Curvature spectra of simple Lie groups

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Abstract The Killing form β of a real (or complex) semisimple Lie group G is a left-invariant pseudo-Riemannian (or, respectively, holomorphic) Einstein metric. Let Ω denote the multiple of its curvature operator, acting on symmetric 2-tensors, with the factor chosen so that $\Omega\beta = 2\beta$. We observe that the result of Meyberg [8], describing the spectrum of Ω in complex simple Lie groups, easily leads to an analogous description for real simple Lie groups. In particular, 1 is not an eigenvalue of Ω in any real or complex simple Lie group G except those locally isomorphic to $\mathrm{SL}(n,\mathbb{C})$ or one of its real forms. As shown in our recent paper [6], the last conclusion implies that, on such simple Lie groups G, nonzero multiples of the Killing form β are isolated among left-invariant Einstein metrics. Meyberg's theorem also allows us to understand the kernel of Λ, which is another natural operator. This in turn leads to a proof of a known, yet unpublished, fact: namely, that a semisimple real or complex Lie algebra with no simple ideals of dimension 3 is essentially determined by its Cartan three-form.

Keywords Simple Lie group · indefinite Einstein metric · left-invariant Einstein metric · Cartan three-form

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1 Introduction

Every real Lie group G carries a distinguished left-invariant torsionfree connection D, defined by $D_x y = [x,y]/2$ for all left-invariant vector fields x and y. In view of the Jacobi

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identity, the curvature tensor of D is D-parallel, and hence so is the Ricci tensor of D, equal to a nonzero multiple of the Killing form β. Our convention about β reads

$$\beta(x,x) = \mathrm{tr}\,[\mathrm{Ad}\,x]^2 \quad \text{for any } x \text{ in the Lie algebra } \mathfrak{g} \text{ of } G. \tag{1.1}$$

Thus, if G is semisimple, β constitutes a bi-invariant, locally symmetric, non-Ricci-flat pseudo-Riemannian Einstein metric on G, with the Levi-Civita connection D. We denote by $\Omega : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\odot 2}$ a specific multiple of the curvature operator of the metric β, acting on symmetric bilinear forms $\sigma : \mathfrak{g}\times\mathfrak{g} \to \mathrm{I\!R}$, so that, whenever $x,y \in \mathfrak{g}$,

$$\begin{array}{ll} \text{a)} & [\Omega\sigma](x,y) = 2\,\mathrm{tr}\,[(\mathrm{Ad}\,x)(\mathrm{Ad}\,y)\Sigma], \quad \text{for } \Sigma : \mathfrak{g}\to\mathfrak{g} \text{ with} \\ \text{b)} & \sigma(x,y) = \beta(\Sigma x, y). \end{array} \tag{1.2}$$

See Remark 2.5. The same formula (1.2) defines the operator Ω in a *complex* semisimple Lie group G, acting on symmetric complex-bilinear forms σ. We then identify Ω with the analogous curvature operator for the ($\mathbb{C}$-bilinear) Killing form β, treating the latter as a holomorphic Einstein metric on the underlying complex manifold of G.

The structure of Ω in complex simple Lie groups is known from the work of Meyberg [8], who showed that Ω is diagonalizable and described its spectrum. For the reader's convenience, we reproduce Meyberg's theorem in an appendix. His result easily leads to a similar description of the spectrum of Ω in real simple Lie algebras $\mathfrak{g}$, which we state as Theorem 4.1 and derive in Section 4 from the fact that, given any such $\mathfrak{g}$,

$$\begin{array}{ll} \text{a)} & \text{either } \mathfrak{g} \text{ is a real form of a complex simple Lie algebra } \mathfrak{h}, \text{ or} \\ \text{b)} & \mathfrak{g} \text{ arises by treating a complex simple Lie algebra } \mathfrak{h} \text{ as real.} \end{array} \tag{1.3}$$

See [7, Lemma 4 on p. 173]. The Lie-algebra isomorphism types of real simple Lie algebras $\mathfrak{g}$ thus form two disjoint classes, characterized by (1.3.a) and (1.3.b).

