Nijenhuis geometry of parallel tensors

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Abstract. A tensor – meaning here a tensor field Θ of any type (p, q) on a manifold – may be called integrable if it is parallel relative to some torsion-free connection. We provide analytical and geometric characterizations of integrability for differential q-forms, $q = 0, 1, 2, n-1, n$ (in dimension n), vectors, bivectors, symmetric $(2, 0)$ and $(0, 2)$ tensors, as well as complex-diagonalizable and nilpotent tensors of type $(1, 1)$. In most cases, integrability is equivalent to algebraic constancy of Θ coupled with the vanishing of one or more suitably defined Nijenhuis-type tensors, depending on Θ via a quasilinear first-order differential operator. For $(p, q) = (1, 1)$, they include the ordinary Nijenhuis tensor.

1. Introduction

We refer to a tensor field Θ of any type on a manifold M as *algebraically constant* when, for any $x, y \in M$, some linear isomorphism $T_xM \to T_yM$ sends Θ_x to Θ_y. The algebraic constancy amounts to being constant for functions, to vanishing nowhere or everywhere in the case of vector fields and 1-forms, and to having constant rank for symmetric or skew-symmetric $(0, 2)$ and $(2, 0)$ tensors.

We call a tensor field Θ *integrable* if some torsion-free connection makes it parallel, and *locally constant* if it has constant components in suitable local coordinates around each point. As one sees using a partition of unity, for integrability of Θ it suffices that such torsion-free connections exist locally. Consequently,

(1.1) the local constancy of Θ implies its integrability (but not conversely),

counterexamples to the converse being nonflat pseudo-Riemannian metrics.

Given an algebraically constant tensor Θ on a manifold M and a distribution $\mathcal{D} \subseteq TM$ naturally associated with it, as $\mathcal{D}$ is obviously ∇-parallel when $\nabla\Theta = 0$,

(1.2) the integrability of Θ implies the distribution-integrability of $\mathcal{D}$.

The local constancy of an algebraically constant tensor is nothing else than integrability, in the sense of [**17**, Prop. 1.1], of the corresponding G-structure (Remark 6.1).

With a $(1, 1)$ tensor Θ on a manifold M one associates its Nijenhuis tensor N, introduced by Nijenhuis [**21**] and studied by many others [**4, 5, 7, 11, 12, 14,**

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16, 19, 28], which sends vector fields v,w to the vector field

$$N(v,w) = \Theta[\Theta v,w] + \Theta[v,\Theta w] - [\Theta v,\Theta w] - \Theta^2[v,w]. \tag{1.3}$$

As pointed out by several authors [**7**, Sect. 2.3], [**4**, Definition 2.2], $N=0$ identically whenever Θ is integrable since, for any torsion-free connection ∇ on M,

$$N(v,w) = [\Theta\nabla_v\Theta - \nabla_{\Theta v}\Theta]w + [\nabla_{\Theta w}\Theta - \Theta\nabla_w\Theta]v. \tag{1.4}$$

Various generalizations of the Nijenhuis tensor have been proposed [**3, 18, 24, 25, 27**]. Below, after stating Proposition F, we elaborate on such generalizations that are of interest to us and have therefore been introduced in this paper.

Complex-diagonalizability of a linear endomorphism of a finite-dimensional real vector space V means, as usual, diagonalizability of its complex-linear extension to the complexification of V. Since any endomorphism of V is, uniquely, the sum of a complex-diagonalizable and a nilpotent one [**15**, Sect. 4.2], it is natural to deal with these two classes of endomorphisms separately.

In Sect. 5, 8–9, 11, 13, 14 and 15–16 we prove our six main results, stated below. We begin with a fact due to Kurita [**19**, Theorem 9], which also easily follows (see Sect. 3) from a theorem of Bolsinov, Konyaev and Matveev [**4**, Theorem 3.2]:

Remark A. For an algebraically constant complex-diagonalizable $(1,1)$ tensor Θ on a manifold M of dimension $n\geq 1$, the vanishing of N is equivalent to the integrability of Θ, as well as to its local constancy.

Algebraically constant tensors Θ of type $(1,1)$ give rise to the vector subbundles $\mathcal{Z}^i = \operatorname{Ker}\Theta^i$ and $\mathcal{B}^i = \operatorname{Im}\Theta^i$ of TM, for integers $i\geq 0$.

Theorem B. *Given an algebraically constant nilpotent* $(1,1)$ *tensor* Θ *on a manifold* M *of dimension* $n\geq 1$*, the following four conditions are equivalent.*

- (i) $N=0$ *and* $\mathcal{Z}^i = \operatorname{Ker}\Theta^i$ *is integrable for every* $i=1,\dots,n$.
- (ii) *In some commuting local frame* $e_1,\dots,e_n$ *around each point,* Θ *has the Jordan normal form, with* $\Theta e_1 = 0$ *and* $\Theta e_i = 0$ *or* $\Theta e_i = e_{i-1}$ *if* $i>1$.
- (iii) Θ *is locally constant.*
- (iv) Θ *is integrable.*

The Jordan normal form of an algebraically constant nilpotent $(1,1)$ tensor Θ may be represented by

(1.5) a weakly decreasing string $d_1\dots d_m$ of positive integers,

each d_q standing for a $d_q\times d_q$ Jordan block matrix with ones immediately above the diagonal and zeros everywhere else. Of interest to us are the Jordan normal forms $d_1\dots d_m$ such that $d_1=\ldots=d_{m-1}$. In other words,

(1.6) either all blocks have the same length, or they represent exactly two different lengths, with the shorter one occurring just once.

We say that a given algebraic type of an algebraically constant nilpotent $(1,1)$ tensor Θ is *controlled by the Nijenhuis tensor* if the vanishing of N implies, on any underlying manifold, the local constancy of Θ.

Theorem C. *Condition* (1.6) *imposed on the Jordan normal form of an algebraically constant nilpotent* $(1,1)$ *tensor* Θ *on a manifold* M *of dimension* $n\geq 1$ *is necessary and sufficient for the algebraic type of* Θ *to be controlled by its Nijenhuis tensor.*

Theorem C would be true as stated even if our definition of being controlled by N referred to integrability rather than local constancy. Namely, Proposition 8.1 – our proof of the necessity of (1.6) – realizes any Θ not satisfying (1.6) as a left-invariant tensor with $N=0$ on a step 2 nilpotent Lie group, which fails the integrability test (1.1) due to having nonintegrable $\mathrm{Ker}\,\Theta^p$ for some integer $p\geq 1$.

For nilpotent $(1,1)$ tensors Θ which are *generic*, that is, $\dim \mathrm{Ker}\,\Theta = 1$ or, equivalently, the Jordan normal form of Θ is the one-term string n (a single Jordan block), the sufficiency of (1.6) in Theorem C is a result of Kobayashi [**16**, Sect. 3]. See also [**12**, Cor. 2.4], [**28**, Theorem 1], [**5**, Theorem 1.3, Cor. 1.5], [**4**, Theorem 4.6]. Kobayashi [**16**, Sect. 5] further illustrated the necessity of (1.6) by an example, with $n=4$ and the Jordan normal form 211, cited in [**4**, Example 2.1].

Sect. 10 exhibits a special case of Theorem C by means of an affine-bundle construction, resulting in nonzero algebraically constant nilpotent $(1,1)$ tensors Θ with $N=0$, satisfying the condition $\Theta^2=0$ (equivalent to $\mathrm{Im}\,\Theta\subseteq\mathrm{Ker}\,\Theta$, that is, to having the Jordan normal form $2\ldots2$ or $2\ldots21\ldots1$).

For the normal form $2\ldots2$, also characterized by the equality $\mathrm{Ker}\,\Theta=\mathrm{Im}\,\Theta$, corresponding to the *almost-tangent structures* [**31**], the assertion of Theorem C is due to Goel [**11**, Theorem 2.4], while our affine-bundle construction becomes that of Crampin and Thompson [**9**]. Our construction is "locally universal" (Theorem 10.1), which generalizes the local version of [**9**, Theorem on p. 69].

We justify the following observation in Sect. 11.

Proposition D. *The closedness of an algebraically constant differential q-form on an n-manifold, $q=0,1,2,n-1,n$, implies its local constancy.*

The converse implication (closedness from integrability, and hence also from local constancy) is obviously true for forms of all degrees.

Even weakened by the replacement of local constancy with integrability – cf. (1.1) – Proposition D fails to hold for differential forms of other degrees: as we verify in Sect. 12, for any dimension $n\geq 5$ and any $q\in\{3,\ldots,n-2\}$, in local coordinates $x^1,\ldots,x^n$, the following formula defines a differential q-form ζ which is algebraically constant and closed, but not integrable:

$$\zeta=(dx^1\wedge dx^2+dx^3\wedge dx^4)\wedge(dx^5+x^1dx^2-x^3dx^4)\wedge dx^6\wedge\ldots\wedge dx^{q+2}. \tag{1.7}$$

Constant-rank (skew)symmetric $(0,2)$ and $(2,0)$ tensors, being bundle morphisms $TM\to T^*M$ or $T^*M\to TM$, have well-defined unique kernels and images. The next displayed condition uses the natural concept of projectability, presented in Sect. 2: for the integrability of a constant-rank symmetric $(0,2)$ tensor g on a manifold, it is necessary and sufficient – as we justify in Sect. 13 – that

(1.8) the distribution $\mathrm{Ker}\,g$ be integrable, and g projectable along $\mathrm{Ker}\,g$.

Condition (1.8), rephrased as $\pounds_v g=0$ for every local section v of $\mathrm{Ker}\,g$, is well known to be an integrability test for g. To the best of our knowledge, this fact goes back to Moisil [**20**] and Vrănceanu [**30**]. See also [**8, 22, 26**], [**10**, Theorem 5.1].

The sweeping recent result of Bandyopadhyay, Dacorogna, Matveev and Troyanov [**1**, Theorem 4.4] provides a characterization of local constancy for $(0,2)$ tensors *without any symmetry/skew-symmetry assumptions.* The criterion (1.8), much more modest in scope, focuses on the symmetric case and integrability (as opposed to local constancy); what we gain is simplicity of the resulting conditions.

In contrast with 1-forms (Proposition D), the local constancy of a vector field obviously follows just from its algebraic constancy. An analogous difference occurs between symmetric $(2,0)$ tensors and symmetric $(0,2)$ tensors: the former – unlike the latter – require no projectability condition to guarantee integrability.

Proposition E. *The integrability of a constant-rank symmetric* $(2,0)$ *tensor* Θ *on a manifold is equivalent to the integrability of* $\mathrm{Im}\,\Theta$*.*

Such tensors Θ can be naturally identified (see Remark 2.3) with *sub-pseudo-Riemannian metrics* [**13**], which include the sub-Riemannian ones [**2**], such as the Galilei spacetime metric.

For a bivector, that is, a skew-symmetric $(2,0)$ tensor Θ, assumed to have constant rank, formula (2.7) defines the *restriction* of Θ to $\mathcal{B} = \mathrm{Im}\,\Theta$, which is a nondegenerate section of $\mathcal{B}^{\wedge 2}$, thus giving rise to its inverse, a section of $[\mathcal{B}^*]^{\wedge 2}$.

Proposition F. *A constant-rank bivector* Θ *on a manifold is locally constant or – equivalently – integrable if and only if the distribution* $\mathrm{Im}\,\Theta$ *is integrable and the inverse of the restriction of* Θ *to* $\mathrm{Im}\,\Theta$ *is closed along each leaf of* $\mathrm{Im}\,\Theta$*.*

The generalizations of the Nijenhuis tensor which are of interest to us are motivated by Remark A, Theorem C and Proposition D: we want to associate with a given tensor Θ one (or more) Nijenhuis-type tensor(s), each depending on Θ via a quasilinear first-order differential operator, in such a way that, if Θ algebraically constant, the vanishing of these tensors is equivalent to the integrability of Θ.

