Killing fields on compact pseudo-Kähler manifolds

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Abstract. We show that a Killing field on a compact pseudo-Kähler ddbar manifold is necessarily (real) holomorphic. Our argument works without the ddbar assumption in real dimension four. The claim about holomorphicity of Killing fields on compact pseudo-Kähler manifolds appears in a 2012 paper by Yamada, and in an appendix we provide a detailed explanation of why we believe that Yamada's argument is incomplete.

Introduction

By a pseudo-Kähler manifold we mean a pseudo-Riemannian manifold (M,g) endowed with a ∇-parallel almost-complex structure J, for the Levi-Civita connection ∇ of g, such that the operator $J_x : T_xM \to T_xM$ is a linear g_x-isometry (or is, equivalently, g_x-skew-adjoint) at every point $x \in M$. We then call (M,g) a *pseudo-Kähler* $\partial\bar\partial$ *manifold* if the underlying complex manifold M has the $\partial\bar\partial$ property, or "satisfies the $\partial\bar\partial$ lemma" (Section 2). The $\partial\bar\partial$ property follows if M is compact and admits a Riemannian Kähler metric.

Theorem A. *Every Killing vector field on a compact pseudo-Kähler* $\partial\bar\partial$ *manifold is real holomorphic.*

We provide two proofs of Theorem A, in Sections 2 and 3. The former is derived directly from the $\partial\bar\partial$ condition; the latter, shorter, relies on the Hodge decomposition, which is equivalent to the $\partial\bar\partial$ property [**2**, p. 296, subsect. (5.21)].

The Riemannian-Kähler case of Theorem A is well known, and straightforward [**1**, the lines following Remark 4.83 on pp. 60–61]. See also Remark 1.2.

For pseudo-Kähler surfaces, our argument yields a stronger conclusion.

Theorem B. *In real dimension four the assertion of Theorem* A *holds without the* $\partial\bar\partial$ *hypothesis.*

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1. Proof of Theorem B

All manifolds, mappings, tensor fields and connections are assumed smooth.

LEMMA 1.1. *Given a connection* ∇ *on a manifold* M, *let a vector field* v *on* M *be affine in the sense that its local flow preserves* ∇. *Then, for any* ∇*-parallel tensor field* Θ *on* M, *of any type, the Lie derivative* $\pounds_v\Theta$ *is* ∇*-parallel as well. If* Θ *happens to be a closed differential form,* $\pounds_v\Theta = d[\Theta(v,\,\cdot\,,\dots,\,\cdot\,)]$.

PROOF. Clearly, $-\pounds_v\Theta$ is the derivative with respect to the real variable t, at $t=0$, of the push-forwards $[d\phi_t]\Theta$ under the local flow $t\mapsto\phi_t$ of v. All $[d\phi_t]\Theta$ being ∇-parallel, so is $\pounds_v\Theta$. For the final clause, use Cartan's homotopy formula $\pounds_v = \imath_v d + d\imath_v$ for $\pounds_v$ acting on differential forms [**4**, Thm. 14.35, p. 372]. □

Lemma 1.1 also follows from the Leibniz rule: $\pounds_v(\nabla\Theta) = (\pounds_v\nabla)\Theta + \nabla(\pounds_v\Theta)$.

Let (M,g) now be a fixed pseudo-Kähler manifold. If v is any vector field on M then, with J and ∇v treated as bundle morphisms $TM\to TM$,

$$\text{(1.1)}\qquad \text{for } B=\nabla v \text{ and } A=\pounds_v J \text{ one has } A=[J,B] \text{ and } JA=-AJ,$$

which is immediate from the Leibniz rule. For the Kähler form $\omega = g(J\cdot\,,\,\cdot\,)$ of (M,g) and any g-Killing vector field v, it follows from (1.1) and Lemma 1.1 that

$$\text{(1.2)}\qquad \begin{array}{rl}\text{i)} & A=\pounds_v J \text{ and } \alpha=\pounds_v\omega \text{ are related by } \alpha=g(A\cdot\,,\,\cdot\,), \text{ while}\\ \text{ii)} & A^*=-A,\quad JA=-AJ,\quad \nabla A=0,\quad \nabla\alpha=0,\quad \text{and } \alpha \text{ is exact.}\end{array}$$

Given an exact p-form α on a compact pseudo-Riemannian manifold (M,g),

$$\text{(1.3)}\quad \alpha \text{ is } L^2\text{-orthogonal to all parallel } p \text{ times covariant tensor fields } \theta \text{ on } M.$$

Namely, $(\theta,\alpha)=(\mu,\alpha)=(\mu,d\beta)=(d^*\mu,\beta)$ for β with $\alpha=d\beta$, the skew-symmetric part μ of θ, and the L^2 inner product $(\,,\,)$, while $d^*\mu=0$, as $\nabla\mu=0$.

