. Thus, we then have $Dw = 0$ if $w = \nabla\psi$ and $\Delta\psi = -2\lambda\psi$.

Namely, (i) follows from (Q.2), as the equality $(\nabla w)^* = -\nabla w$ gives $\delta w = 0$. That (0.1) yields $Dw = 0$ is clear from (3.5.ii) and (2.14.ii). Next, if g is a Kähler metric and w is holomorphic, $[J, \nabla w] = 0$ (see Remark L.1(b)), so that $J_p^k w^p{}_{,q}{}^q = J_q^p w^k{}_{,p}{}^q$, which, by (H.1.a), equals $-J_p^k R_l^p w^l$, proving (iii). Finally, (iv) is immediate from (Q.3).

About the relation between D and the Hodge Laplacian, see Remark Q.8 below.

Lemma Q.2. *On any compact Kähler manifold* (M, g), *the operator* D *with* (Q.1) *is nonnegative, and its kernel consists of all holomorphic vector fields. In addition, for every* C^2 *vector field* w *on* M, *the* L^2 *norm of* $\pounds_w J$ *is given by*

$$\|\pounds_w J\|^2 = 2(Dw, w)\,. \tag{Q.7}$$

In fact, for any vector field w on M, setting $A = \nabla w$ we have $\pounds_w J = [J, A]$ (see Remark L.1(b)), and so $|\pounds_w J|^2 = \operatorname{tr}[J,A][J,A]^* = 2\operatorname{tr} JAJA^* + 2\operatorname{tr} AA^*$. As $\operatorname{tr} AA^* = |\nabla w|^2$, we now obtain (Q.7) by integration, using (Q.5), (H.2.ii) and (2.15). Our assertion then follows from Remark Q.1(iii).

Remark Q.3. Inspired by Lemma Q.2, one might define the space of "holomorphic" vector fields on any compact Riemannian manifold (M, g) to be the kernel of D. However, as observed by Yano, cf. [80, p. 93], for any C^2 vector field w on a compact Riemannian manifold (M, g), we have

$$(\text{Q.8}) \qquad 2(Dw, w) \;=\; \|\pounds_w g\|^2 \;-\; 2\|\delta w\|^2,$$

since $|\pounds_w g|^2 = (w_{j,k} + w_{k,j})(w^{j,k} + w^{k,j}) = 2(w_{j,k} + w_{k,j})w^{j,k}$, which differs from $-2(w_{j,k}+w_{k,j})^{,k}w^j$ by a divergence, and so (Q.8) follows from (Q.2) by integration.

Thus, nonnegativity of D fails in general: examples with $(Dw, w) < 0$ are non-Killing conformal vector fields w in dimensions $n > 2$, for which $n\pounds_w g = 2(\delta w)g$, and so (Q.8) gives $n(Dw, w) = (2-n)\|\delta w\|^2 < 0$. Further such examples arise from Remark Q.1(iv): for instance, on a sphere S^n of constant curvature K, choosing an eigenfunction ψ of $-\Delta$ for the lowest positive eigenvalue nK, and noting that $\lambda = (n-1)K$, we get $Dw = (2-n)w$ for $w = \nabla\psi$.

On the other hand, D provides a characterization of Killing fields w on compact Riemannian manifolds by a pair of scalar equations: $Dw = 0$ and $\delta w = 0$. This is clear from (Q.8) and Remark Q.1(i).

For a function $\psi : M \to \mathbf{R}$ on a compact Riemannian manifold (M, g),

$$(\text{Q.9}) \qquad \mu\|w\|^2 \;=\; (Dw, w) \;+\; 2\int_M \mathrm{r}(w, w)\,\mathrm{d}g \quad \text{if} \;\; w = \nabla\psi \;\text{ and }\; \Delta\psi = -\mu\psi.$$

In fact, by (2.20.a), $\mu\|w\|^2 = -\mu(\psi, \Delta\psi) = \|\Delta\psi\|^2$. Bochner's formula (2.22), with $\varphi = \psi$, thus yields $\mu\|w\|^2 = \|\nabla w\|^2 + \int_M \mathrm{r}(w, w)\,\mathrm{d}g$, and (Q.5) gives (Q.9).

