NONEXISTENCE OF INVARIANT EINSTEIN METRICS ON SPECIAL LINEAR GROUPS

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ABSTRACT. We prove the fact named in the title, which is a special case of Alekseevsky’s conjecture.

1. INTRODUCTION

In [1] Alekseevsky conjectured that, whenever $G/K$ is a simply connected noncompact homogeneous nonflat Einstein manifold, $K$ must be a maximal compact connected subgroup of the (connected) Lie group $G$.

In the special case where $G = \text{SL}(n, \mathbb{R})$ and $K$ is the trivial group, Alekseevsky’s conjecture becomes a nonexistence statement, proved in this note:

Theorem A. There exists no left-invariant Riemannian Einstein metric on $\text{SL}(n, \mathbb{R})$ for any $n \geq 2$.

The assertion of Theorem A new for $n \geq 3$ and Einstein metrics with negative Einstein constants, stands in some contrast to the result of Leite and Dotti de Miatello [8] stating that $\text{SL}(n, \mathbb{R})$, for every $n \geq 3$, does admit a left-invariant metric of negative Ricci curvature.

We obtain Theorem A in §5 as an easy consequence of Corollary C (see below) combined with the well-known fact that the adjoint-action orbit of any definite symmetric bilinear form on $\text{sl}(n, \mathbb{R})$ is unbounded (Lemma 5.1). Corollary C is in turn obvious from the following result:

Theorem B. Given a unimodular Lie group $G$ with the Lie algebra $\mathfrak{g}$, let $\mathcal{M}$ be the set of all left-invariant Riemannian Einstein metrics with the Einstein constant $-1$ on $G$, and let $\mathcal{L}$ be the set of Levi-Civita connections of all such metrics. Then

(i) $\mathcal{L}$ is a bounded subset of the vector space $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$,
(ii) $\mathcal{M}$ is a bounded set in $[\mathfrak{g}^*]^{\otimes 2}$,
(iii) both $\mathcal{M}$ and $\mathcal{L}$ are unions of some, possibly empty, families of orbits for the actions of $\text{Aut} G$ on $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ and $[\mathfrak{g}^*]^{\otimes 2}$.
Corollary C. Let $G$ be a unimodular Lie group such that, for every negative-definite symmetric bilinear form $\rho$ on its Lie algebra $g$, the $\text{ad} G$ orbit of $\rho$ is unbounded in $[g^*]^{\otimes 2}$. Then $G$ admits no left-invariant Einstein metric with negative Einstein constant.

We prove Theorem B in §4 using an argument based on dealing primarily with left-invariant connections rather than metrics. Specifically, we exhibit the following explicit construction of such connections: If $\nabla$ is the Levi-Civita connection of a non-Ricci-flat left-invariant Riemannian Einstein metric on a unimodular Lie group $G$, then the formula $\nabla_v w = [v, w] - \nabla_v w$, for left-invariant vector fields $v, w$ on $G$, defines a left-invariant torsion-free connection on $G$, the Ricci tensor of which is symmetric and positive semidefinite. This implies a uniform bound on all such $\nabla$ with negative Ricci curvature: due to their fixed difference (and hence distance), $\nabla$ and $-\nabla$, as points in $g^* \otimes g^* \otimes g$, cannot approach infinity, since their Ricci tensors would then become asymptotically equal, rather than opposite in sign.

The italicized statement above, although useful in producing a contradiction needed to prove Theorems A and B, does not seem to be an interesting result in its own right.

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2. Preliminaries

Manifolds, mappings, connections and tensor fields of all types are always assumed to be of class $C^\infty$.

The Ricci tensor $\rho = \rho \nabla$ of a torsionfree connection $\nabla$ on a manifold satisfies the Bochner identity (cf. [5, formula (4.39) on p. 449]):

$$\rho (v, w) = \text{div}[\nabla w] - d (\text{div} w)$$

for any vector field $w$.

In fact, the coordinate form of (2.1), $R_{jk}^l w^k = w^k,_{jk} - w^k,_{kj}$, arises by contraction in $l = k$ from the Ricci identity $w^l,_{jk} - w^l,_{kj} = R_{jks}^l w^s$, which in turn is nothing else that the definition of the curvature tensor $R$. Evaluated on a vector field $v$, (2.1) implies that $\rho (v, w)$ equals

$$\text{div}[\nabla_v w - (\text{div} w)v] + (\text{div} v)(\text{div} w) - \text{tr} [\nabla v][\nabla w],$$

as one sees differentiating by parts. Here $\text{div} w = \text{tr} \nabla w$, with $\nabla w$ treated as the endomorphism of the tangent bundle acting on vector fields $v$ by $v \mapsto \nabla_v w$. 
We always identify the Lie algebra $\mathfrak{g}$ of a given Lie group $G$ with the space of left-invariant vector fields on $G$. Whenever a connection $\nabla$ (or, a twice-covariant symmetric tensor field $\sigma$) on $G$ is invariant under left translations, we treat it as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ (or, $[\mathfrak{g}^*]^{\otimes 2}$), that is, a bilinear mapping $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (or, respectively, $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, which is in addition symmetric).