REMARK 1.2. By (1.2-ii) and (1.3), for a Killing field v on a compact *Riemannian* Kähler manifold, $\pounds_v\omega$ is L^2-orthogonal to itself, and so, as a consequence of (1.2-i), v must be real holomorphic.

Let (M,g) be, again, a pseudo-Kähler manifold. The vector-bundle morphisms $C: TM\to TM$ having $C^*=-C$ (that is, g_x-skew-adjoint at every point $x\in M$) constitute the sections of

$$\text{(1.4)}\qquad \text{the vector subbundle } \mathfrak{so}(TM) \text{ of } \mathrm{End}_{\mathbb{R}}(TM)=\mathrm{Hom}_{\mathbb{R}}(TM,TM).$$

We denote by $\mathcal{E}$ the vector subbundle of $\mathfrak{so}(TM)$, the sections C of which are also complex-antilinear (so that $JC = -CJ$, in addition to $C^* = -C$). Then

(1.5) $\mathcal{E}$ is a complex vector bundle of rank $m(m-1)/2$, where $m = \dim_{\mathbb{C}} M$, with a pseudo-Hermitian fibre metric having the real part induced by g.

In fact, $C \mapsto JC$ provides the complex structure for $\mathcal{E}$. Nondegeneracy of g restricted to $\mathcal{E}$ follows from g-orthogonality of the decomposition $\mathrm{End}_{\mathbb{R}}(TM) = \mathrm{End}_{\mathbb{C}}(TM) \oplus \mathcal{E} \oplus \mathcal{D}$, the sections C of the subbundle $\mathcal{D}$ being characterized by $JC = -CJ$ and $C^* = C$, with $\mathrm{End}_{\mathbb{C}}(TM)$ orthogonal to $\mathcal{E} \oplus \mathcal{D}$ since any antilinear morphism $C : TM \to TM$ is conjugate, via J, to $-C$, and so $\mathrm{tr}_{\mathbb{R}} C = 0$. The pseudo-Hermitian fibre metric in $\mathcal{E}$ arises by restricting $\langle \cdot, \cdot \rangle - i \langle J \cdot, \cdot \rangle$ to $\mathcal{E}$, for the pseudo-Riemannian fibre metric $\langle \cdot, \cdot \rangle$ in $\mathrm{End}_{\mathbb{R}}(TM)$ induced by g. The rank $m(m-1)/2$ follows since $\mathfrak{so}(TM) = \mathfrak{u}(TM) \oplus \mathcal{E}$, with $\mathfrak{u}(TM) \subseteq \mathfrak{so}(TM)$ characterized by having sections $C : TM \to TM$ that commute with J (which, due to their g-skew-adjointness, makes them also g^{c}-skew-adjoint, for $g^{\mathrm{c}} = g - i\omega$): $\mathfrak{so}(TM)$ and $\mathfrak{u}(TM)$ have the real ranks $m(2m-1)$ and m^2.

Proof of Theorem B. By (1.5), with $m = 2$, the pseudo-Hermitian fibre metric in the *line* bundle $\mathcal{E}$ must be positive or negative definite. Hence so is its g-induced real part. For any Killing field v, (1.2-ii) implies that $A = \pounds_v J$ is a section of $\mathcal{E}$ which, due to (1.2) – (1.3), is L^2-orthogonal to itself, and so $\pounds_v J = 0$. ☐

The above proof does not extend to compact pseudo-Kähler manifolds (M, g) of complex dimensions $m > 2$ with indefinite metrics. Namely, if the pair (j, k) represents the metric signature of g, with j minuses and k pluses (both j, k even, $j + k = 2m$), then the analogous signature of the real part (induced by g) of the pseudo-Hermitian fibre metric in $\mathcal{E}$ is $(jk/2,\, [j^2 + k^2 - 2(j+k)]/4)$, with both components (indices) positive unless $jk = 0$ or $j = k = 2$.

One easily verifies this last claim, about the signature, by using a J_x-invariant timelike-spacelike orthogonal decomposition of $T_x M$, at any $x \in M$, to obtain obvious three-summand orthogonal decompositions of both $\mathfrak{so}(TM)$ and $\mathfrak{u}(TM)$ at x, two summands being spacelike, and one timelike.