In the following theorem, the inequality $\mathrm{r} \geq \lambda g$ means that $\mathrm{r} - \lambda g$ is positive semidefinite at every point, r being, as usual, the Ricci tensor; in other words, λ is assumed to be a lower bound on the Ricci curvature.

Theorem Q.4. *Let (M, g) be a compact Kähler manifold such that*

$$(\text{Q.10}) \qquad \mathrm{r} \;\geq\; \lambda g \quad \text{with a constant } \; \lambda > 0\,.$$

Then $\mu \geq 2\lambda$ for every positive eigenvalue μ of $-\Delta$.

If, in addition, $\mathrm{r} = \lambda g$, that is, g is a Kähler-Einstein metric with the Einstein constant $\lambda > 0$, then the assignment $\psi \mapsto \nabla\psi$ defines a linear isomorphism between the space of all functions $\psi : M \to \mathbf{R}$ with $\Delta\psi = -2\lambda\psi$ and the space of all holomorphic gradient vector fields on M.

Proof. That $\mu \geq 2\lambda$ is obvious from (Q.9) and Lemma Q.2. Now let $\mathrm{r} = \lambda g$. If $\psi : M \to \mathbf{R}$ and $\Delta\psi = -2\lambda\psi$, (Q.9) with $\mu = 2\lambda$ gives $(Dw, w) = 0$ for $w = \nabla\psi$, and so, by (Q.7), w is a holomorphic gradient. Thus, the operator $\psi \mapsto \nabla\psi$ is valued in the required space, and it is also injective, as ψ can be constant only if $\psi = 0$. Finally, let w be any holomorphic gradient, so that $w = \nabla\psi$ for some $\psi : M \to \mathbf{R}$. Since $Dw = 0$ (see Remark Q.1(iii)), assertion (iv) in Remark Q.1 shows that $\Delta\psi + 2\lambda\psi$ is constant and, adding a constant to ψ, we may assume that $\Delta\psi = -2\lambda\psi$, as required. $\square$

A weaker form of Theorem Q.4 holds when (M, g), rather than being Kähler, is just assumed to be a compact Riemannian manifold of any real dimension

n. Condition (Q.10) then implies the *Lichnérowicz inequality* $\mu \geq (n-1)^{-1}n\lambda$ for every positive eigenvalue μ of $-\Delta$. (Proof: if $\Delta\psi = -\mu\psi$ and $\mu\|\psi\| > 0$, the Schwarz inequality $(\Delta\psi)^2 = \langle g, \nabla d\psi\rangle^2 \leq n|\nabla d\psi|^2$ implies, for $w = \nabla\psi$, that $(\delta w)^2 - \operatorname{tr}(\nabla w)^2 = (\Delta\psi)^2 - |\nabla d\psi|^2 \leq (n-1)(\Delta\psi)^2/n$, and so (2.22) gives $(n-1)^{-1}n\lambda\|w\|^2 \leq (n-1)^{-1}n\int_M \mathrm{r}(w,w)\,\mathrm{d}g \leq \|\Delta\psi\|^2$. Since $\mu\|\psi\|^2 = -(\psi,\Delta\psi) = \|w\|^2$ by (2.20.a), we now get $(n-1)^{-1}n\lambda\mu\|\psi\|^2 = (n-1)^{-1}n\lambda\|w\|^2 \leq \|\Delta\psi\|^2 = \mu^2\|\psi\|^2$, as required.)

The following is an obvious consequence of Theorem Q.4:

Corollary Q.5. *In a compact Kähler-Einstein manifold (M,g), with a positive Einstein constant λ,*

(i) $\mu \geq 2\lambda$ *for every positive eigenvalue* μ *of* $-\Delta$,

(ii) 2λ *is an eigenvalue of* $-\Delta$ *if and only if* M *admits a nontrivial holomorphic gradient vector field.*

The assertion of Corollary Q.5(ii) remains valid even if the word 'gradient' is dropped, as one easily sees using Theorem Q.6(d) below, due to Matsushima [92], along with (Q.11).