For both real and complex semisimple Lie groups G, studying Ω can be further motivated as follows. Let 'metrics' on G be, by definition, pseudo-Riemannian or, respectively, holomorphic, and $\mathscr{E}$ denote the set of Levi-Civita connections of left-invariant Einstein metrics on G. Then, as shown in [6, Theorem 12.3], whenever a semisimple Lie group G has the property that 1 is *not* an eigenvalue of Ω, the Levi-Civita connection D of its Killing form β is an isolated point of $\mathscr{E}$. The converse implication holds except when G is locally isomorphic to $\mathrm{SU}(n)$, with $n \geq 3$. See [6, Theorems 22.2 and 22.3].

In a real/complex Lie algebra $\mathfrak{g}$, we define $\Lambda : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\wedge 4}$ by

$$(\Lambda\sigma)(x,y,z,z') = \sigma([x,y],[z,z']) + \sigma([y,z],[x,z']) + \sigma([z,x],[y,z']). \tag{1.4}$$

Thus, Λ is a real/complex-linear operator, sending symmetric bilinear forms σ on $\mathfrak{g}$ to exterior 4-forms on $\mathfrak{g}$. The Killing form β has $\beta([x,y],[z,z']) = \beta([[x,y],z],z')$, as $\mathrm{Ad}\,z$ is β-skew-adjoint, and so, by the Jacobi identity and (1.1) – (1.2.a),

$$\text{i)} \quad \Lambda\beta = 0, \qquad \text{ii)} \quad \text{if } \mathfrak{g} \text{ is semisimple}, \quad \Omega\beta = 2\beta. \tag{1.5}$$

For semisimple Lie algebras $\mathfrak{g}$ there is also the operator $\Pi : [\mathfrak{g}^*]^{\otimes 4} \to [\mathfrak{g}^*]^{\otimes 2}$ such that

$$\Pi(\xi\otimes\xi'\otimes\eta\otimes\eta') = \beta([x,x'],\cdot)\otimes\beta([y,y'],\cdot), \tag{1.6}$$

whenever $\xi,\xi',\eta,\eta' \in \mathfrak{g}^*$, with $x,x',y,y' \in \mathfrak{g}$ characterized by $\xi = \beta(x,\cdot)$, $\xi' = \beta(x',\cdot)$, $\eta = \beta(y,\cdot)$, $\eta' = \beta(y',\cdot)$. According to formula (3.1) below, $\Pi([\mathfrak{g}^*]^{\wedge 4}) \subset [\mathfrak{g}^*]^{\odot 2}$.

Our first main result, established in Section 3, relates Ω to $\Pi\Lambda : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\odot 2}$, the composite of Λ and the restriction of Π to the subspace $[\mathfrak{g}^*]^{\wedge 4} \subset [\mathfrak{g}^*]^{\otimes 4}$.

Theorem A *Let* Ω, Λ *and* Π *be the operators defined by* (1.2)*,* (1.4) *and* (1.6) *for a given real/complex semisimple Lie algebra* $\mathfrak{g}$*. Then* $2\Pi\Lambda = -(\Omega + \mathrm{Id})(\Omega - 2\,\mathrm{Id})$.

Next, in Section 5, we use Meyberg's result and Theorem A, both to show that

$$\mathrm{Ker}\,\Lambda = \mathrm{Ker}\,(\Omega - 2\,\mathrm{Id}) \oplus \mathrm{Ker}\,(\Omega + \mathrm{Id}) \quad \text{in any real/complex simple Lie algebra,} \tag{1.7}$$

and to obtain the following explicit description of $\mathrm{Ker}\,\Lambda$ for semisimple Lie algebras, which also provides a crucial step in our proof of Theorem C (see below).