As an example, N given by (1.3) serves in this capacity for complex-diagonalizable $(1,1)$ tensors and nilpotent $(1,1)$ tensors with the property (1.6); its quasi-linearity is immediate from (1.4). In the case of differential q-forms ζ in dimension n, where $q \in \{0,1,2,n-1,n\}$ (but not – see (1.7) – for other degrees), the exterior derivative $d\zeta$ is a Nijenhuis-type tensor in our sense, while an analogous role for vector fields is played by the zero tensor.

For any symmetric $(0,2)$ tensor g of constant rank r on a manifold M, we introduce two Nijenhuis-type tensors N' and N'', both of type $(0,2r+3)$, defined as follows: N' (or, N'') sends vector fields $v, v_1, \ldots, v_r$ (or, $w, u, v_1, \ldots, v_r$) to the $(r+2)$-form, or $(r+1)$-form

$$
\text{(1.9)}\qquad
\begin{array}{ll}
\text{a)} & N'(v, v_1, \ldots, v_r) = d[g(v,\,\cdot\,)] \wedge g(v_1,\,\cdot\,) \wedge \ldots \wedge g(v_r,\,\cdot\,),\\
\text{b)} & N''(w, u, v_1, \ldots, v_r) = \{[\pounds g](w,u)\} \wedge g(v_1,\,\cdot\,) \wedge \ldots \wedge g(v_r,\,\cdot\,),
\end{array}
$$

$[\pounds g](w,u)$ being treated here, formally, as a 1-form sending any vector field v to the function $[\pounds_v g](w,u)$. The word ‘formally’ reflects the fact that $[\pounds g](w,u)$ is not tensorial in v. Nevertheless, in Sect. 16 we point out that N' and N'' are well-defined tensors, and prove the following result.

Theorem G. *The vanishing of both* N' *and* N'' *is necessary and sufficient for the integrability of the given symmetric* $(0,2)$ *tensor* g *of constant rank* r*.*

The lines following formula (15.3) in Sect. 15 provide a set of d_1 Nijenhuis-type tensors for an algebraically constant nilpotent $(1,1)$ tensor Θ with the Jordan normal form $d_1 \ldots d_m$. This set consists of N – see (1.3) – and $d_1 - 1$ additional tensors responsible for integrability of $\mathrm{Ker}\,\Theta^i$, $1 \le i < d_1$ (which makes them redundant in the case (1.6), due to Theorem C).

Finally, formulae (15.4) – (15.5) in Sect. 15 define, for (skew)symmetric $(2,0)$ tensors Θ of constant rank r, a Nijenhuis-type $(2r+3,0)$ tensor $\widehat{N}$ such that

$\widehat{N}=0$ if and only if $\operatorname{Im}\Theta$ is integrable. When Θ is symmetric, vanishing of $\widehat{N}$ thus amounts, by Proposition E, to integrability of Θ. However, for a rank r *bivector* Θ, the condition $\widehat{N}=0$, despite still being necessary, is not sufficient in order that Θ be integrable. Obvious examples illustrating the last claim arise, cf. Proposition F, on a product manifold $M=\varSigma\times\varSigma'$, with Θ obtained as the trivial extension to M of the inverse of a nonclosed nondegenerate 2-form on $\varSigma$.

2. Preliminaries

Manifolds (by definition connected) and mappings, including sections of bundles, are always assumed to be smooth. Tensor fields will usually be referred to as *tensors.* All vector spaces are real (except in Sect. 3) and finite-dimensional.

Given a manifold M and vector subbundles $\mathcal{D},\mathcal{D}',\mathcal{D}''$ of TM, we write

$$[\mathcal{D},\mathcal{D}']\subseteq\mathcal{D}'' \tag{2.1}$$

when $[w,w']$ is a local section of $\mathcal{D}''$ for any local sections w of $\mathcal{D}$ and w' of $\mathcal{D}'$.

Lemma 2.1. *For $M,\mathcal{D},\mathcal{D}'$ as above, suppose that $\mathcal{D}$ contains $\mathcal{D}'$ with codimension one, and $[\mathcal{D},\mathcal{D}']\subseteq\mathcal{D}$. Then $\mathcal{D}$ is integrable.*

Proof. As $[\mathcal{D}',\mathcal{D}']\subseteq\mathcal{D}$, the relation $[\mathcal{D},\mathcal{D}]\subseteq\mathcal{D}$ follows if we note that, locally, sections of $\mathcal{D}$ have the form $v+\phi w$ for various sections v of $\mathcal{D}'$, functions ϕ, and one fixed section w of $\mathcal{D}$. □

Let $\pi:M\to\varSigma$ be a mapping between manifolds. We say that a vector field w (or, a distribution $\mathcal{Z}$) on M is π-*projectable* if $d\pi_x w_x=u_{\pi(x)}$ or, respectively, $d\pi_x(\mathcal{Z}_x)=\mathcal{W}_{\pi(x)}$ for all $x\in M$ and some vector field u (or, distribution $\mathcal{W}$) on $\varSigma$. If this is the case,

$$\text{the integrability of } \mathcal{Z} \text{ implies that of } \mathcal{W}, \tag{2.2}$$

since π restricted to any leaf of $\mathcal{Z}$ is, locally, a submersion onto an integral manifold of $\mathcal{W}$. We also define π-projectability of a $(0,q)$ tensor field Θ on M by requiring Θ to be the π-pullback of a $(0,q)$ tensor field on $\varSigma$.

Given an integrable distribution $\mathcal{V}$ on a manifold M, every point of M has a neighborhood U such that, for some manifold $\varSigma$, the leaves of $\mathcal{V}$ restricted to U are the fibres of a bundle projection $\pi:U\to\varSigma$.

Let $\mathcal{V}$ be an integrable distribution on a manifold M. By $\mathcal{V}$-*projectability* of a vector field on an open set $U'\subseteq M$ (or, of a distribution on U', or of a $(0,q)$ tensor field on U') we mean its π-projectability for any $\pi,U,\varSigma$ as in the last paragraph such that $U\subseteq U'$. Then, for a vector field w on M,

$$\begin{array}{l} w \text{ is } \mathcal{V}\text{-projectable if and only if, for every section}\\ v \text{ of } \mathcal{V}, \text{ the Lie bracket } [v,w] \text{ is also a section of } \mathcal{V}.\end{array} \tag{2.3}$$

(This is obvious in local coordinates for M turning π as above into a Cartesian-product projection.) It is also clear that, given a $(0,q)$ tensor field Θ,

$$\begin{array}{l}\Theta \text{ is } \mathcal{V}\text{-projectable if and only if } d_v[\Theta(w_1,\dots,w_q)]=0 \text{ for all sec-}\\ \text{tions } v \text{ of } \mathcal{V} \text{ and all } \mathcal{V}\text{-projectable local vector fields } w_1,\dots,w_q.\end{array} \tag{2.4}$$

Lemma 2.2. *For an integrable distribution $\mathcal{V}$ and any distribution $\mathcal{Z}$ on an n-dimensional manifold M, the following two conditions are equivalent.*

(a) *$\mathcal{Z}$ is $\mathcal{V}$-projectable,*

(b) $\mathcal{Z}\cap\mathcal{V}$ *has a constant dimension and* $[\mathcal{V},\mathcal{Z}]\subseteq\mathcal{V}+\mathcal{Z}$.

Under the additional assumption that $\mathcal{V}\subseteq\mathcal{Z}$,

(c) $\mathcal{Z}$ *is* $\mathcal{V}$*-projectable if and only if* $[\mathcal{V},\mathcal{Z}]\subseteq\mathcal{Z}$.

Proof. The equivalence of (a) and (b), once established, trivially implies (c) when $\mathcal{V}\subseteq\mathcal{Z}$. We proceed, however, by first proving (c), in the case where $\mathcal{V}\subseteq\mathcal{Z}$. It will then clearly follow from (c) that (a) and (b) are equivalent, since $\mathcal{V}$-projectability of $\mathcal{Z}$ amounts to $\mathcal{V}$-projectability of $\mathcal{V}+\mathcal{Z}$, and $\mathcal{V}\subseteq\mathcal{V}+\mathcal{Z}$.

If $\mathcal{V}\subseteq\mathcal{Z}$ and $\mathcal{Z}$ is $\mathcal{V}$-projectable, onto some distribution $\mathcal{W}$ on a local leaf space of $\mathcal{V}$, then $\mathcal{Z}$ is spanned by $\mathcal{V}$-projectable sections obtained as lifts of sections of $\mathcal{W}$ (including sections of $\mathcal{V}$, which are lifts of 0). As any section of $\mathcal{Z}$ is a functional combination of $\mathcal{V}$-projectable ones, (2.3) yields $[\mathcal{V},\mathcal{Z}]\subseteq\mathcal{Z}$.

Conversely, let $\mathcal{V}\subseteq\mathcal{Z}$ and $[\mathcal{V},\mathcal{Z}]\subseteq\mathcal{Z}$. We choose local coordinates $x^1,\dots,x^n$ such that $\mathcal{V}$ is spanned by the coordinate vector fields ∂_i, $i=1,\dots,m$, and a local trivialization of the subbundle $\mathcal{Z}$ of TM having the form $\partial_1,\dots,\partial_m,w_{m+1},\dots,w_s$. Using the index ranges $1\le i,j,k\le m<a,b,c\le s$, we obtain, since $[\mathcal{V},\mathcal{Z}]\subseteq\mathcal{Z}$,

$$[\partial_i,w_a]\,=\,\Gamma_{ia}^j\partial_j\,+\,\Gamma_{ia}^b w_b \tag{2.5}$$

for some functions Γ_{ia}^j, Γ_{ia}^b. The w_b-component of the Jacobi identity $[\partial_i,[\partial_j,w_a]]=[\partial_j,[\partial_i,w_a]]$, with $[\partial_i,\partial_j]=0$, now implies *symmetry of* $\partial_i\Gamma_{ja}^b+\Gamma_{ic}^b\Gamma_{ja}^c$ *in* i,j. This symmetry amounts to the vanishing of the curvature, that is, flatness, for the linear connection with the components Γ_{ia}^b in a rank $s-m$ vector bundle over a manifold with the coordinates x^i, $i=1,\dots,m$. The equations $\partial_i\psi^b+\Gamma_{ic}^b\psi^c=0$, stating that ψ^a, with $m<a\le s$, are the components of a parallel section ψ, is thus locally solvable with any prescibed initial value at a given point z. Let us choose such a section ψ_a, for $a=m+1,\dots,s$, with the initial value $(\delta_a^1,\dots,\delta_a^m)$ at z, so that

$$\partial_i\psi_a^b\,+\,\Gamma_{ic}^b\psi_a^c\,=\,0,\quad \psi_a^b(z)=\delta_a^b. \tag{2.6}$$

Setting $u_a=\psi_a^b w_b$, we obtain a new local trivialization $\partial_1,\dots,\partial_m,u_{m+1},\dots,u_s$ of $\mathcal{Z}$ while, by (2.5) and (2.6), $[\partial_i,u_a]$ are sections of $\mathcal{Z}$. Therefore, due to (2.3), our new local trivialization of $\mathcal{Z}$ consists of $\mathcal{V}$-projectable sections, which makes $\mathcal{Z}$ itself $\mathcal{V}$-projectable. □

Given a symmetric or skew-symmetric $(2,0)$ tensor Θ in a vector space V, let $\operatorname{Ker}\Theta$ and $\operatorname{Im}\Theta$ be the kernel and image of $V^*\ni\xi\mapsto\Theta(\xi,\,\cdot\,)\in V$. By the *restriction* of Θ to $W=\operatorname{Im}\Theta$ we mean the $(2,0)$ tensor Θ_W in W given by

$$\begin{array}{l}\Theta_W(\eta,\eta')=\Theta(\xi,\xi')\ \text{ for any }\ \eta,\eta'\in W^*\text{ and}\\ \text{any extensions }\ \xi,\xi':V\to\mathbb{R}\ \text{ of }\ \eta,\eta'\ \text{ to }\ V.\end{array} \tag{2.7}$$

As ξ and ξ' are unique up to adding elements of $W'=\operatorname{Ker}\Theta\subseteq V^*$, the polar space of W, the restriction is well defined. In other words, since $W'=\operatorname{Ker}\Theta$, the bilinear form Θ on V^* descends to one on $V^*/W'=W^*$, which is our Θ_W. (The natural identification of W^* with V^*/W' sends $\eta\in W^*$ to the W'-coset of any extension of η to V.) In addition, Θ is the image of Θ_W under the linear operator $W^{\odot 2}\to V^{\odot 2}$ or $W^{\wedge 2}\to V^{\wedge 2}$ induced by the inclusion $W\to V$. Finally,

$$\text{the restriction }\ \Theta_W\ \text{ is nondegenerate,} \tag{2.8}$$

as any $\eta\in W\smallsetminus\{0\}$ has an extension ξ to V not lying in $W'=\operatorname{Ker}\Theta$, and hence $\Theta(\xi,\xi')\neq 0$ for some $\xi'\in V^*$.