2. Proof of Theorem A

We denote by $\Omega^{p,q} M$ the space of complex-valued differential (p, q) forms on a complex manifold M. On such M, as $\bar{\partial} \zeta = 0$ whenever $d\zeta = 0$,

(2.1) closedness of a $(p, 0)$ form ζ implies its holomorphicity.

Conversely, according to [**2**, p. 296, subsect. (5.21)] and [**6**, p. 101, Corollary 9.5], on a compact complex $\partial\bar{\partial}$ manifold,

(2.2) all holomorphic differential forms are closed.

Every compact complex manifold admitting a Riemannian Kähler metric has the following $\partial\bar{\partial}$ *property*, also referred to as *the* $\partial\bar{\partial}$ *lemma* [**5**, Prop. 6.17 on p. 144]: any closed ∂-exact or $\bar{\partial}$-exact (p,q) form equals $\partial\bar{\partial}\lambda$ for some $(p-1,q-1)$ form λ. Since many expositions do not state what happens when p or q is 0, we note that – as Fangyang Zheng pointed out to us – the $\partial\bar{\partial}$ property for $(p,0)$ forms easily follows from the case where p and q are positive.

Lemma 2.1. *On a compact complex manifold M with the "positive p,q version" of the $\partial\bar{\partial}$ property, if $\xi\in\Omega^{p,0}M$, for $p\geq 1$, and $\partial\xi$ is closed, then $\partial\xi=0$.*

Proof. As $0=d\partial\xi=\bar{\partial}\partial\xi=-\partial\bar{\partial}\xi$, the "positive" $\partial\bar{\partial}$ lemma applied to the closed $\bar{\partial}$-exact $(p,1)$ form $\bar{\partial}\xi$ gives $\bar{\partial}\xi=\bar{\partial}\partial\eta$ for some $\eta\in\Omega^{p-1,0}M$. Being thus holomorphic, $\xi-\partial\eta\in\Omega^{p,0}M$ is closed by (2.2), and $0=\partial(\xi-\partial\eta)=\partial\xi$. □

Lemma 2.1 implies, via complex conjugation, its analog for $(0,q)$ forms. Also by Lemma 2.1, on a compact complex manifold M with the $\partial\bar{\partial}$ property,

(2.3) the only exact $(p,0)$ form ζ on M is $\zeta=0$,

since exactness of $\zeta\in\Omega^{p,0}M$ amounts to its ∂-exactness and implies its closedness.

For a pseudo-Kähler manifold (M,g), a bundle morphism $A:TM\to TM$, and the corresponding twice-covariant tensor field $\alpha=g(A\cdot\,,\,\cdot)$, one clearly has

$$\alpha(J\cdot\,,J\cdot)=\pm\alpha \text{ if and only if } JA=\pm AJ, \quad\text{with either sign } \pm\,. \tag{2.4}$$

Given a pseudo-Kähler manifold (M,g), vector fields u,v on M and sections A,C of $\mathfrak{so}(TM)$, cf. (1.4), may be used to represent a complex-valued 1-form ξ and 2-form ζ on M, as follows,

$$\xi=u+iv,\quad \zeta=A+iC, \tag{2.5}$$

meaning that $\xi=g(u,\,\cdot)+ig(v,\,\cdot)$ and $\zeta=g(B\cdot\,,\,\cdot)+ig(C\cdot\,,\,\cdot)$. We prefer not to think of (2.5) as sections of the complexifications of TM or $\mathfrak{so}(TM)$. For a vector field v treated via (2.5) as a real 1-form, and $B=\nabla v$, our factor convention for the exterior derivative gives

$$dv\,=\,B\,-\,B^*,\quad\text{ and so } d(Jv)\,=\,\nabla(Jv)\,-\,[\nabla(Jv)]^*\,=\,JB+B^*J. \tag{2.6}$$

Remark 2.2. On a complex manifold, a real-valued 2-form α is the real part of a complex-bilinear complex-valued 2-form ζ if and only if $\alpha(J\cdot\,,J\cdot)=-\alpha$, and then necessarily $\zeta=\alpha-i\alpha(J\cdot\,,\,\cdot)$. (This clearly remains valid for arbitrary twice-covariant tensor fields, without skew-symmetry.)

