That any linearly independent pair A, B in $\mathfrak{sl}(\Pi)$ with $[A, B] = A$ has the form (7.1) in some basis w, w' of Π can be seen as follows. We have $A \in [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^\perp$, where $\mathfrak{g} = \operatorname{span}\{A, B\}$, and so A is $\langle\,,\rangle$-null, that is, $\operatorname{tr} A = \det A = 0$. In a basis of Π containing an element of $\operatorname{Ker} A$, the matrix representing A is therefore triangular, with zeros on the diagonal, so that $A^2 = 0$, while $A \neq 0$. Thus, $A(\Pi) \subset \operatorname{Ker} A$ and, as both spaces are one-dimensional, $A(\Pi) = \operatorname{Ker} A$. The relation $[A, B] = A$ implies in turn that $\operatorname{Ker} A$ is invariant under B, and so B has real characteristic roots. Since $\operatorname{tr} B = 0$, the two roots must be nonzero, or else we would have $\operatorname{Ker} B = \operatorname{Ker} A$ and, in a basis containing an element of $\operatorname{Ker} A$, the matrices of both A and B would be triangular, with zeros on the diagonal, contradicting the linear independence of A and B. Thus, B is diagonalizable, with some nonzero eigenvalues $\pm c$ such that $\operatorname{Ker} A = \operatorname{Ker}(B + c)$. Choosing a basis w, w' of Π diagonalizing B with $w' \in \operatorname{Ker} A$, we may rescale w so that $Aw = w'$ (since $A(\Pi) = A(\operatorname{Ker}(B - c)) = \operatorname{Ker} A$). Applying $[A, B] = A$ to w we now get $c = 1/2$, which yields (7.1), proving (b). □

Appendix B. Local Lie-group structures

In this appendix we state and prove Theorem B.1, a well-known result, included here to provide a convenient reference for the proof of Lemma 8.1(i).

Given a real/complex vector space $\mathfrak{h}$ of sections of a real/complex vector bundle $\mathcal{V}$ over a manifold Σ, we will say that $\mathfrak{h}$ *trivializes* $\mathcal{V}$ if, for every $y \in \Sigma$, the evaluation operator $\psi \mapsto \psi_y$ is an isomorphism $\mathfrak{h} \to \mathcal{V}_y$. This amounts to requiring that $\dim \mathfrak{h}$ coincide with the fibre dimension of $\mathcal{V}$ and each $v \in \mathfrak{h}$ be either identically zero, or nonzero at every point of Σ. In other words, some (or any) basis of $\mathfrak{h}$ should form a trivialization of $\mathcal{V}$.

Theorem B.1. *Let a Lie algebra $\mathfrak{h}$ of vector fields on a simply connected manifold Σ trivialize its tangent bundle $T\Sigma$, and let $\Psi : \mathfrak{h} \to \mathfrak{g}$ be any Lie-algebra isomorphism between $\mathfrak{h}$ and the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on a Lie group G. Then there exists a mapping $F : \Sigma \to G$ such that every $v \in \mathfrak{h}$ is F-projectable onto Ψv. Any such mapping F is, locally, a diffeomorphism, and Ψ determines F uniquely up to compositions with left translations in G.*

Proof. Given $(y, z) \in \Sigma \times G$, let $K_{y,z} : T_y\Sigma \to T_{(y,z)}(\Sigma \times G) = T_y\Sigma \times T_zG$ be the linear operator with $K_{y,z}u = (u, (\Psi u')_z)$ for $u' \in \mathfrak{h}$ characterized by $u'_y = u \in T_y\Sigma$. Since $\Psi u'$ is left-invariant, the formula $\mathcal{H}_{(y,z)} = K_{y,z}(T_y\Sigma)$ defines a vector subbundle of $T(\Sigma \times G)$, invariant under the left action of G on $\Sigma \times G$. Thus, $\mathcal{H}$ is (the horizontal distribution of) a G-connection in the trivial G-principal bundle over Σ with the total space $\Sigma \times G$.

The distribution $\mathcal{H}$ on $\Sigma \times G$ is integrable, that is, our G-connection is flat. In other words, the $\mathcal{H}$-horizontal lift operation $v \mapsto \tilde{v}$, applied to vector fields v, w on Σ is a Lie-algebra homomorphism. In fact, $\tilde{v}_{(y,z)} = (v_y, (\Psi v')_z)$, with $v' \in \mathfrak{h}$ such that $v'_y = v_y$. Choosing in $\mathfrak{h}$ a basis e_j, $j = 1, \ldots, n$, we have $v = v^j e_j$,

$w = w^j e_j$, $[e_j, e_k] = c_{jk}^l e_l$ and $[\Psi e_j, \Psi e_k] = c_{jk}^l \Psi e_l$ for some real numbers c_{jk}^l and functions v^j, w^j. (The indices $j, k, l = 1, \ldots, n$, if repeated, are summed over.) Thus, $\tilde{v} = (v, v^j \Psi e_j)$, that is, $\tilde{v}_{(y,z)} = (v_y, v^j(y)(\Psi e_j)_z)$, and similarly for w. Hence $[\tilde{v}, \tilde{w}] = ([v, w], (d_v w^l - d_w v^l + v^j w^k c_{jk}^l)\Psi e_l)$, as required: namely, $[v, w] = [v^j e_j, w^k e_k] = (d_v w^l - d_w v^l + v^j w^k c_{jk}^l) e_l$, and so $[v, w]^l = d_v w^l - d_w v^l + v^j w^k c_{jk}^l$.