Remark 2.3. Constant-rank symmetric $(2,0)$ tensors Θ on a manifold M are naturally identified with sub-pseudo-Riemannian metrics on M, that is, pseudo-Riemannian fibre metrics h on vector subbundles $\mathcal{B}$ of TM. In fact, one may set $\mathcal{B} = \mathrm{Im}\,\Theta$ and, using (2.7) – (2.8), declare h to be the inverse of the restriction of Θ to $\mathcal{B}$. Thus, cf. the lines preceding (2.8), Θ is the image of the inverse of h under the bundle morphism $\mathcal{B}^{\odot 2} \to [TM]^{\odot 2}$ induced by the inclusion $\mathcal{B} \to TM$.

The $(r-1)$-fold contraction of two $(r,0)$-tensors Θ, Π on a manifold with a fixed Riemannian metric g, appearing in (i) below, is

(2.9) the $(2,0)$ tensor β given by $\beta^{ij} = \Theta^{ii_2\ldots i_r}\Pi^{jj_2\ldots j_r}g_{i_2j_2}\ldots g_{i_rj_r}$.

Remark 2.4. Let V be a Euclidean n-space with the inner product $\langle\cdot,\cdot\rangle$.

(i) The $(r-1)$-fold contraction (2.9) against itself of a nonzero decomposable r-vector $v_1 \wedge \ldots \wedge v_r \in V^{\wedge r}$ yields a $(2,0)$ tensor which, viewed with the aid of $\langle\cdot,\cdot\rangle$ as an endomorphism of V, equals a nonzero multiple of the orthogonal projection onto the span of $v_1, \ldots, v_r$. (To see this, we are free to assume that $v_1, \ldots, v_r$ are orthonormal.)

(ii) If V is oriented, $*(e_1 \wedge \ldots \wedge e_r) = e_{r+1} \wedge \ldots \wedge e_n$ for the Hodge star $* : V^{\wedge r} \to V^{\wedge(n-r)}$ and any positive orthonormal basis $e_1, \ldots, e_n$ of V.

Remark 2.5. In an $s \times (n-s)$ product n-dimensional manifold M with global product coordinates x^i, x^a (index ranges $1 \le i \le s < a \le n$), let the component functions g_{ij}, Θ^{ij} and $\Gamma_{ij}^k = \Gamma_{ji}^k$ represent families of $(0,2)$ tensors, $(2,0)$ tensors and torsion-free connections on the leaves of the integrable distribution spanned by the coordinate vector fields ∂_i. Suppose that each tensor is parallel relative to the corresponding connection on the leaf: $\partial_i g_{jk} = \Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{jl}$ and $\partial_i \Theta^{jk} = -\Gamma_{il}^j \Theta^{lk} - \Gamma_{il}^k \Theta^{jl}$. Setting $g_{\lambda\mu} = \Theta^{\lambda\mu} = \Gamma_{\lambda\mu}^\nu = 0$ whenever at least one of the indices $\lambda, \mu, \nu \in \{1, \ldots, n\}$ is in the a range, we extend the above data to their analogs defined on M, namely, a $(0,2)$ tensor g, a $(2,0)$ tensor Θ and a torsion-free connection ∇, in such a way that, obviously, $\nabla g = \nabla\Theta = 0$.

3. The complex-diagonalizable case

By (1.3), for a $(1,1)$ tensor field Θ and any $a \in \mathbb{R}$,

(3.1) Θ and $\Theta - a\mathrm{Id}$ have the same Nijenhuis tensor.

To justify Remark A, we invoke a result of Bolsinov, Konyaev and Matveev [**4**, Theorem 3.2]. It states that, if a $(1,1)$ tensor Θ with $N = 0$ on a manifold M has complex characteristic roots of constant (algebraic) multiplicities, then M and Θ are, locally, decomposed into Cartesian products of factor manifolds/tensors with $N = 0$, where each factor corresponds to (and realizes) a real eigenvalue function of Θ, or a conjugate pair of its (nonreal) complex characteristic-root functions.

Under the assumption made in Remark A, the complex characteristic roots of Θ are all constant. Let the symbols M and Θ now stand for one of of factor manifolds/tensors with $N = 0$, mentioned above.

If the (constant) eigenvalue realized by this Θ is real, our claim follows: Θ equals a constant multiple of Id.

Otherwise, the characteristic roots realized by Θ are $a \pm bi$, with $a, b \in \mathbb{R}$ and $b \ne 0$. Let $J = b^{-1}(\Theta - a\mathrm{Id})$. By (3.1), J still has $N = 0$, while the characteristic roots of J (and hence those of its complexification $\hat{J}$) are i and $-i$. As $\hat{J}$ is diagonalizable – due to our assumption – we get $\hat{J}^2 = -\mathrm{Id}$. Thus, $J^2 = -\mathrm{Id}$, and the local constancy of Θ follows from the Newlander-Nirenberg theorem.

The more modest goal of establishing a weaker version of Remark A, with integrability of Θ replacing its local constancy, is easily achieved as follows. Rather than invoking the Newlander-Nirenberg theorem, one shows that an almost-complex structure J or, more generally, a $(1,1)$ tensor J with $N=0$ and $J^2 = c\mathrm{Id}$, where $c \in \mathrm{I\!R} \smallsetminus \{0\}$, has $\hat{\nabla} J = 0$ for some torsion-free connection $\hat{\nabla}$.

We start from any torsion-free connection ∇. By (1.4), $4c[B_v w - B_w v] = N(v,w)$ for $B_v w$ given by $4cB_v w = 2J[\nabla_{\!v} J]w + J[\nabla_{\!w} J]v + [\nabla_{\!Jw} J]v$, that is, $4cB_v w = (J[\nabla_{\!v} J]w + J[\nabla_{\!w} J]v) + (J[\nabla_{\!v} J]w + [\nabla_{\!Jw} J]v)$. Thus, the vanishing of N for J amounts to symmetry of $B_v w$ in v,w, while $[J, B_v] = \nabla_{\!v} J$ since, J^2 being parallel, $\nabla_{\!v} J$ anticommutes with J. This is precisely the relation $\hat{\nabla} J = 0$ for the torsion-free connection $\hat{\nabla}$ characterized by $\hat{\nabla}_{\!v} = \nabla_{\!v} + B_v$.

The assignment $\nabla \mapsto \hat{\nabla} = \nabla + B$ appearing above is a natural projection of the affine space of all torsion-free connections on the manifold in question onto the affine subspace formed by those connections which make J parallel.

The above conclusion is due to Clark and Bruckheimer [**7**, Theorem 6]. Our argument is a concise version of one used, in a more general situation, by Hernando, Reyes and Gadea [**14**, Theorems 3.4 and 7.1].

4. Tensors of type $(1,1)$

For the reader's convenience, we repeat here the definition, due to Nijenhuis [**21**], of the Nijenhuis tensor (1.3) associated with a $(1,1)$ tensor Θ on a manifold:

$$N(v,w) \,=\, \Theta[\Theta v, w] \,+\, \Theta[v, \Theta w] \,-\, [\Theta v, \Theta w] \,-\, \Theta^2[v,w]. \tag{4.1}$$

Applying Θ^i to both sides, with any integer $i \geq 0$, one obviously obtains

$$\Theta^{i+1}(\Theta[v,w] - [v,\Theta w]) \,=\, \Theta^i(\Theta[\Theta v, w] - [\Theta v, \Theta w]) \,-\, \Theta^i[N(v,w)]. \tag{4.2}$$

Let $N=0$. For any vector fields v,w and integers $i,j \geq 0$,

$$\text{if } \Theta^i v = 0, \text{ then } \Theta^i \text{ also annihilates } \Theta^j[v,w] - [v, \Theta^j w]. \tag{4.3}$$

Namely, let $R(i,j)$ be the assertion (4.3), and $R(j)$ the claim that $R(i,j)$ holds for all $i \geq 1$. Now $R(1,1)$ is immediate from (4.1), while, assuming $R(i,1)$, and choosing any v with $0 = \Theta^{i+1} v = \Theta^i \Theta v$, we get, from $R(i,1)$ for Θv (not v), zero on the right-hand side of (4.2), and hence also on the left-hand side, which yields $R(i+1,1)$ and, by induction on i, establishes $R(i,1)$ for all $i \geq 1$, that is $R(1)$. If we now assume $R(j)$, and use any $i \geq 1$, we see that $\Theta^i[v, \Theta^{j+1} w] = \Theta^{i+1}[v, \Theta^j w]$ when $\Theta^i v = 0$ (from $R(i,1)$ applied to $\Theta^j w$ rather than w), which in turn equals $\Theta^{i+j+1}[v,w]$ (due to $R(i+1,j)$, a consequence of $R(j)$). One thus has $R(j+1)$, which completes the proof of (4.3).

When, again, $N=0$ in (4.1) and i,j,k are nonnegative integers,

$$\begin{array}{l}\text{a) } [\mathcal{B}^i, \mathcal{B}^i] \subseteq \mathcal{B}^i, \quad \text{b) } [\mathcal{Z}^i, \mathcal{B}^j] \subseteq \mathcal{Z}^i + \mathcal{B}^j, \quad \text{c) } [\mathcal{Z}^i, \mathcal{Z}^j] \subseteq \mathcal{Z}^{i+j}, \\ \text{d) } [\mathcal{Z}^i, \mathcal{Z}^k] \subseteq \mathcal{Z}^k \ \text{ if } \mathcal{Z}^i \text{ is integrable and } k \geq i\end{array} \tag{4.4}$$

– notation of (2.1) – with Θ assumed algebraically constant. In fact, (4.4-a), that is, the integrability of each $\mathcal{B}^i$, follows via induction on i, from (4.1) with $N=0$ and with v,w replaced by $\Theta^i v, \Theta^i w$. (The third Lie bracket in (4.1) then is a section of $\mathcal{B}^{i+1}$, once we assume that $[\mathcal{B}^i, \mathcal{B}^i] \subseteq \mathcal{B}^i$.) For (4.4-b), note that the Lie bracket of sections v of $\mathcal{Z}^i$ and $\Theta^j w$ of $\mathcal{B}^j$ equals, by (4.3), $\Theta^j[v,w]$ plus a section of $\mathcal{Z}^i$. Finally, (4.4-c) and (4.4-d) are further consequences of (4.3): given sections v of $\mathcal{Z}^i$ and w of $\mathcal{Z}^j$, (4.3) reads $\Theta^{i+j}[v,w]=0$, while (4.3) with $j=k-i$, for sections v of $\mathcal{Z}^i$ and w of $\mathcal{Z}^k$ (which makes $\Theta^j w$ and $[v,\Theta^j w]$ sections of $\mathcal{Z}^i$ – the latter due to the assumed integrability of $\mathcal{Z}^i$), yields $\Theta^k[v,w]=0$.