One calls a torsionfree connection $\nabla$ on an $m$-dimensional manifold equiaffine if the connection induced by $\nabla$ in the $m$th exterior power of the tangent bundle is flat, or, on other words, if the manifold admits, locally, $\nabla$-parallel volume forms. Equiaffinity of $\nabla$ is equivalent to symmetry of its Ricci tensor $\rho$. This well-known fact, cf. [4, p. 567], is stated here just as a comment, and will not be used in our argument.

If $\nabla$ is a left-invariant torsionfree connection on a Lie group $G$, we will say that $\nabla$ is unimodular if some/any left-invariant volume form on $G$ is $\nabla$-parallel.

Unimodularity of $\nabla$ obviously implies its equiaffinity.

**Lemma 2.1.** Let $\nabla$ be a left-invariant torsionfree connection on a Lie group $G$ with the Lie algebra $\mathfrak{g}$.

(a) $\nabla$ is unimodular if and only if $\nabla_v : \mathfrak{g} \in \text{End} \mathfrak{g}$ is traceless for every $v \in \mathfrak{g}$.

(b) $\nabla$ is unimodular whenever there exists a nondegenerate left-invariant $\nabla$-parallel twice-covariant symmetric tensor field on $G$, that is, whenever $\nabla$ is the Levi-Civita connection of some left-invariant pseudo-Riemannian metric.

(c) If $G$ is unimodular, then unimodularity of $\nabla$ is equivalent to the condition $\text{div} w = 0$ for every $w \in \mathfrak{g}$.

(d) If both $G$ and $\nabla$ are unimodular, then the Ricci tensor $\rho = \rho^\nabla$ is given by $\rho(v, w) = -\text{tr} [\nabla v][\nabla w]$ for $v, w \in \mathfrak{g}$.

**Proof.**

3. **RICCI-PARALLEL MULTIPLICATIONS**

Let $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a multiplication (bilinear operation) in a real vector space $\mathfrak{g}$ with $\dim \mathfrak{g} < \infty$. For $v, w \in \mathfrak{g}$, we will write $\nabla_v w$ instead of $\nabla(v, w)$, and

$$\nabla_v (\text{or, } \nabla w) : \mathfrak{g} \to \text{End} \mathfrak{g}$$

denote by $\nabla_v (\text{or, } \nabla w)$ the operator sending $w$ (or, respectively, $v$) to $\nabla_v w$. 


We also define the Ricci pairing \( \rho = \rho^\nabla \) of \( \nabla \), its skew-symmetrization \( A \), and its symmetrization \( S \), to be the symmetric bilinear form \( \rho : g \times g \to \mathbb{R} \) and the multiplications \( S, A \) in \( g \), given, for any \( v, w \in g \), by

\[
(3.2) \quad \rho(v, w) = -\text{tr}[\nabla v][\nabla w]
\]

(so that \( -\rho^\nabla \) is the pullback, under the operator \( w \mapsto \nabla w \), of a natural pseudo-Euclidean inner product in \( \text{End} \, g \), and

\[
(3.3) \quad 2A_vw = \nabla_v w - \nabla_w v, \quad 2S_vw = \nabla_v w + \nabla_w v
\]

We call a multiplication \( \nabla \) in \( g \) Ricci-parallel if

\[
(3.4) \quad \nabla_v \text{ is skew-adjoint relative to } \rho^\nabla \text{ for every } v \in g
\]
or, equivalently, if

\[
(3.5) \quad \rho(\nabla_v w, w) = 0 \text{ for } \rho = \rho^\nabla \text{ and all } v, w \in g.
\]

**Example 3.1.**

**Example 3.2.** In any real Lie algebra \( g \), the formula \( \nabla_v w = [v, w]/2 \) defines a skew-symmetric Ricci-parallel multiplication \( \nabla \). In fact, (3.5) amounts here to \( \text{Ad} \)-invariance of the Killing form of \( g \).

**Example 3.3.**

### 4. Negative Ricci Curvature

We write \( \rho > 0 \), \( \rho < 0 \) or \( \rho \geq 0 \) to express positive/negative definiteness or semidefiniteness of a symmetric bilinear form \( \rho \).