Theorem B *Given a real/complex semisimple Lie algebra* $\mathfrak{g}$ *with a direct-sum decomposition* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$ *into simple ideals,* $s \geq 1$*, let* Λ *and* Λ_i *denote the operator defined by* (1.4) *for* $\mathfrak{g}$ *and, respectively, its analog for the* i*th summand* $\mathfrak{g}_i$*.*

- (i) $\mathrm{Ker}\,\Lambda = \mathrm{Ker}\,\Lambda_1 \oplus \ldots \oplus \mathrm{Ker}\,\Lambda_s$*, where* $[\mathfrak{g}_i^*]^{\odot 2} \subset [\mathfrak{g}^*]^{\odot 2}$ *via trivial extensions.*
- (ii) $\Lambda = 0$ *if* $\dim \mathfrak{g} = 3$.
- (iii) $\dim \mathrm{Ker}\,\Lambda = 12$ *if* $\mathfrak{g}$ *is simple and* $\dim \mathfrak{g} = 6$*, which happens only when* $\mathfrak{g}$ *is real and isomorphic to the underlying real Lie algebra of* $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})$*, while* $\mathrm{Ker}\,\Lambda$ *then consists of the real parts of all symmetric* $\mathbb{C}$*-bilinear functions* $\mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$.
- (iv) $\dim \mathrm{Ker}\,\Lambda \in \{1,2\}$ *whenever* $\mathfrak{g}$ *is simple and* $\dim \mathfrak{g} \notin \{3,6\}$*, while* $\mathrm{Ker}\,\Lambda$ *is then spanned either by the Killing form* β*, or by* $\mathrm{Re}\,\beta^{\mathfrak{h}}$ *and* $\mathrm{Im}\,\beta^{\mathfrak{h}}$*, with* $\beta^{\mathfrak{h}}$ *denoting the Killing form of the complex simple Lie algebra* $\mathfrak{h}$ *in* (1.3)*. The former case occurs if* $\mathfrak{g}$ *is complex, or real of type* (1.3.a)*, the latter if* $\mathfrak{g}$ *is real of type* (1.3.b).

Finally, one defines the *Cartan three-form* $C \in [\mathfrak{g}^*]^{\wedge 3}$ of a Lie algebra $\mathfrak{g}$ by

$$C = \beta([\,\cdot\,,\,\cdot\,],\,\cdot\,), \qquad \text{where} \quad \beta \quad \text{denotes the Killing form.} \tag{1.8}$$

The following result has been known for decades, although no published proof of it seems to exist [4]. By an *isomorphism of the Cartan three-forms* we mean here a vector-space isomorphism of the Lie algebras in question, sending one three-form onto the other.

Theorem C *Let* $\mathfrak{g}$ *be a real/complex semisimple Lie algebra with a fixed direct-sum decomposition into simple ideals, which we briefly refer to as the "summands" of* $\mathfrak{g}$*.*

- (i) *If* $\mathfrak{h}$ *is a real/complex Lie algebra, the Cartan three-forms of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are isomorphic and, in the real case,* $\mathfrak{g}$ *has no summands of dimension* 3*, then* $\mathfrak{h}$ *is isomorphic to* $\mathfrak{g}$.
- (ii) *If* $\mathfrak{g}$ *contains no summands of dimension* 3 *or* 6*, then every automorphism of the Cartan three-form of* $\mathfrak{g}$ *is a Lie-algebra automorphism of* $\mathfrak{g}$ *followed by an operator that acts on each summand as the multiplication by a cubic root of* 1.
- (iii) *If* $\mathfrak{g}$ *is the underlying real Lie algebra of a complex simple Lie algebra and* $\dim \mathfrak{g} \neq 6$*, then every automorphism of the Cartan three-form of* $\mathfrak{g}$ *is complex-linear or antilinear.*

Conversely, if $\mathfrak{g}$ *has* k *summands of dimension* 3 *and* l *summands of dimension* 6*, then the Lie-algebra automorphisms of* $\mathfrak{g}$ *form a subgroup of codimension* $5k + 10l$ *in the automorphism group of the Cartan three-form.*

We derive Theorem C from Theorem B, in Section 7.