The conclusion (4.4-a) is due to Bolsinov, Konyaev and Matveev [**4**, Cor. 2.5.].

5. Proof of Theorem B

We use induction on the dimension, with the following induction step.

Lemma 5.1. *Let Θ be an algebraically constant $(1,1)$ tensor with $N=0$ in* (1.3) *on a manifold M such that the distribution $\mathcal{Z}=\operatorname{Ker}\Theta$ is integrable.*

- (a) *Θ-images of $\mathcal{Z}$-projectable local vector fields in M are themselves $\mathcal{Z}$-projectable, so that Θ naturally descends to a $(1,1)$ tensor $\widehat{\Theta}$ on any local leaf space Σ of $\mathcal{Z}$, and $\widehat{\Theta}$ also has $N=0$.*
- (b) *If local vector fields v,w, and hence also $\Theta v,\Theta w$, are $\mathcal{Z}$-projectable and the projected images of $v,w,\Theta v,\Theta w$ all commute, then $[\Theta v,\Theta w]=0$.*
- (c) *Nilpotency of Θ, or integrability of the distributions $\operatorname{Ker}\Theta^i$ for all $i\geq 1$, implies the same property for $\widehat{\Theta}$.*

Proof. Applying (1.3) to v with $\Theta v=0$ and w projectable along $\mathcal{Z}$, we obtain $\Theta[v,\Theta w]=0$, as $\Theta[v,w]$, and hence $\Theta^2[v,w]$, vanishes due to projectability of w and (2.3). By (2.3), this proves the first part of (a), with an obvious definition of $\widehat{\Theta}$. Evaluating (1.3) on projectable vector fields, or applying Θ to them, we get $N=0$ for $\widehat{\Theta}$ or, respectively, the claim about nilpotency in (c).

Under the assumptions of (b), $[\Theta v,w],[v,\Theta w]$ and $\Theta[v,w]$ are – by (a) – projectable onto 0, which makes them sections of $\mathcal{Z}=\operatorname{Ker}\Theta$, so that (1.3) with $N=0$ yields $[\Theta v,\Theta w]=\Theta([\Theta v,w]+[v,\Theta w]-\Theta[v,w])=0$.

If the distributions $\mathcal{Z}^i=\operatorname{Ker}\Theta^i$ are all integrable, projectable vector fields that project onto sections of $\operatorname{Ker}\widehat{\Theta}^i$ span the distribution $\mathcal{Z}^{i+1}$ (the Θ^i-preimage of the vertical distribution $\mathcal{Z}$). Projectability of each $\mathcal{Z}^{i+1}$, immediate from that of Θ, or from (4.4-d) and Lemma 2.2(c), combined with (2.2), proves (c). □

The assertion $N=0$ in (a) is also a special case of [**4**, Prop. 2.4].

Proof of Theorem B. As the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) are obvious – the last two from from (1.1), (1.2) and (1.4) – we now just proceed to show that (ii) holds whenever (i) does, using induction on $n\geq 1$. The case $n=1$ being trivial, we now fix $n>1$ and assume that (i) implies (ii) in dimensions less than n, while (i) is satisfied on an n-manifold M, with $\Theta\neq 0$. Using $\widehat{\Theta}$ and a local leaf space Σ arising from Lemma 5.1(a), and replacing M by a suitable neighborhood of a given point, we get a bundle projection $\pi:M\to\Sigma$ with the vertical distribution $\mathcal{Z}=\operatorname{Ker}\Theta$, while (i), and hence (ii), holds for $\widehat{\Theta}$, on Σ, since $\dim\Sigma<n$. The resulting commuting Jordan-form frame for $\widehat{\Theta}$ is split into $\widehat{\Theta}$-*orbits* $u_1,\dots,u_d$ of various *lengths* $d\geq 1$, with the *initial* vector (field) u_1 lying in $\operatorname{Ker}\widehat{\Theta}$, the *final* vector u_d outside of $\operatorname{Im}\widehat{\Theta}$, and $u_i=\widehat{\Theta}^{d-i}u_d$ for $i=1,\dots,d$.

We now associate with every given $\widehat{\Theta}$-orbit $u_1,\dots,u_d$ the corresponding Θ-orbit $v_0,v_1,\dots,v_d$ of length $d+1$ in M. First, we choose each final vector field v_d, on M, so that it projects onto u_d under π, and set $v_i=\Theta^{d-i}v_d$, $i=0,\dots,d-1$. We call v_0 the *pre-initial* vector. Our v_d is only unique up to adding sections of

$\mathcal{Z} = \operatorname{Ker}\Theta$ (and will be modified later); this is also the obvious reason why *the nonfinal vectors* $v_0, v_1, \dots, v_{d-1}$ *are uniquely determined.* Due to π-projectability of Θ onto $\widehat{\Theta}$ in Lemma 5.1(a), the resulting Θ-orbit, with the pre-initial vector removed, projects onto the original $\widehat{\Theta}$-orbit, while the pre-initial vectors are sections of $\mathcal{Z} \cap \operatorname{Im}\Theta$, projecting onto zero. Also, by Lemma 5.1(b), *the nonfinal vectors from the union of all the* Θ*-orbits commute with one another* (which includes the pre-initial ones). Denoting by k the total number of these commuting vectors, we see that they generate

(5.1) a free local action of $\mathrm{I\!R}^k$ in M.

The union of all the Θ-orbits forms a linearly independent system at every point: the non-pre-initial ones project onto a frame in Σ, which makes them linearly independent over $\mathcal{Z} = \operatorname{Ker}\Theta$ (meaning linear independence of their images in $TM/\mathcal{Z}$), while the pre-initial ones, lying in $\mathcal{Z}$, are linearly independent, being the Θ-images of the initial vectors, linearly independent over $\mathcal{Z}$.

Next, we modify – as announced above – the final vectors e_a chosen in M, and augment the union of all the Θ-orbits with some sections e_λ, so as to obtain a commuting frame in M which, automatically, will be a Jordan-form frame for $\widehat{\Theta}$. (The indices a, λ have some appropriate ranges.) To this end, we identify M, locally, with a Cartesian product of a horizontal factor (our leaf space Σ) and a vertical factor, tangent to $\mathcal{Z}$. Our e_a and e_λ are suitable systems of commuting vector fields on the factor manifolds, trivially extended to vector fields in M (which causes e_a to commute with e_λ). In a first step, for e_a we choose the final vectors of our Jordan-form frame for $\widehat{\Theta}$, and for e_λ some vertical coordinate vector fields chosen, locally, so as to be linearly independent over $\mathcal{Z} \cap \operatorname{Im}\Theta$ and represent, under the quotient-bundle projection, a local trivialization of $\mathcal{Z}/(\mathcal{Z} \cap \operatorname{Im}\Theta)$. Let Q now be one leaf of the integrable distribution spanned by all e_a and e_λ. Thus, Q has codimension k in M and is transverse to the orbits of the local free action (5.1). We now modify all e_a and e_λ further, by using the action (5.1) to spread their restrictions to Q from Q to a neighborhood of Q in M. Due to equivariance of π relative to the action (5.1) and the analogous free action in Σ generated by the nonfinal vectors from the union of all the $\widehat{\Theta}$-orbits, and the invariance of the final vectors in Σ under the latter action, the modified e_a still project onto the final vectors (and e_λ onto 0, as the action leaves $\mathcal{Z}$ invariant). Finally, e_a and e_λ commute both with the nonfinal vectors from the union of the Θ-orbits, and with one another: the former follows from their $\mathrm{I\!R}^k$-invariance, the latter since their restrictions to Q commute. This completes the proof. □

6. Algebraic constancy and connections

Given a real vector bundle E of rank k over a manifold M and integers $p, q \ge 0$, we say that a smooth section Θ of $E^{\otimes p} \otimes [E^*]^{\otimes q}$ is *algebraically constant* when, for any $x, y \in M$, some linear isomorphism $E_x \to E_y$ sends Θ_x to Θ_y. In this case, fixing $z \in M$ and an ordered basis $\mathbf{e} = (e_1, \dots, e_k)$ of E_z, let us

(6.1) denote by $\mathbf{e}\Theta_z$ the system of components of Θ_z in the basis $\mathbf{e}$,

that is, the (p, q) tensor in $\mathrm{I\!R}^k$ arising as the image of Θ_z under the linear isomorphism $E_z \to \mathrm{I\!R}^k$ associated with $\mathbf{e}$. We now define two objects, the first being the matrix group $G \subseteq \mathrm{GL}(k, \mathrm{I\!R})$ formed by all transition matrices between $\mathbf{e}$ and all ordered bases $\bar{\mathbf{e}}$ of E_z such that $\bar{\mathbf{e}}\Theta_z = \mathbf{e}\Theta_z$. In other words, G is the isotropy group of $\mathbf{e}\Theta_z$ for the obvious action of $\mathrm{GL}(k,\mathrm{I\!R})$ on (p,q) tensors in $\mathrm{I\!R}^k$.

The second one, a G-principal bundle P over M, is contained in the $\mathrm{GL}(k,\mathrm{I\!R})$-principal bundle Q over M naturally associated with E, and the fibre of P over any $x\in M$ consists of the ordered bases $\tilde{\mathbf{e}}$ of E_x having $\tilde{\mathbf{e}}\Theta_x = \mathbf{e}\Theta_z$.

Smoothness of P follows since P is the preimage of the point $\mathbf{e}\Theta_z$ under the submersion $\varPhi : Q \to \varSigma$ sending any ordered basis $\hat{\mathbf{e}}$ of E_x, at any $x\in M$, to $\hat{\mathbf{e}}\Theta_x$, with $\varSigma$ denoting the $\mathrm{GL}(k,\mathrm{I\!R})$-orbit of $\mathbf{e}\Theta_z$ viewed, again, as a (p,q) tensor in $\mathrm{I\!R}^k$. The submersion property of $\varPhi$ is obvious: even the restriction of $\varPhi$ to any fibre Q_x of Q is a submersion, diffeomorphically equivalent to the projection $\mathrm{GL}(k,\mathrm{I\!R}) \to \mathrm{GL}(k,\mathrm{I\!R})/G$.

Thus, a smooth section Θ of $E^{\otimes p}\otimes[E^*]^{\otimes q}$ is parallel relative to some linear connection ∇ in E if and only if it is algebraically constant [**19**, Theorems 1-2], the 'only if' (or, 'if') claim being obvious since M is assumed connected or, respectively, since ∇ induced by any principal G-connection in P clearly makes Θ parallel. Such connections are precisely the linear connections in E characterized by vanishing of their *inner torsion* in the sense of [**23**, Sect. 5].

REMARK 6.1. Our construction depends on the choice of $z\in M$ and an ordered basis $\mathbf{e}$ of E_z. However, different choices lead to equivariantly equivalent objects. The case of importance to us is $E = TM$, where P is the G-structure associated with the given algebraically constant (p,q) tensor Θ. When $(p,q)=(1,1)$ and Θ is nilpotent, *we will always use* z *and* $\mathbf{e}$ *realizing the Jordan normal form* $d_1\ldots d_m$ *of* Θ, defined as in (1.5).