Remark 2.3. For a complex-valued 2-form ζ on a complex manifold M, having bidegree $(2,0)$, or $(0,2)$, or $(1,1)$ clearly amounts to its being complex-bilinear, or bi-antilinear or, respectively, J-invariant: $\zeta(J\cdot\,,J\cdot)=\zeta$. Sums ζ of $(2,0)$ and $(0,2)$ forms are similarly characterized by J-anti-invariance: $\zeta(J\cdot\,,J\cdot)=-\zeta$.

Thus, by (2.4), in the pseudo-Kähler case, $\zeta = A + iC$ in (2.5) is a $(1,1)$ form if and only if A and C commute with J.

LEMMA 2.4. *For a Killing vector field v on a pseudo-Kähler manifold (M,g), using the notation of* (2.5)*, we have*

$$\begin{aligned} &\xi \in \Omega^{1,0}M, \quad \zeta \in \Omega^{2,0}M, \quad \partial\xi = \zeta, \quad \bar{\partial}\xi = i(JBJ - B), \quad \text{where} \\ &\xi = Jv - iv, \quad \zeta = A - iAJ, \quad \text{with} \quad A = [J,B] \quad \text{for} \quad B = \nabla v. \end{aligned} \tag{2.7}$$

PROOF. First, $JBJ - B$, as well as $A = [J,B]$ and AJ, are g_x-skew-adjoint at every point $x \in M$, since so is $B = \nabla v$, and A anticommutes with J, cf. (1.1). Thus, ξ, ζ and $\gamma = i(JBJ - B)$ are indeed differential forms of degrees 1, 2, 2.

Furthermore, ξ is complex-linear, and ζ complex-bilinear. This is immediate for ξ. For ζ, note that $\zeta = \alpha - i\alpha(J\,\cdot\,,\,\cdot\,)$, where $\alpha = g(A\,\cdot\,,\,\cdot\,)$, while (1.1) and (2.4) give $\alpha(J\,\cdot\,,J\,\cdot\,) = -\alpha$. Now we can use Remark 2.2.

Thus, $\xi \in \Omega^{1,0}M$. Also, according to Remark 2.3, $\zeta \in \Omega^{2,0}M$ and $\gamma \in \Omega^{1,1}M$, since $JBJ - B$ obviously commutes with J. Finally, for $A = [J,B]$, (2.6) with $B^* = -B$ gives $d\xi = A - 2iB = [A - i(JBJ + B)] + i(JBJ - B)$, while the summands $A - i(JBJ + B) = A - iAJ = \zeta$ and $i(JBJ - B) = \gamma$ lie in $\Omega^{2,0}M$ and $\Omega^{1,1}M$, which completes the proof. □

PROOF OF THEOREM A. By (1.2) and (2.4), the ∂-exact $(2,0)$ form $\zeta = \partial\xi$ in (2.7) is parallel, and hence closed. Lemma 2.1 now gives $\zeta = 0$, so that $\pounds_v J = A = 0$ due to (1.1) and (2.7). □

3. Another proof of Theorem A

On a compact complex manifold M with the $\partial\bar{\partial}$ property, every cohomology space $H^k(M,\mathbb{C})$ has the Hodge decomposition [**2**, p. 296, subsect. (5.21)]:

$$H^k(M,\mathbb{C}) = H^{k,0}M \oplus H^{k-1,1}M \oplus \ldots \oplus H^{1,k-1}M \oplus H^{0,k}M, \tag{3.1}$$

with each $H^{p,q}M$ consisting of cohomology classes of closed (p,q) forms. The complex conjugation of differential forms descends to a real-linear involution of $H^k(M,\mathbb{C})$, the fixed points of which obviously are the real cohomology classes (those containing real closed differential forms). In terms of the decomposition (3.1), a complex cohomology class

(3.2) is real if and only if, for all p and q, its $H^{q,p}$ component equals the conjugate of its $H^{p,q}$ component.

The standard formula $N(u,v) = [u,v] + J[Ju,v] + J[u,Jv] - [Ju,Jv]$, for the Nijenhuis tensor N of an almost-complex structure J on a manifold M and any vector fields u, v, clearly becomes

$$N(u,v) = [\nabla_{Jv}J]u - [\nabla_{Ju}J]v + J[\nabla_u J]v - J[\nabla_v J]u \tag{3.3}$$

when one uses any fixed torsionfree connection ∇ on M. We call ∇ a *Kähler connection* for the given almost-complex structure J if it is torsionfree and $\nabla J = 0$. By (3.3), J then must be integrable.