Theorem Q.6. *Given a compact Einstein manifold (M,g), let $\lambda, \mathfrak{h}, \mathfrak{g}$ and $\mathfrak{p}$ be the Einstein constant of g, the kernel of the operator D given by* (Q.1)*, the Lie algebra of all Killing fields on (M,g) and, respectively, the space of all gradient vector fields w on M with $Dw = 0$. Then we have an L^2-orthogonal decomposition*

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}. \tag{Q.11}$$

In particular, $\mathfrak{g} \subset \mathfrak{h}$. Furthermore,

(a) $\mathfrak{h} = \mathfrak{g} = \mathfrak{p} = \{0\}$ *if* $\lambda < 0$,

(b) $\mathfrak{p} = \{0\}$ *and* $\mathfrak{h} = \mathfrak{g}$ *is the space of all parallel vector fields, if* $\lambda = 0$.

(c) *In the case where $\lambda > 0$, the $\mathfrak{g}$ and $\mathfrak{p}$ components of any $w \in \mathfrak{h}$, relative to the decomposition* (Q.11)*, are $w + (2\lambda)^{-1}\nabla\delta w$ and $-(2\lambda)^{-1}\nabla\delta w$, while $\mathfrak{p}$ consists of the gradients of all functions $\psi : M \to \mathbf{R}$ with $\Delta\psi = -2\lambda\psi$.*

(d) *If, in addition, (M,g) is a Kähler manifold and $\lambda \neq 0$, then $\mathfrak{h}$ coincides with the space $\mathfrak{h}(M)$ of all holomorphic vector fields on M, and $\mathfrak{p} = J\mathfrak{g}$.*

Proof. That $\mathfrak{g} \subset \mathfrak{h}$ is obvious from from Remark Q.1(i), while L^2-orthogonality of the spaces $\mathfrak{g}$ and $\mathfrak{p}$ follows from formula (2.18), stating that Killing fields are L^2-orthogonal to gradients. Next, (a) and (b) are immediate from (Q.5), and (Q.11) is trivially satisfied when $\lambda \leq 0$. Let us therefore suppose that $\lambda > 0$. We claim that $u = 2\lambda w + \nabla\delta w$ is a Killing field whenever $w \in \mathfrak{h}$. In fact, $|\pounds_u g|^2 = 2u^{j,k}(u_{j,k} + u_{k,j})$ (cf. the line following (Q.8)), and so, since the same holds for w rather than u, we get $|\pounds_u g|^2/4 = 2\lambda^2 w^{j,k}(w_{j,k} + w_{k,j}) + 4\lambda w^{j,k}w^l{}_{,ljk} + w_p{}^{,pjk}w^l{}_{,ljk}$, that is, $|\pounds_u g|^2/4 = \lambda^2|\pounds_w g|^2 + 4\lambda\langle\nabla w, \nabla d\phi\rangle + |\nabla d\phi|^2$, and so $\|\pounds_u g\|^2/4 = \lambda^2\|\pounds_w g\|^2 + 4\lambda(\nabla w, \nabla d\phi) + \|\nabla d\phi\|^2$, where $\phi = \delta w$. Relation (Q.4) with $Dw = 0$ and $\mathrm{r} = \lambda g$ gives $\Delta\phi = -2\lambda\phi$. (From now on, ϕ stands for δw.) Thus, (2.22) with $\mathrm{r} = \lambda g$ and $\Delta\phi = -2\lambda\phi$ implies that $\|\nabla d\phi\|^2 = 4\lambda^2\|\phi\|^2 - \lambda\|\nabla\phi\|^2$, that is, $\|\nabla d\phi\|^2 = 2\lambda^2\|\phi\|^2$ (since $\|\nabla\phi\|^2 = 2\lambda\|\phi\|^2$ by (2.20.a)). Also, by (Q.8) with $Dw = 0$, we have $\|\pounds_w g\|^2 = 2\|\phi\|^2$. Next, (Q.6) with $Dw = 0$ and $\phi = \delta w$ reads $2(\nabla w, \nabla d\phi) = -\|\nabla\phi\|^2 = -2\lambda^2\|\phi\|^2$. Combining these equalities, we see that $\|\pounds_u g\|^2 = 0$, as required. Thus, (Q.11) holds also when $\lambda > 0$, and each $w \in \mathfrak{h}$ has the $\mathfrak{g}$ and $\mathfrak{p}$ components described in (c). Also, if $w = \nabla\psi \in \mathfrak{p}$, Remark Q.1(iv) with $Dw = 0$ shows that $\Delta\psi + 2\lambda\psi$ is constant, and hence may be assumed equal to 0. This proves (c).