7. The Lie brackets of a local Jordan frame

Recall our convention (1.5) about representing the Jordan normal forms of nilpotent $(1,1)$ tensors in dimension n as weakly decreasing strings $d_1\ldots d_m$ of positive integers, so that $d_1+\ldots+d_m = n$, and $\Theta = 0$ has the Jordan normal form $1\ldots 1$, each 1 being the 1×1 block matrix $[0]$, while n is the single-block Jordan normal form of a *generic* nilpotent $(1,1)$ tensor in dimension n. The Jordan normal form $2\ldots 2$ characterizes the case $\mathrm{Ker}\,\Theta = \mathrm{Im}\,\Theta$.

If an algebraically constant nilpotent $(1,1)$ tensor Θ on an n-manifold has the Jordan normal form $d_1\ldots d_m$, then each subbundle $\mathrm{Ker}\,\Theta^i$ clearly has the fibre dimension $\min(i,d_1)+\ldots+\min(i,d_m)$, and hence

$$\mathrm{rank}\,\Theta^i \,=\, n - \min(i,d_1) - \ldots - \min(i,d_m). \tag{7.1}$$

Let us fix an algebraically constant nilpotent $(1,1)$ tensor Θ on an n-manifold M and a local frame field realizing the Jordan normal form $d_1\ldots d_m$ of Θ. (See Remark 6.1.) We focus on three (not necessarily distinct) Θ-orbits

$$(e_1,\ldots,e_p),\ (\tilde{e}_1,\ldots,\tilde{e}_q),\ (\hat{e}_1,\ldots,\hat{e}_r), \tag{7.2}$$

by which we mean portions of our frame field corresponding to three of the entries $d_1,\ldots,d_m$. Setting $e_i = \tilde{e}_j = \hat{e}_k = 0$ for nonpositive integers i,j,k, we obtain $(\Theta e_i, \Theta\tilde{e}_j, \Theta\hat{e}_k) = (e_{i-1}, \tilde{e}_{j-1}, \hat{e}_{k-1})$ for all integers i,j,k not exceeding, respectively, p,q or r. Finally, we denote by $C_{i,j}^k$ the coefficient of $\hat{e}_k$ in the expansion of the Lie bracket $[e_i,\tilde{e}_j]$ as a (functional) combination of our fixed frame, and also set $C_{i,j}^k = 0$ if $k>r$ or one of i,j,k is nonpositive; thus $C_{i,j}^k$ is well defined for integers i, j, k with $i \le p$ and $j \le q$. Now $N = 0$ in (1.3) if and only if

$$C^k_{i,j} + C^{k-2}_{i-1,j-1} = C^{k-1}_{i-1,j} + C^{k-1}_{i,j-1} \text{ whenever } k \ge 3,\ i \le p \text{ and } j \le q, \tag{7.3}$$

with $C^k_{i,j} = 0$ for $k > r$. Namely, $N(e_i, \tilde{e}_j)$ evaluated from (1.3), and then projected onto the span of $(\hat{e}_1, \dots, \hat{e}_r)$, obviously equals

$$(C^k_{i-1,j} + C^k_{i,j-1})\hat{e}_{k-1} - C^k_{i-1,j-1}\hat{e}_k - C^k_{i,j}\hat{e}_{k-2} \text{ summed over all } k \le r. \tag{7.4}$$

The vanishing of the terms involving $\hat{e}_r$ (or $\hat{e}_{r-1}$, if $r \ge 2$) means that $C^r_{i-1,j-1} = 0$ (or, respectively, $C^{r-1}_{i-1,j-1} = C^r_{i-1,j} + C^r_{i,j-1}$), both of which are special cases of (7.3), with $k \in \{r+1, r+2\}$. Leaving these terms aside, we see that the equality in (7.3) multiplied by $\hat{e}_{k-2}$ follows if we shift the summation index from k or $k-1$ to $k-2$ in the first two terms of (7.4), which yields (7.3) as $\hat{e}_{k-2} = 0$ unless $k \ge 3$. In terms of $E^s_{i,j} = C^{i+j-s+1}_{i,j}$, or $C^k_{i,j} = E^{i+j-k+1}_{i,j}$, (7.3) can be rewritten as

$$E^s_{i,j} + E^s_{i-1,j-1} = E^s_{i-1,j} + E^s_{i,j-1} \text{ if } i+j \ge s+2,\ i \le p \text{ and } j \le q, \tag{7.5}$$

while $E^s_{i,j} = 0$ whenever $i + j \ge s + r$. The reason why we prefer to switch from the integer variables i, j, k to i, j, s with $s = i + j - k + 1$, or $k = i + j - s + 1$, is that (7.5) uses a fixed value of s, allowing us to treat different values of s as completely unrelated. Our conclusions may be summarized as follows.

Lemma 7.1. *Given Θ and the frame field as above, the Nijenhuis tensor* (1.3) *vanishes if and only if, for any ordered triple of not necessarily distinct Θ-orbits* (7.2)*, one has* (7.5) *along with*

$$\begin{array}{l} E^s_{i,j} = 0 \text{ if } i+j < s, \text{ or } i+j \ge s+r, \text{ or } i \le 0, \text{ or } j \le 0, \text{ and} \\ \text{our } E^s_{i,j} \text{ are defined for all } i,j,s \in \mathbb{Z} \text{ such that } i \le p \text{ and } j \le q. \end{array} \tag{7.6}$$

Proof. We already saw that (7.5) is equivalent to (7.3), while (7.6) is clearly nothing else than the obvious boundary conditions ($C^k_{i,j} = 0$ if $k \le 0$, or $k > r$, or $i \le 0$, or $j \le 0$) coupled with our convention about when $C^k_{i,j}$ makes sense. □

Next, $\mathcal{Z}^l = \operatorname{Ker}\Theta^l$ is integrable (which may or may not be the case) if and only if $C^k_{i,j} = 0$ whenever $i, j \le l < k$, that is,

$$E^s_{i,j} = 0 \text{ for all } i,j,s \text{ with } i,j \le l \text{ and } i+j \ge s+l, \tag{7.7}$$

and for all ordered triples of (not necessarily distinct) Θ-orbits (7.2). With the last clause repeated, the integrability of $\mathcal{Z}^l = \operatorname{Ker}\Theta^l$ for *all* $l \ge 0$ clearly amounts to

$$E^s_{i,j} = 0 \text{ whenever } i,j \ge s, \tag{7.8}$$

since the condition $i, j \le i + j - s$ is nothing else than $i, j \ge s$.

Remark 7.2. The equality in (7.5) obviously holds, for all $i, j, s \in \mathbb{Z}$, if $E^s_{i,j}$ is a function of i alone, or of j alone, or equals $i + j$ plus a function of s.

8. Proof of Theorem C: the necessity of (1.6)

For the Jordan normal form of an algebraically constant nilpotent $(1,1)$ tensor, *not* being of type (1.6) clearly means that it

(8.1) contains three different Jordan blocks of lengths p, q, r with $p \le q < r$.

Proposition 8.1. *In any dimension* $n \geq 1$, *the condition* (8.1), *imposed on the Jordan normal form of an algebraically constant nilpotent* $(1,1)$ *tensor* Θ, *implies that the algebraic type of* Θ *is not controlled by the Nijenhuis tensor* (1.3). *More precisely,* Θ *can be realized as a left-invariant* $(1,1)$ *tensor on a Lie group, in such a way that* $N = 0$, *but* $\operatorname{Ker}\Theta^p$ *is nonintegrable for some integer* $p \geq 1$. *One may choose* p *to be the shortest block length in the Jordan normal form of* Θ.

Proof. We identify a local frame field for Θ, chosen as in Sect. 7, with a basis of a Lie algebra $\mathfrak{g}$ formed by left-invariant vector fields on a Lie group G. This is achieved by requiring (7.3) and the boundary conditions ($C_{i,j}^k = 0$ if $k \leq 0$, or $k > r$, or $i \leq 0$, or $j \leq 0$) to be satisfied by *constants* $C_{i,j}^k$ or, equivalently, finding constants $E_{i,j}^s$ with (7.5) – (7.6). (Our choice will cause all brackets to lie in the center, thus implying the Jacobi identity.) Our Θ then becomes a left-invariant $(1,1)$ tensor field on G acting as an endomorphism of the tangent bundle which sends each frame vector field either to the preceding one, or to zero. As a consequence of (8.1), our local frame contains

(8.2) three different Θ-orbits (7.2) of lengths p, q, r with $p \leq q < r$.

Fixing such Θ-orbits, we now set, in the discussion of Sect. 7, $E_{i,j}^s = 0$ for all integers i, j, s, with the exception of (i, j, p) from the set $[1, p] \times [1, q] \times \{p\}$ contained in the range $[1, p] \times [1, q] \times [1, r]$ corresponding to our three Θ-orbits (8.2).

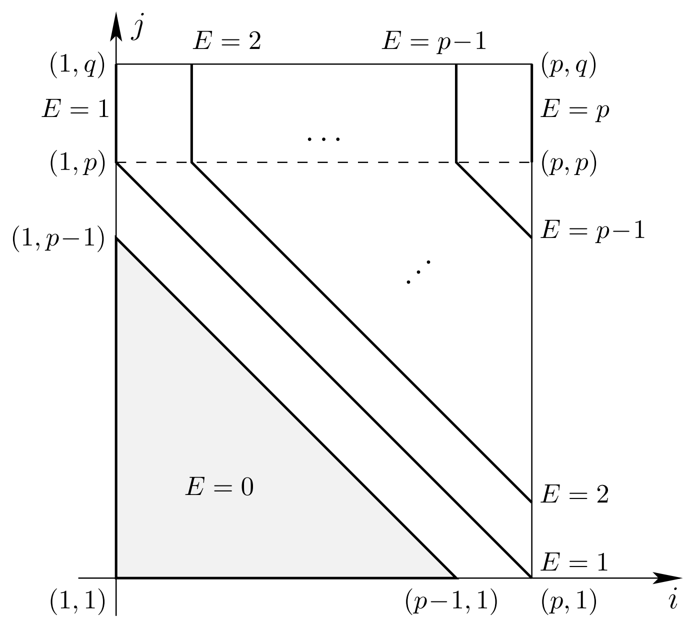

Figure 1. Values of $E = E_{i,j}^p$

Given integers $i \leq p$ and $j \leq q$, we define $E_{i,j}^p$ by

$$E_{i,j}^p = \max(0,\, i + \min(0, j - p)). \tag{8.3}$$

Speaking below of rectangles, triangles, lines and line segments, we always mean their intersections with $\mathbb{Z}^2$, while (sub)rectangles are occasionally reduced to segment or single points. Restricted to (i,j) ranging over the rectangle $[1,p]\times[1,q]$, our $E^p_{i,j}$ equals 0 on the triangle with vertices $(1,1),(1,p-1),(p-1,1)$ (treated as the empty set when $p=1$, or the single point $(1,1)$ for $p=2$), and $E^p_{i,j}=p$ on the segment $\{p\}\times[p,q]$ (a point when $p=q$); the latter claim is obvious, the former immediate from the equality

$$i\,+\,\min(0,j-p)\;=\;\min(i,i+j-p). \tag{8.4}$$

If $p>1$, then, for any $l\in\{1,\dots,p-1\}$, we have $E^p_{i,j}=l$ on the two-segment broken line joining the points $(l,q),(l,p),(p,l)$ (reduced to a segment when $p=q>1$); cf. (8.4). (This is particularly simple for $p=q=1$, with $E^1_{1,1}=1$.)