Therefore, as Σ is simply connected, $\Sigma \times G$ is the disjoint union of the leaves of $\mathcal{H}$, and the projection $\pi : \Sigma \times G \to \Sigma$ maps each leaf N diffeomorphically onto Σ (cf. [12, Vol. I, Corollary 9.2, p. 92]). On the other hand, one easily sees that a mapping F has the properties claimed in our assertion if and only if $d\Xi_y = K_{y,F(y)}$ for all $y \in \Sigma$, where $\Xi : \Sigma \to \Sigma \times G$ is given by $\Xi(y) = (y, F(y))$. Equivalently, Ξ is required to be an $\mathcal{H}$-horizontal section of the G-bundle $\Sigma \times G$, that is, the inverse diffeomorphism $\Sigma \to N$ of $\pi : N \to \Sigma$ for some leaf N of $\mathcal{H}$. The existence of F and its uniqueness up to left translations are now immediate, while such F is, locally, a diffeomorphism in view of the inverse maping theorem. This completes the proof. □

Appendix C. Lagrangians and Hamiltonians

A more detailed exposition of the topics oulined here can be found in [16].

We use the same symbol $\mathcal{V}$ for the total space of a vector bundle $\mathcal{V}$ over a manifold Σ as for the bundle itself, identifying each fibre $\mathcal{V}_y$, $y \in \Sigma$, with the submanifold $\pi^{-1}(y)$ of $\mathcal{V}$, where $\pi : \mathcal{V} \to \Sigma$ is the bundle projection. (Thus, $T\Sigma$ and $T^*\Sigma$ are manifolds.) As a set, $\mathcal{V} = \{(y, \psi) : y \in \Sigma, \;\; \psi \in \mathcal{V}_y\}$.

The identity mapping $\Pi \to \Pi$ in a real vector space Π with $\dim \Pi < \infty$, treated as a vector field on Π, is called the *radial vector field* on Π. On the total space $\mathcal{V}$ of any vector bundle over a manifold Σ we have the *radial vector field*, denoted here by $\mathbf{x}$, which is vertical (tangent to the fibres) and, restricted to each fibre of $\mathcal{V}$, coincides with the radial field on the fibre.

By a *Lagrangian* $L : U \to \mathbf{R}$, or, respectively, a *Hamiltonian* $H : U_* \to \mathbf{R}$ in a manifold Σ one means a function on a nonempty open set $U \subset T\Sigma$ or $U_* \subset T\Sigma$. The *Legendre mapping* $U \to T^*\Sigma$, or $U_* \to T\Sigma$, associated with L or H, is defined by requiring that, for each $y \in \Sigma$, it send any $v \in U \cap T_y\Sigma$ or $\xi \in U_* \cap T_y^*\Sigma$ to the differential of $L : U \cap T_y\Sigma \to \mathbf{R}$ (or, of $H : U_* \cap T_y\Sigma \to \mathbf{R}$) at v (or at ξ), which is an element of $T_v^*(U \cap T_y\Sigma) = T_y^*\Sigma \subset T^*\Sigma$ or, respectively, of $T_\xi^*(U_* \cap T_y^*\Sigma) = T_y\Sigma \subset T\Sigma$. We call such a Lagrangian $L : U \to \mathbf{R}$ or Hamiltonian $H : U_* \to \mathbf{R}$ in Σ *nonsingular* if the associated Legendre mapping is a diffeomorphism $U \to U_*$, or $U \to U_*$ (then referred to as the Legendre *transformation*), for some open set $U_* \subset T^*\Sigma$ or, respectively, $U \subset T\Sigma$. Nonsingular Lagrangians L in Σ are in a natural bijective correspondence with nonsingular Hamiltonians H in Σ. Namely, if $L : U \to \mathbf{R}$ is nonsingular, we define $H : U \to \mathbf{R}$ by $H = d_{\mathbf{x}}L - L$, for the radial vector field $\mathbf{x}$ mentioned above, and then use the Legendre transformation to identify U with U_*, so that H becomes a function $U_* \to \mathbf{R}$. A nonsingular Hamiltonian $H : U_* \to \mathbf{R}$ similarly gives rise to $L : U_* \to \mathbf{R}$ with $L = d_{\mathbf{x}}H - H$