Finally, under the assumptions of (d), $\mathfrak{h} = \mathfrak{h}(M)$ by Lemma Q.2, and $J\mathfrak{p} \subset \mathfrak{g}$ (that is, $\mathfrak{p} \subset J\mathfrak{g}$) in view of Remark L.1(d). Conversely, $J\mathfrak{g} \subset \mathfrak{p}$. In fact, for any $u \in \mathfrak{g}$, (Q.11) gives $Ju = w + v$ with $w \in \mathfrak{g}$ and $v \in \mathfrak{p}$, while Ju is, locally, a gradient (Remark L.1(d)). Thus, ∇w is both self-adjoint and skew-adjoint at every point, that is, $\nabla w = 0$, and (2.12.b) yields $w = 0$, as $\mathrm{r} = \lambda g$ and $\lambda \neq 0$. Hence $Ju = v \in \mathfrak{p}$, which completes the proof. □

Corollary Q.7. *For any compact Kähler-Einstein manifold* (M, g)*, the identity component* $\mathrm{Isom}^{\mathrm{o}}(M, g)$ *of the isometry group of* (M, g) *is a maximal compact connected Lie subgroup of the biholomorphism group* $\mathrm{Aut}(M)$*.*

Proof. Suppose, on the contrary, that there exists a vector field $w \in \mathfrak{h}$ such that $w \notin \mathfrak{g}$ and w belongs to the Lie algebra, containing $\mathfrak{g}$, of a compact Lie group G of biholomorphisms of M. Replacing w by its $\mathfrak{p}$ component relative to the decomposition (Q.11), we may assume that $w = \nabla\psi$ for some $\psi : M \to \mathbf{R}$. Thus, $d_w\psi = |w|^2$ is nonnegative everywhere and positive somewhere in M. Hence $\int_M d_w\psi\, \mathrm{d}g' > 0$ for any fixed G-invariant Riemannian metric g' on M, which contradicts (2.18), as w is a Killing field on (M, g'). □

Remark Q.8. If (M, g) is an Einstein manifold and λ is its Einstein constant, then $D = H - 2\lambda$, where $H = -d\delta - \delta d$ is the Hodge Laplacian acting on 1-forms ξ (identified with vector fields w, so that $\xi = \imath_w g$). Thus, $Dw = 0$ if and only if $Hw = 2\lambda w$. Note that the decomposition of w in Theorem Q.6 coincides with the Hodge decomposition of the eigenform $\xi = \imath_w g$ of the Hodge Laplacian.

Appendix R. The real and complex Monge-Ampère equations

in preparation

Appendix S. More on the Koiso-Cao construction

in preparation

Appendix T. Maximal compact subgroups of Lie groups

The following classical result, due to Malcev [91], was later generalized by Iwasawa [76]. The word 'maximal' means here *having the largest possible dimension.*

Theorem T.1. *Let* K *and* $\hat{K}$ *be maximal compact connected Lie subgroups of a Lie group* G*. Then* K *is conjugate to* $\hat{K}$ *in* G*.*

Proof.

in preparation

□

Appendix U. Ricci solitons and the Ricci flow

A C^∞ curve $t \mapsto g(t)$ of Riemannian metrics on a manifold M is called a solution to the *Ricci-flow equation* [61] if $dg/dt = -2\mathrm{r}$, that is, at every time t the derivative dg/dt equals -2 times the Ricci tensor $\mathrm{r} = \mathrm{r}_{g(t)}$ of the metric $g(t)$. Hamilton [61] proved that, on a compact manifold M, the Ricci-flow equation with any prescribed initial metric $g(0)$ has a unique solution on some maximal time interval $[0, T)$, where $0 < T \le \infty$.

A vast literature dealing with the Ricci flow begins with Hamilton's paper [61] and includes the preprints [101] – [103] in which Perelman proves the Poincaré conjecture by a Ricci-flow argument, implementing a program designed by Hamilton.