The corresponding Nijenhuis tensor (1.3) vanishes identically, by Lemma 7.1, since – as we now proceed to show – our $E^p_{i,j}$ satisfy (7.5) and (7.6). First, (7.6) holds, as nonpositivity of i,j or $i+j-s=i+j-p$ in (8.3) yields $E^p_{i,j}=0$, by (8.4), and the remaining implication is vacuous: $i+j\le p+q=s+q<s+r$.

Next, (7.5) "essentially" follows from Remark 7.2: $E^p_{i,j}=\max(0,i)$, which is a function of i, on the subrectangle $[1,p]\times[p,q]$ (a segment when $p=q$). On $[1,p]\times[1,p]$, (8.3) in turn gives $E^p_{i,j}=\max(0,\,i+j-p)$, which coincides with $i+j-p$ on the subtriangle given by $i+j\ge s+2=p+2$.

To dispel any doubts, we now establish (7.5) rigorously, for $s=p$. Of interest to us are integers i,j with $i+j\ge p+2$, $i\le p$ and $j\le q$. We are also free to assume that $i,j\ge 1$, since otherwise, by (7.6), all four terms in (7.5) equal 0. If $j>p$, (8.3) gives $E^p_{i,j}=E^p_{i,j-1}=i$ for all $i\ge 0$. Now the four terms in (7.5) are $i,i-1,i-1,i$ (whenever $i\ge 1$), and the required equality follows. When $j\le p$ (and hence $j-1<p$), given $i\in\mathbb{Z}$, (8.3) reads $E^p_{i,j}=\max(0,\,i+j-p)$ and, similarly, $E^p_{i,j-1}=\max(0,\,i+j-p-1)$ with j replaced by $j-1$. In the case of interest to us, $i+j\ge p+2$ (see the beginning of this paragraph), so that $E^p_{i,j}=i+j-p$ and $E^p_{i,j-1}=i+j-p-1$, for all i, and we get the equality in (7.5): $(i+j-p)+(i-1+j-1-p)=(i-1+j-p)+(i+j-1-p)$.

Finally, since $E^p_{p,p}=p\ne 0$, (7.7) applied to $i=j=s=l=p$ shows that $\operatorname{Ker}\Theta^p$ is not integrable. □

9. Proof of Theorem C: the sufficiency of (1.6)

We now show that, given an algebraically constant nilpotent $(1,1)$ tensor Θ on a manifold M of dimension $n\ge 1$, with $N=0$, and with the Jordan normal form $d_1\dots d_m$ satisfying condition (1.6), Θ must also have the property (i) in Theorem B, and hence be locally constant. To this end, we choose a local frame field realizing the Jordan normal form of Θ. See Remark 6.1.

In the first case of (1.6), $d_1=\ldots=d_m=d$ for some $d\ge 1$, and our local frame field splits into disjoint Θ-orbits of the form $v_1,\dots,v_d$, all of length d, while $v_i=\Theta^{d-i}v_d$ for $i=1,\dots,d-1$, and the final vector v_d lies outside of $\operatorname{Im}\Theta$. Thus, $\mathcal{Z}^i$, $i\ge 0$, is obviously equal to either TM (when $i\ge d$), or to $\mathcal{B}^{d-i}$ (if $1\le i<d$), and (4.4-a) yields our claim.

Consider now the second case of (1.6): $d_1=\ldots=d_{m-1}=d>d'=d_m$ for some $d,d'\ge 1$, with $m>1$, leading to Θ-orbits $v_1,\dots,v_d$ of length d, of which there are $m-1$, and to one Θ-orbit of length d'.

We first prove the integrability of $\mathcal{Z}^i$ when $1 \le i \le d'$, using induction on i. As $\mathcal{Z}^1$ is spanned by the m initial vectors from all Θ-orbits, taken one from each, and $\mathcal{B}^{d-1}$ by the $m-1$ initial vectors from all Θ-orbits of length d, the latter subbundle of TM is contained in the former with codimension one. Thus, (4.4-b) and Lemma 2.1 yield the integrability of $\mathcal{Z}^1$. For the induction step, if $1 \le i < d'$ and $\mathcal{Z}^i$ is integrable, $\mathcal{Z}^{i+1}$ is spanned by $m(i+1)$ vectors: $v_1, \ldots, v_{i+1}$ from all the Θ-orbits combined (if one writes the Θ-orbits as $v_1, \ldots, v_d$ or $v_1, \ldots, v_{d'}$), and so $\mathcal{Z}^{i+1}$ contains, with codimension one, the span $\mathcal{Z}^i + \mathcal{B}^{d-i-1}$ of $\mathcal{Z}^i$ and $\mathcal{B}^{d-i-1}$. By (4.4-b) and (4.4-d) $[\mathcal{Z}^i + \mathcal{B}^{d-i-1}, \mathcal{Z}^{i+1}] \subseteq \mathcal{Z}^{i+1} + \mathcal{B}^{d-i-1} \subseteq \mathcal{Z}^{i+1}$, and Lemma 2.1 completes the induction step.

Finally, let $d' < i < d$ and $k = d' - 1$. This time $\mathcal{Z}^i$ contains $\mathcal{Z}^k + \mathcal{B}^{d-i}$ with codimension one: the former is spanned by $(m-1)i + d'$ vectors (the initial i ones from all Θ-orbits of length d, plus the whole Θ-orbit of length d'), the latter – by the same vectors except the last one in the length d' orbit. Once again, (4.4-b) and (4.4-d) give $[\mathcal{Z}^k + \mathcal{B}^{d-i}, \mathcal{Z}^i] \subseteq \mathcal{Z}^i + \mathcal{B}^{d-i} \subseteq \mathcal{Z}^i$, and we can use Lemma 2.1.

10. Generalized almost-tangent structures

The following construction provides – as shown below – a local description of all algebraically constant $(1,1)$ tensors Θ such that $\Theta^2 = 0$ and the Nijenhuis tensor (1.3) vanishes identically.

Given a distribution $\mathcal{D}$ on a manifold Σ, let M be the total space of an affine bundle over Σ associated with the quotient vector bundle $T\Sigma/\mathcal{D}$. Using the bundle projection $\pi : M \to \Sigma$ and the quotient-bundle projection morphism $T\Sigma \ni v \mapsto [v] \in T\Sigma/\mathcal{D}$, we define a $(1,1)$ tensor Θ on M by

(10.1) $$\Theta_x v = [d\pi_x v] \in T_y\Sigma/\mathcal{D}_y = T_x M_y, \text{ if } x \in M_y = \pi^{-1}(y),$$

whenever $x \in M$ and $v \in T_x M$. Then $\Theta^2 = 0$, since all Θ-images are vertical. Also, $N = 0$ in (1.3). In fact, $\mathrm{Im}\,\Theta$ is the vertical distribution $\mathcal{V} = \mathrm{Ker}\,d\pi$. Evaluating (1.3), withous loss of generality, on π-projectable vector fields, we see that, by (2.3), the first, second and fourth terms on the right-hand side of vanish as $\Theta^2 = 0$. So does the third term: Θv, Θw restricted to each fibre are affine-space translations, and consequently commute.

Theorem 10.1. *Every algebraically constant* $(1,1)$ *tensor* Θ *with* $\Theta^2 = 0$ *and vanishing Nijenhuis tensor* (1.3) *arises, locally, from the above construction, and the fibre dimension of* $\mathcal{D}$ *equals the codimension of* $\mathrm{Im}\,\Theta$ *in* $\mathrm{Ker}\,\Theta$*, while*

(10.2) Θ is integrable if and only if so is the distribution $\mathcal{D}$.

Proof. Suppose that $\Theta^2 = 0$ and $N = 0$ in (1.3). By (4.4-a), $\mathrm{Im}\,\Theta$ is an integrable distribution, while $\mathrm{Im}\,\Theta \subseteq \mathrm{Ker}\,\Theta$. Due to (2.3) and (1.3) with $\Theta^2 = 0$,

(10.3) any two $(\mathrm{Im}\,\Theta)$-projectable vector fields have commuting Θ-images.

By (4.4-b) for $i = j = 1$ and Lemma 2.2(c), on an open set $M' \subseteq M$ with a bundle projection $\pi : M' \to \Sigma$ having $\mathrm{Im}\,\Theta$ as the vertical distribution, $\mathrm{Ker}\,\Theta$ is π-projectable onto a distribution $\mathcal{D}$ on Σ, with (10.2) obvious from Theorem B and (2.2). Any π-projectable lift, along the fibre $\pi^{-1}(y)$, of any vector w tangent to Σ at $y \in \Sigma$, is mapped by Θ onto the "vertical lift" of w, a vector field tangent to $\pi^{-1}(y)$, which vanishes precisely when w is tangent to $\mathcal{D}$. By (10.3) the vertical lifts of any $w, w' \in T_y\Sigma$ commute. This turns $\pi^{-1}(y)$, locally, into an affine space having the translation vector space $T_y\Sigma/\mathcal{D}_y$, with Θ given by (10.1). □

Theorem 10.1 illustrates a special case of Theorem C: the condition $\Theta^2 = 0$ corresponds to the Jordan normal forms $2\dots2$ and $2\dots21\dots1$ (plus $1\dots1$, for $\Theta = 0$). Of these, only $2\dots2$, $2\dots21$ and $1\dots1$ satisfy (1.6), reflecting the fact that $\mathcal{D}$ is necessarily integrable only if it has the fibre dimension $0, 1$ or $\dim\Sigma$.

When $\operatorname{Ker}\Theta = \operatorname{Im}\Theta$, that is, $\mathcal{D}$ is the zero distribution, our construction gives rise to what is referred to as *almost-tangent structures* [**31, 11**], and Theorem 10.1 becomes the local version of [**9**, Theorem on p. 69].

11. Differential q-forms on an n-manifold, $q = 0, 1, 2, n-1, n$

We now prove Proposition D. Let ζ be an algebraically constant differential q-form on an n-dimensional manifold, $q = 0, 1, 2, n-1, n$, with $d\zeta = 0$ (the last condition being obviously redundant if $q = n$ or – as ζ is constant – if $q = 0$).

The cases $q = 0$ and $q = 1$ are obvious: the 1-form ζ (if nonzero), being locally exact, equals dx^1 in suitable local coordinates $x^1, \dots, x^n$.

When $q = 2$, algebraic constancy amounts to constant rank, and our claim follows as Darboux's theorem [**6**, p. 40] gives $\zeta = dx^1\wedge dx^2 + \dots + dx^{2r-1}\wedge dx^{2r}$ in some local coordinates $x^1, \dots, x^n$, with $2r = \operatorname{rank}\zeta \ge 0$.

If $q = n$ and $\zeta \ne 0$, we have, in suitable local coordinates $x^1, \dots, x^n$,

$$\zeta = dx^1\wedge dx^2\wedge\dots\wedge dx^n, \text{ where } x^2, \dots, x^n \text{ can be arbitrary,} \tag{11.1}$$

as long as $dx^2\wedge\dots\wedge dx^n \ne 0$. In fact, starting from $\zeta = \phi\, dx^1\wedge dx^2\wedge\dots\wedge dx^n$ for a function ϕ without zeros, and choosing ψ with $\partial_1\psi = \phi$, we see that $d\psi$ equals $\phi\, dx^1$ plus a functional combination of $dx^2, \dots, dx^n$ and so $\zeta = d\psi\wedge dx^2\wedge\dots\wedge dx^n$.