**Lemma 4.1.** Let \( S, A \in g^* \otimes g^* \otimes g \) be multiplications in a finite-dimensional real vector space \( g \) such that \( A \) is skew-symmetric, \( S \) is symmetric, while, setting \( \nabla = A + S \) and \( \rho = \rho^\nabla \), we have \( \{\nabla, \nabla\} \circ \nabla = 0 \) and either \( \rho > 0 \), or \( \rho < 0 \). Then \( \rho^{A-S} \geq 0 \).

In fact, as \( \nabla_w = (A + S)_w = Sw - Aw \) is, for every \( w \in g \), skew-adjoint relative to the Euclidean inner product \( \pm \rho^\nabla \), cf. (3.4), our claim is immediate from the definition of \( \rho^{A-S} \) (see (3.2)).

**Lemma 4.2.** If \( A \in g^* \otimes g^* \otimes g \) is a fixed multiplication in a real vector space \( g \) with \( \dim g < \infty \), and \( w \in g \) is a fixed vector, then the set

\[
(4.1) \quad \{S \in g^* \otimes g^* \otimes g : \rho^{A+S}(w, w) < 0 \leq \rho^{A-S}(w, w)\}
\]

is bounded in \( g^* \otimes g^* \otimes g \).
Proof. Suppose that, on the contrary, (4.1) contains a sequence of elements $S$ with $|S| \to \infty$ for some norm $|\cdot|$ in $g^* \otimes g^* \otimes g$. The ratio $[\rho^{A-S}(w, w)]/[\rho^{A+S}(w, w)]$ is nonpositive for such $S$, due to the definition of (4.1), yet at the same time, restricted to a subsequence, it tends to 1 as $|S| \to \infty$. (The limit is 1 since the ratio does not change when $A$ and $S$ are both divided by $|S|$, the dependence of $\rho_\nabla$ on $\nabla$ in (3.2) being homogeneous quadratic.) The resulting contradiction completes the proof. □

Proof of Theorem B. As the connections $\nabla \in N$ are torsionfree, decomposing each of them into the sum $A + S$ of its skew-symmetric and symmetric parts we see that we see that $A = [\cdot, \cdot]/2$ is fixed. Thus, $N$ is bounded, since, by Lemma 4.1, it is contained in the set (4.1) for every $w \in g \setminus \{0\}$. Assertion (ii) is in turn immediate from (i), since $\rho_\nabla$ is a quadratic polynomial function of $\nabla$. □

5. Proof of Theorem A

We begin with a well-known lemma.

Lemma 5.1. Any definite symmetric bilinear form $\rho$ on $g = sl(n, \mathbb{R})$ has an unbounded $ad G$ orbit in $[g^*]^{\otimes 2}$, for $G = SL(n, \mathbb{R})$ and $n \geq 2$.

Proof. Let us fix a vector $x \in \mathbb{R}^n$ and a linear functional $\theta \in [\mathbb{R}^n]^*$, both nonzero, such that $\theta(x) = 0$, and a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ with $x = e_2$ and $\theta = e_1$, where $e_1, \ldots, e_n$ is the basis of $[\mathbb{R}^n]^*$ dual to $e_1, \ldots, e_n$. For any scalar $c > 0$, defining $F \in SL(n, \mathbb{R})$ by $Fe_1 = c^{-1}e_1$, $Fe_2 = ce_2$, and $Fe_j = e_j$ if $j > 2$, we have $Fx = cx$ and $F^*\theta = c^{-1}\theta$. For the tensor product $w = \theta \otimes x \in sl(n, \mathbb{R})$, this gives $(ad F)w = FwF^{-1} = [(F^{-1})^*\theta] \otimes Fx = c^2\theta \otimes x = c^2w$ and hence $[(ad F)^*\rho](w, w) = c^4\rho(w, w)$. Consequently, the $ad G$ orbit of $\rho$ is unbounded, as it has an unbounded image under the linear functional that sends $\sigma \in [g^*]^{\otimes 2}$ to $\sigma(w, w)$. □

To prove Theorem A, suppose that, on the contrary, $SL(n, \mathbb{R})$ admits a left-invariant Einstein metric with Einstein constant $\kappa$.

First, $\kappa$ cannot be positive, as Myers’s theorem [5, pp. 606–607] would then imply compactness of $SL(n, \mathbb{R})$.

The case $\kappa = 0$ is in turn excluded by the result of Alekseevsky and Kimelfeld [2], who showed that a homogeneous Ricci-flat Riemannian manifold is necessarily flat. (According to Remark......, no left-invariant flat metric exists on $SL(n, \mathbb{R})$.)

Finally, if $\kappa < 0$, Corollary C contradicts Lemma 5.1.
REFERENCES


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