Given a compact Ricci soliton (M, g), with (0.1) for some fixed w and λ, the solution to the Ricci-flow equation with the initial condition $g(0) = g$, defined on the time interval $(-\infty, (2\lambda)^{-1})$, can be described quite explicitly (modulo solving ordinary differential equations). Namely, $g(t) = (1 - 2\lambda t)\varphi_\theta^* g$ with $\theta = -\lambda^{-1}\log(1 - 2\lambda t)$, where $\theta \mapsto \varphi_\theta$ is the flow of w. (That $dg/dt = -2\mathrm{r}_{g(t)}$ is clear since $d(\varphi_\theta^* g)/d\theta = \varphi_\theta^* \pounds_w g$ for a vector field w with the flow $\theta \mapsto \varphi_\theta$.)

Conclusions about compact Ricci solitons can in this way be derived from more general results on the Ricci flow; this is how Ivey [71] originally excluded the case $n = 3$ in Theorem 4.4. As an illustration, here is one step in Ivey's proof: a compact 3-dimensional Ricci soliton (M, g) with positive Ricci curvature must have constant sectional curvature, since, according to a theorem of Hamilton [61], the solution $t \mapsto g(t)$ to the Ricci-flow equation in a compact 3-manifold with an initial metric $g(0)$ of positive Ricci curvature, on the maximal time interval $[0, T)$, rescaled so as to have time-independent volume, converges uniformly (as $t \to T$) to a metric of positive constant sectional curvature. Cf. also [42, pp. 90–91].

Let $t \mapsto g(t)$ be a fixed solution to the Ricci-flow equation on a compact manifold. If the initial metric $g(0)$ is a Ricci soliton, then, according to the above formula for $g(t)$, the metrics $g(t)$ are all mutually homothetic (isometric up to a constant factor). Conversely, Perelman [101, pp. 6–9] has shown that, if $g(t)$ and $g(t')$ are homothetic for some t, t' with $t \ne t'$, then all $g(t)$ are mutually homothetic Ricci solitons. This is the precise meaning of the statement about 'fixed points' in the second paragraph of the introduction.

Appendix W. Another meaning of 'quasi-Einstein'

Goldberg and Vaisman [56] as well as other authors [44], [47] define a *quasi-Einstein metric* g on a manifold M by requiring its Ricci tensor to have the form

$$\text{(W.1)} \qquad \mathrm{r} \;=\; \alpha g \;+\; \beta\xi\otimes\xi \quad \text{for some functions } \alpha, \beta \text{ and a 1-form } \xi$$

(in coordinates: $R_{jk} = \alpha g_{jk} + \beta\xi_j\xi_k$). In other words, at each point, either r is a multiple of g, or it has exactly two eigenvalues, of multiplicities 1 and $\dim M - 1$.

For obvious algebraic reasons, (W.1) is satisfied in any Riemannian manifold of cohomogeneity 1 whose isometry group, restricted to each principal orbit, has an irreducible isotropy representation at every point. This is, for instance, the case for any metric g on S^n which, although not $\mathrm{SO}(n+1)$-invariant, is preserved by the subgroup $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$ keeping two antipodal points fixed. More generally, (W.1) is easily verified to hold for any warped-product Riemannian manifold with a one-dimensional base (totally geodesic factor) and an Einstein fibre.

There is a conformal relation between a special case of condition (W.1) and equation (6.1), characterizing gradient Ricci solitons. Specifically, let g be a Riemannian metric on a manifold M of dimension $n \geq 3$. Then g satisfies (6.1) if and only if $(n-2)\,\tilde{\mathrm{r}} = \gamma\tilde{g} + df \otimes df$ with $\gamma = \tilde{\Delta} f + (n-2)\lambda e^{2f/(n-2)}$ for the conformally related metric $\tilde{g} = e^{-2f/(n-2)}g$, where $\tilde{\mathrm{r}}$ and $\tilde{\Delta}$ correspond to $\tilde{g}$. (In fact, $(n-2)\,\tilde{\mathrm{r}} = (n-2)(\mathrm{r} + \nabla df) + df \otimes df + [\Delta f - g(\nabla f, \nabla f)]g$ and $\tilde{\Delta} f = [\Delta f - g(\nabla f, \nabla f)]e^{2f/(n-2)}$ whenever $\tilde{g} = e^{-2f/(n-2)}g$.) By (6.3), $\Delta f - g(\nabla f, \nabla f) = c - 2\lambda f$ with $c \in \mathbf{R}$, so that γ is a specific function of f.

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