Finally, let $q = n-1$. Assuming ζ to be nonzero, and fixing a nonzero n-form ω, we get $\zeta = \omega(v, \cdot, \dots, \cdot)$, for a unique (nonzero) vector field v. Then, by (11.1), $\omega = dx^1\wedge dx^2\wedge\dots\wedge dx^n$ in some local coordinates $x^1, \dots, x^n$, with $x^2, \dots, x^n$ chosen so that $dx^2(v) = \dots = dx^n(v) = 0$. Now $\zeta = \chi\, dx^2\wedge\dots\wedge dx^n$ for $\chi = dx^1(v)$, and $\partial_1\chi = 0$ as $d\zeta = 0$. Our ζ, being thus a top-degree form in $n-1$ variables, equals, by (11.1), $dy^2\wedge\dots\wedge dy^n$ in suitable coordinates $y^1, \dots, y^n$.

12. Differential forms of other degrees

We now proceed to verify the statement preceding formula (1.7). The algebraic constancy of ζ is clear as $\zeta = (\xi^1\wedge\xi^2 + \xi^3\wedge\xi^4)\wedge\xi^5\wedge\dots\wedge\xi^{q+2}$, with linearly independent 1-forms $\xi^1, \dots, \xi^{q+2}$, and its closedness since $d\zeta$ is the exterior product of $(dx^1\wedge dx^2 + dx^3\wedge dx^4)\wedge(dx^1\wedge dx^2 - dx^3\wedge dx^4)$ (obviously equal to 0) and $dx^6\wedge\dots\wedge dx^{q+2}$. Being algebraically constant, ζ gives rise to the vector subbundle $\mathcal{F}$ of T^*M such that the sections of $\mathcal{F}$ are those 1-forms ξ for which $\xi\wedge\zeta = 0$. The sections ξ of $\mathcal{F}$ also coincide with functional combinations of the 1-forms

$$\eta,\ dx^6, \dots,\ dx^{q+2}, \text{ where } \eta = dx^5 + x^1dx^2 - x^3dx^4. \tag{12.1}$$

In fact, writing $\xi = \xi_i\, dx^i$, we see that $\xi\wedge\zeta$ contains no contributions from the terms $\xi_i\, dx^i$ (no summation) with $6 \le i \le q+2$ (making $\xi_6, \dots, \xi_{q+2}$ completely arbitrary) while for $\theta = (dx^1\wedge dx^2 + dx^3\wedge dx^4)\wedge(dx^5 + x^1dx^2 - x^3dx^4)$ one has

$$\theta = dx^1\wedge dx^2\wedge dx^5 + dx^3\wedge dx^4\wedge dx^5 - x^3dx^1\wedge dx^2\wedge dx^4 + x^1dx^2\wedge dx^3\wedge dx^4.$$

and so each term $\xi_i dx^i$ (no summation, again) with $q+2 < i \le n$ contributes to $\xi\wedge\zeta$ the expression $\xi_i dx^i\wedge\theta\wedge dx^6\wedge\ldots\wedge dx^{q+2}$ (no summation) comprising all the terms in $\xi\wedge\zeta$ involving the factor dx^i. Linear independence of the differentials $dx^1,\ldots,dx^n$ now gives $\xi_i = 0$ whenever $q+2 < i \le n$. Finally, the exterior products of $\xi_1 dx^1$, $\xi_2 dx^2$, $\xi_3 dx^3$, $\xi_4 dx^4$, $\xi_5 dx^5$ with θ are

$$\begin{aligned}
&\xi_1(dx^1\wedge dx^3\wedge dx^4\wedge dx^5 + x^1 dx^1\wedge dx^2\wedge dx^3\wedge dx^4),\\
&\xi_2(dx^2\wedge dx^3\wedge dx^4\wedge dx^5),\\
&\xi_3(dx^1\wedge dx^2\wedge dx^3\wedge dx^5 - x^3 dx^1\wedge dx^2\wedge dx^3\wedge dx^4),\\
&\xi_4(dx^1\wedge dx^2\wedge dx^4\wedge dx^5),\\
&\xi_5(x^3 dx^1\wedge dx^2\wedge dx^4\wedge dx^5 - x^1 dx^2\wedge dx^3\wedge dx^4\wedge dx^5).
\end{aligned}\tag{12.2}$$

The condition $\xi\wedge\zeta = 0$ means, after the cancellation of $dx^6\wedge\ldots\wedge dx^{q+2}$, that the sum of the five lines of (12.2) equals 0. Writing $[ijkl]$ for $dx^i\wedge dx^j\wedge dx^k\wedge dx^l$, we see that [1345] and [1235] occur just once each, giving $\xi_1 = \xi_3 = 0$, while the sum of the remaining three lines equals $(\xi_4+\xi_5 x^3)[1245]+(\xi_2-\xi_5 x^1)[2345]$. Thus, $\xi_4+\xi_5 x^3 = 0 = \xi_2-\xi_5 x^1$, and the sum of $\xi_i dx^i$ over $i = 1,\ldots,5$ equals a function times the 1-form η in (12.1), proving our claim about (12.1).

If ζ were integrable, so would be – according to (1.2) – the simultanous kernel of the 1-forms (12.1) (that is, of all sections of $\mathcal{F}$), naturally determined by ζ. This is not the case, as $d\eta\wedge\eta\wedge dx^6\wedge\ldots\wedge dx^{q+2}$ is nonzero, being equal to $(dx^1\wedge dx^2\wedge dx^5 - dx^3\wedge dx^4\wedge dx^5 - x^3 dx^1\wedge dx^2\wedge dx^4 - x^1 dx^2\wedge dx^3\wedge dx^4)\wedge dx^6\wedge\ldots\wedge dx^{q+2}$.

13. Symmetric (0,2) and (2,0) tensors

Necessity and sufficiency of (1.8). Let g be integrable, with $\nabla g = 0$ for a fixed torsion-free connection ∇. The integrability of the distribution $\mathcal{V} = \operatorname{Ker} g$, due to (1.2), allows us to choose local coordinates and index ranges for i, a, λ, μ, ν as in Remark 2.5, so that $\mathcal{V}$ is spanned by the coordinate vector fields ∂_a. As $\mathcal{V}$ is obviously ∇-parallel, $\Gamma_{ia}^k = \Gamma_{ab}^k = 0$, while $g_{ia} = g_{ab} = 0$, so that $\partial_a g_{ij} = \Gamma_{ai}^k g_{kj} + \Gamma_{aj}^k g_{ik} = 0$, and projectability of g along $\mathcal{V}$ follows from (2.4).

Conversely, suppose that g is projectable along the integrable distribution $\mathcal{V} = \operatorname{Ker} g$. As before, we invoke Remark 2.5, selecting local coordinates with index ranges for i, a, λ, μ, ν so as to make $\mathcal{V}$ the span of the coordinate fields ∂_a. As projectability of g along $\mathcal{V}$ gives $\partial_a g_{ij} = 0$, while $g_{ia} = g_{ab} = 0$, the components g_{ij} represent a pseudo-Riemannian metric in the factor manifold with the coordinates x^i. Denoting by Γ_{ij}^k the components of its Levi-Civita connection, we now use Remark 2.5 to define the required torsion-free connection ∇ with $\nabla g = 0$. □

Proof of Proposition E. Integrability of the former implies that of the latter by (1.2). Conversely, let the distribution $\mathcal{B} = \operatorname{Im}\Theta$ be integrable. Using Remark 2.5, we fix local coordinates and index ranges for i, a, λ, μ, ν so that $\mathcal{B}$ is the span of the coordinate fields ∂_i. Thus, $\Theta^{ia} = \Theta^{ab} = 0$, as the 1-forms dx^a annihilate each ∂_i, and hence are sections of the subbundle $\operatorname{Ker}\Theta \subseteq T^*M$. On each leaf of $\mathcal{B}$, the restriction of Θ is nondegenerate – see (2.8) – and so it is the reciprocal of a pseudo-Riemannian metric on the leaf. Its Levi-Civita connection, with the components Γ_{ij}^k (possibly depending on the variables x^a), makes the restriction of Θ parallel. Thus, we may again invoke Remark 2.5 to obtain a torsion-free connection ∇ such that $\nabla\Theta = 0$. □

14. Local constancy of bivector fields

Proof of Proposition F. The ‘only if’ part is immediate: for a torsion-free connection ∇ on the given manifold having $\nabla\Theta = 0$, the distribution $\mathcal{B} = \operatorname{Im}\Theta$ is ∇-parallel and hence integrable, cf. (1.2), and the torsion-free connections induced by ∇ on the leaves of $\mathcal{B}$ make the restriction of Θ to each leaf parallel, which implies the same (and hence also closedness) for their inverses.

Let us now assume that $\mathcal{B} = \operatorname{Im}\Theta$ is integrable and the inverses of the restrictions of Θ to the leaves of $\mathcal{B}$ are all closed. These inverses are symplectic forms ζ on the leaves, and the Darboux theorem with parameters [**1**, Lemma 3.10] allows us to choose functions x^i which, restricted to each leaf, form local coordinates with $\zeta = dx^1 \wedge dx^2 + \ldots + dx^{2r-1} \wedge dx^{2r}$, where $2r = \operatorname{rank}\zeta = \operatorname{rank}\Theta$. We may also choose functions x^a, with the index ranges $1 \le i \le 2r < a \le \dim M$, such that the differentials dx^a form a local trivialization of the subbundle $\operatorname{Ker}\Theta \subseteq T^*M$. In the resulting product coordinates x^i, x^a the components of Θ are all constant: $\Theta^{ia} = \Theta^{ab} = 0$, while $\Theta^{ij} = 1$ (or, $\Theta^{ij} = -1$) if (j,i), or (i,j), is one of the pairs $(1,2), (3,4), \ldots, (2r-1, 2r)$, and $\Theta^{ij} = 0$ otherwise. $\square$

15. Integrability of the kernels and images

For any vector bundle $\mathcal{L}$ over a manifold M and a vector-bundle morphism $\Theta : TM \to \mathcal{L}^*$ of constant rank r into its dual $\mathcal{L}^*$, the resulting dual morphism $\Theta^* : \mathcal{L} \to T^*M$, which also has $\operatorname{rank}\Theta^* = r$, gives rise to a tensor-like object $\widetilde{N}$ (specifically, a section of $\operatorname{Hom}(\mathcal{L}\otimes\mathcal{L}^{\wedge r}, [T^*M]^{\wedge(r+2)})$), sending sections $v, v_1, \ldots, v_r$ of $\mathcal{L}$ to the $(r+2)$-form

$$\widetilde{N}(v, v_1, \ldots, v_r) = [d(\Theta^* v)] \wedge \Theta^* v_1 \wedge \ldots \wedge \Theta^* v_r. \tag{15.1}$$

Here $d[\Theta^*(fv)] = f d(\Theta^* v) + df \wedge \Theta^* v$ for a function f. However, $\widetilde{N}$ itself is tensorial: the nontensorial term $df \wedge \Theta^* v$ in the last equality has zero exterior product with $\Theta^* v_1 \wedge \ldots \wedge \Theta^* v_r$, since $\operatorname{rank}\Theta^* = r$. Furthermore,

$$\widetilde{N} = 0 \ \text{ identically if and only if } \operatorname{Ker}\Theta \text{ is integrable.} \tag{15.2}$$

In fact, as $\operatorname{Ker}\Theta$ is the simultanous kernel of the 1-forms $\Theta^* v$, for all sections v of $\mathcal{L}$, its integrability amounts to d-closedness of the ideal generated by all such 1-forms which, as $\operatorname{rank}\Theta^* = r$, is nothing else than the vanishing of $\widetilde{N}$.