LEMMA 3.1. *For any ∇-parallel real 2-form α on a complex manifold M with a Kähler connection ∇, such that $\alpha(J\cdot\,,J\cdot) = -\alpha$, the complex-valued 2-form $\zeta = \alpha - i\alpha(J\cdot\,,\,\cdot)$ is holomorphic. If, in addition, M is also compact and has the $\partial\bar\partial$ property, while α is exact, then $\alpha = 0$.*

PROOF. The relation $\alpha(J\cdot\,,J\cdot) = -\alpha$ amounts to complex-bilinearity of ζ, and so $\zeta \in \varOmega^{2,0}M$ (Remarks 2.2 – 2.3). Being ∇-parallel, ζ is closed, and hence holomorphic due to (2.1). The final clause: exactness of α makes $[i\zeta] \in \varOmega^{2,0}M$ a real cohomology class, so that, by (3.2), ζ is exact, and (2.3) gives $\zeta = 0$. □

ANOTHER PROOF OF THEOREM A. Given a Killing field v, the differential 2-form $\alpha = \pounds_v\omega$ is parallel and exact by (1.2), while (1.2) gives $JA = -AJ$ for $A = \pounds_v J$, related to α via $\alpha = g(A\cdot\,,\,\cdot)$, and so $\alpha(J\cdot\,,J\cdot) = -\alpha$ due to (2.4). Lemma 3.1 and (1.2-i) now yield $\pounds_v\omega = \alpha = 0$ and $\pounds_v J = 0$. □

We do not know whether – aside from Theorem B and the Riemannian case – Theorem A remains valid without the $\partial\bar\partial$ hypothesis. For possible future reference, let us note that, as shown above, one has the following conclusions about a Killing field v on a compact pseudo-Kähler manifold, whether or not the $\partial\bar\partial$ property is assumed. First, for $\alpha = \pounds_v\omega$, the complex-valued 2-form $\zeta = \alpha - i\alpha(J\cdot\,,\,\cdot)$ is parallel and holomorphic (see the preceding proof and Lemma 3.1). Also, by (1.2), α is exact, while $A = \pounds_v J : TM \to TM$ is parallel and complex-antilinear, as well as nilpotent at every point. This last conclusion follows since the constant function $\mathrm{tr}_{\mathrm{I\!R}} A^k$, with any integer $k \ge 1$, has zero integral as a consequence of (1.3) applied to $\alpha = g(A\cdot\,,\,\cdot)$ and $\theta = g(A^{k-1}\cdot\,,\,\cdot)$.

Appendix: Yamada's argument

Yamada's claim [**7**, Proposition 3.1] that on a compact pseudo-Kähler manifold, Killing fields are real holomorphic, has a proof which reads, *verbatim*,

(A.1) Let X be a Killing vector field. From Propositions 1.2 and 2.12, $Z = X - \sqrt{-1}\,JX$ is holomorphic. Because the real part of a holomorphic vector field is an infinitesimal automorphism of the complex structure, we have our proposition.

Proposition 1.2 of [**7**], cited from Kobayashi's book [**3**], amounts to the well-known *harmonic-flow condition* satisfied by Killing fields v on pseudo-Riemannian manifolds. Thus, 2.12 in (A.1) should read 2.14, since Propositions 1.2 and 2.14 refer to the Ricci tensor quite prominently, while 2.12 does not mention it at all; also, Proposition 2.14 contains, in its second part, a holomorphicity conclusion.

In the ninth line of the proof of the second part of Proposition 2.14, it is established – correctly – that, for every $(1,0)$ vector field Y, and Z in (A.1), $\nabla''Z$ is L^2-orthogonal to $\nabla''Y$. Then an attempt is made to conclude that $\nabla''Z = 0$, arguing by contradiction: if $\nabla''Z \neq 0$ at some point z_0, one can – again correctly – find Y having $g(\nabla''Z, \nabla''Y) \neq 0$ everywhere in some neighborhood of z_0. As a next step, it is claimed that a contradiction arises: cited *verbatim*,

(A.2) By considering a cut-off function, we see that there exists a complex vector field Y such that $\int_M g(\nabla''Z, \nabla''Y)\, dv \neq 0$.

It is here that the argument seems incomplete: such a cut-off function φ equals 1 on some small "open ball" B centered at z_0, and vanishes outside a larger "concentric ball" B', and after the original choice of Y has been replaced by φY, there is no way to control the integral of $g(\nabla''Z, \nabla''(\varphi Y))$ over $B' \setminus B$ (while the integrals over B and $M \setminus B'$ have fixed values). More precisely, the sum of the three integrals must be zero, $\nabla''Z$ being L^2-orthogonal to all $\nabla''Y$.

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