In the case of $\mathcal{L} = TM$ and a (possibly nonsymmetric) $(0,2)$ tensor g of constant rank r on M, treated as a morphism $\Theta : TM \to T^*M$ sending a vector field w to the 1-form $g(\,\cdot\,, w)$, the dual Θ^* acts via $v \mapsto g(v, \,\cdot\,)$. Then

$$\widetilde{N} \text{ in (15.1) becomes } N' \text{ in (1.9-a), so that } N' \text{ is tensorial.} \tag{15.3}$$

Let Θ now be an algebraically constant nilpotent $(1,1)$ tensor on an n-manifold with the Jordan normal form $d_1 \ldots d_m$, cf. (1.5). Integrability of Θ, as well as its local constancy, is equivalent, by Theorem B, to the simultaneous vanishing of the Nijenhuis tensor N in (1.3) along with further $d_1 - 1$ Nijenhuis-type tensors N^i, where $1 \le i < d_1$, such that $N^i = 0$ if and only if $\mathcal{Z}^i = \operatorname{Ker}\Theta^i$ is integrable. Specifically, this follows from (15.2) if we define N^i to be $\widetilde{N}$ in (15.1) with $\mathcal{L} = TM$ and Θ replaced by Θ^i, where r equals (7.1), and a fixed Riemannian metric on M has been used to identify TM with T^*M, thus turning each Θ^i separately into a vector-bundle morphism $TM \to T^*M = \mathcal{L}^*$.

Finally, given a (skew)symmetric $(2,0)$ tensor Θ of constant rank r on a manifold M, we associate with Θ a Nijenhuis-type $(2r+3,0)$ tensor $\widehat{N}$, testing the integrability of the image distribution $\mathcal{V} = \mathrm{Im}\,\Theta \subseteq TM$. (Note that Θ is a bundle morphisms $T^*M \to TM$ acting via $\xi \mapsto \Theta\xi = \Theta(\,\cdot\,,\xi)$.) To define $\widehat{N}$, we again fix a Riemannian metric on M, which allows us to use contractions and the Hodge star operator $*$ (as the latter enters our formula quadratically, M need not be oriented). With $\Theta\xi = \Theta(\,\cdot\,,\xi)$ as above, for 1-forms ξ on M, we set

$$\widehat{N}(\xi,\xi^1,\dots,\xi^r,\eta,\eta^1,\dots,\eta^r) \;=\; \Omega[\Theta\xi,\Theta\eta], \tag{15.4}$$

where $\xi,\xi^1,\dots,\xi^r,\eta,\eta^1,\dots,\eta^r$ are any 1-forms on M, and

(15.5) Ω denotes the result of an $(r-1)$-fold contraction (2.9) of $*(\Theta\xi^1\wedge\ldots\wedge\Theta\xi^r)$ against $*(\Theta\eta^1\wedge\ldots\wedge\Theta\eta^r)$.

Clearly, at points where the r-tuples $\Theta\xi^1,\dots,\Theta\xi^r$ and $\Theta\eta^1,\dots,\Theta\eta^r$ of vector fields are both linearly independent, Ω is, by Remark 2.4, a nonzero functional multiple of the orthogonal projection onto the orthogonal complement of $\mathcal{V}$ and, applied to the Lie brackets $[\Theta\xi,\Theta\eta]$, tests the integrability of $\mathcal{V}$.

16. Twice-covariant symmetric tensors

The tensoriality of N' in (1.9-a) was established in (15.3). For N'', since

$$[\pounds_{\phi v}g](w,u) = \phi[\pounds_v g](w,u) + (d_w\phi)g(v,u) + (d_u\phi)g(w,v) \tag{16.1}$$

for any function ϕ on the given manifold M, the resulting nontensorial contribution to (1.9-b) equals the sum $(d_w\phi)g(u,\,\cdot\,) + (d_u\phi)g(w,\,\cdot\,)$ of the last two terms in (16.1). Its exterior product with $g(v_1,\,\cdot\,)\wedge\ldots\wedge g(v_r,\,\cdot\,)$ vanishes, being a sum of $(r+1)$-fold exterior products of sections of a rank r subbundle of T^*M, namely, the image of the morphism sending each vector field v to $g(v,\,\cdot\,)$.

Proof of Theorem G. We derive our conclusion from (1.8), by showing that the vanishing of N' (or N''), is equivalent to the integrability of the distribution $\mathcal{V} = \mathrm{Ker}\,g$ (or, respectively, to projectability of g along $\mathcal{V}$).

The first of these claims is obvious from (15.2) and (15.3). It thus obviously suffices to show that *the second equivalence holds if $\mathcal{V}$ is integrable.*

Clearly, with $\mathcal{V} = \mathrm{Ker}\,g$ from now on assumed integrable,

(16.2) $N'' = 0$ if and only if $N''(w,u,v_1,\dots,v_r) = 0$ for all local vector fields $w,u,v_1,\dots,v_r$ projectable along $\mathcal{V}$.

Although $[\pounds g](w,u)$ in (1.9-b) is not a genuine 1-form on the manifold M in question, we now artificially turn it into one, by fixing a local trivialization of TM, containing a local trivialization of $\mathcal{V}$, and declaring $[\pounds g](w,u)$ to be 1-form acting by $v \mapsto [\pounds_v g](w,u)$ on our selected (finitely many) vector fields v trivializing TM. As $[\pounds_v g](w,u) = d_v[g(w,u)] - g([v,w],u) - g(w,[v,u])$, projectability of w,u and (2.3) imply that

$$[\pounds_v g](w,u) = d_v[g(w,u)] \text{ whenever } v \text{ is a section of } \mathcal{V} = \mathrm{Ker}\,g. \tag{16.3}$$

If $N'' = 0$, the $(r+1)$-form $\zeta = N''(w,u,v_1,\dots,v_r)$ vanishes, and hence so does $d_v[g(w,u)]$ in (16.3), for sections v of $\mathcal{V}$, as $\zeta(v,\,\cdot\,,\dots,\,\cdot\,)$ equals $d_v[g(w,u)]$ times the the exterior product $g(v_1,\,\cdot\,)\wedge\ldots\wedge g(v_r,\,\cdot\,)$ (and the latter r-form may be chosen nonzero since $\mathrm{rank}\,g = r$). Thus, by (2.4), g is projectable along $\mathcal{V}$.

Conversely, let us assume projectability of g along $\mathcal{V}$. Now in (16.2) – (16.3) $d_v[g(w,u)] = 0$, and hence $[\pounds_v g](w,u) = 0$ for all sections v of $\mathcal{V}$. The 1-form $[\pounds g](w,u)$ vanishes on $\mathcal{V} = \mathrm{Ker}\, g$, and so obviously do $g(v_1,\,\cdot\,),\ldots,g(v_r,\,\cdot\,)$ in (16.2). As $\mathrm{rank}\, g = r$, the 1-forms vanishing on $\mathcal{V}$ constitute a vector subbundle of fibre dimension r in T^*M. Thus, $N'' = 0$ by (16.2). □

References

[1] S. Bandyopadhyay, B. Dacorogna, V. S. Matveev and M. Troyanov, *Bernhard Riemann 1861 revisited: existence of flat coordinates for an arbitrary bilinear form*, Math. Z. **305** (2023), art. 12, 25 pp.

[2] A. Bejancu, *Curvature in sub-Riemannian geometry*, J. Math. Phys. **53** (2012), art. 023513, 25 pp.

[3] A. M. Blaga and A. Nannicini, *Statistical structures, α-connections and generalized geometry*, Riv. Math. Univ. Parma (N.S.) **13** (2022) 283–296,

[4] A.V. Bolsinov, A.Y. Konyaev and V. S. Matveev, *Nijenhuis geometry*, Adv. Math. **394** (2022), art. 108001, 52 pp.

[5] C. Boubel, *An integrability condition for fields of nilpotent endomorphisms*, preprint, 2011.

[6] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt and P. A. Griffiths, *Exterior Differential Systems*, Math. Sci. Res. Inst. Publ. **18**, Springer-Verlag, New York, 1991.

[7] R. S. Clark and M. Bruckheimer, *Tensor structures on a differentiable manifold*, Ann. Mat. Pura Appl. (4) **54**(1961), 123–141.

[8] M. Crampin, *On differentiable manifolds with degenerate metrics*, Proc. Cambridge Philos. Soc. **64** (1968), 307–316,

[9] M. Crampin and G. Thompson, *Affine bundles and integrable almost tangent structures*, Math. Proc. Cambridge Philos. Soc. **98** (1985), 61–71.

[10] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and its Applications, **364**, Kluwer Academic Publishers Group, Dordrecht, 1996.

[11] D. S. Goel, *Almost tangent structures*, Kodai Math. Sem. Rep. **26** (1974/75), 187–193.

[12] J. Grifone and M. Mehdi, *Existence of conservation laws and characterization of recursion operators for completely integrable systems*, Trans. Amer. Math. Soc. **349** (1997), 4609–4633.

[13] M. Grochowski and W. Kryński, *On contact sub-pseudo-Riemannian isometries*, ESAIM Control Optim. Calc. Var. **23** (2017), 1751–1765.

[14] J. M. Hernando, E. Reyes and P. M. Gadea, *Integrability of tensor structures of electromagnetic type*, Publ. Inst. Math. (Beograd) (N.S.), **37** (1985). 113–122.

[15] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 3rd edition, Grad. Texts in Math. **9**, Springer-Verlag, New York-Berlin, 1980.

[16] E. T. Kobayashi, *A remark on the Nijenhuis tensor*, Pacific J. Math. **12** (1962), 963–977.

[17] S. Kobayashi, *Transformation Groups in Differential Geometry*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[18] Y. Kosmann-Schwarzbach, *Beyond recursion operators*, in: Geometric Methods in Physics XXXVI, Trends Math., 167–180, Birkhäuser/Springer, Cham, 2019.

[19] M. Kurita, *Tensor fields and their parallelism*, Nagoya Math. J. **18** (1961), 133–151.

[20] G. C. Moisil, *Sur les geodésiques des espaces de Riemann singuliers*, Bull. Math. Soc. Roumaine Sci. **42** (1940), 33–52.

[21] A. Nijenhuis, *X_{n-1}-forming sets of eigenvectors*, Indag. Math., **13** (1951), 200–212. Nederl. Akad. Wetensch. Proc. Ser. A 54.

[22] V. Oproiu, *Degenerate Riemannian and degenerate conformal connexions*, An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **16** (1970), 357–376.

[23] P. Piccione and D.V. Tausk, *An existence theorem for G-structure preserving affine immersions*, Indiana Univ. Math. J. **57** (2008), 1431–1465.

[24] D. Reyes, P. Tempesta and G. Tondo, *Generalized Nijenhuis torsions and block-diagonalization of operator fields*, J. Nonlinear Sci. **33** (2023) art. 35, 24 pp.

[25] V. Studený, *General Nijenhuis tensor. An example of a secondary invariant*, J. Bureš et al. (eds.), Proceedings of the Winter School on geometry and physics, Srní, Czech Republic,

January 1994. Palermo: Circolo Matematico di Palermo, Suppl. Rend. Circ. Mat. Palermo, II, Ser. 39, 133–141 (1996).
[26] K. Sulik, *A note on degenerate pseudo-Riemannian manifolds*, Univ. Iagel. Acta Math. **24** (1984), 67–75.
[27] P. Tempesta and G. Tondo, *Haantjes algebras and diagonalization*, J. Geom. Phys. **160** (2021), art. 103968, 21 pp.
[28] G. Thompson, *The integrability of a field of endomorphisms*, Math. Bohem. **127** (2002), 605–611.
[29] P. Tondeur, *Affine Zusammenhänge auf Mannigfaltigkeiten mit fast-symplektischer Struktur*, Comment. Math. Helv. **36** (1961), 234–244.
[30] G. Vrănceanu, *Sur les invariants des espaces de Riemann singuliers*, Disquisit. Math. Phys. **2** (1942), 253–281.
[31] K. Yano and E, T. Davies, *Differential geometry on almost tangent manifolds*, Ann. Mat. Pura Appl. (4) **103** (1975), 131–160.

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