2 Preliminaries

Suppose that $\mathfrak{g}$ is the underlying real Lie algebra of a complex Lie algebra $\mathfrak{h}$. We denote by β and C the Killing form and Cartan three-form of $\mathfrak{g}$, cf. (1.1) and (1.8), by Λ the operator in (1.4) associated with $\mathfrak{g}$, and use the symbols $\beta^{\mathfrak{h}}, C^{\mathfrak{h}}, \Lambda^{\mathfrak{h}}$ for their counterparts corresponding to $\mathfrak{h}$. Obviously, whenever $\sigma : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is a symmetric $\mathbb{C}$-bilinear form,

$$\text{i)}\;\; \beta = 2\,\mathrm{Re}\,\beta^{\mathfrak{h}}, \qquad \text{ii)}\;\; C = 2\,\mathrm{Re}\,C^{\mathfrak{h}}, \qquad \text{iii)}\;\; \Lambda(\mathrm{Re}\,\sigma) = \mathrm{Re}\,(\Lambda^{\mathfrak{h}}\sigma). \tag{2.1}$$

For (2.1.i), see also [6, formula (13.1)]. With $\mathfrak{g}$ and $\mathfrak{h}$ as above, it is clear from (2.1.i) that

$$\begin{array}{l}\mathrm{Re}\,\beta^{\mathfrak{h}} \text{ and } \mathrm{Im}\,\beta^{\mathfrak{h}} \text{ span the real space of symmetric bilinear forms } \sigma \text{ on } \mathfrak{g} \text{ arising}\\ \text{via (1.2.b) from linear endomorphisms } \Sigma \text{ which are complex multiples of } \mathrm{Id}.\end{array} \tag{2.2}$$

Furthermore, (2.1.i) also implies, for dimensional reasons, that

$$\begin{array}{l}\text{the real parts of symmetric } \mathbb{C}\text{-bilinear functions } \mathfrak{g}\times\mathfrak{g}\to\mathbb{C} \text{ form the image}\\ \text{under (1.2.b) of the space of } \mathbb{C}\text{-linear } \beta^{\mathfrak{h}}\text{-self-adjoint endomorphisms of } \mathfrak{h},\end{array} \tag{2.3}$$

as the former space obviously contains the latter.

Let $\mathfrak{g}$ now be a Lie algebra over the scalar field $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$. A fixed basis of $\mathfrak{g}$ allows us to represent elements x,y of $\mathfrak{g}$, symmetric bilinear forms σ on $\mathfrak{g}$, and the Lie-algebra bracket operation $[\,,]$ by their components x^i, y^i, σ_{ij} and $C_{ij}{}^k$ (the structure constants of $\mathfrak{g}$), so that $\sigma(x,y)=\sigma_{ij}x^iy^j$ and $[x,y]^k = C_{ij}{}^k x^i y^j$. Repeated indices are summed over. The Cartan three-form C with (1.8) has the components $C_{ijk}=C_{ij}{}^r\beta_{kr}$, where β is the Killing form. The definition (1.1) of β, its bi-invariance, and the Jacobi identity now read

$$\begin{array}{ll}\text{i)}\;\; \beta_{ij} = C_{ip}{}^q C_{jq}{}^p, & \text{ii)}\;\; C_{ijk} \text{ is skew-symmetric in } i,j,k,\\ \text{iii)}\;\; C_{ij}{}^q C_{qk}{}^l + C_{jk}{}^q C_{qi}{}^l + C_{ki}{}^q C_{qj}{}^l = 0. & \end{array} \tag{2.4}$$

In the remainder of this section $\mathfrak{g}$ is also assumed to be semisimple. We can thus lower and raise indices using the components β_{ij} of the Killing form β and β^{ij} of its reciprocal: $C^k{}_p{}^q = \beta^{kr}C_{rp}{}^q$, and $C_j{}^{sp}=\beta^{sk}C_{jk}{}^p$. For any $x,y,z\in\mathfrak{g}$ and the Cartan three-form C given by (1.8), one has $2\,\mathrm{tr}\,[(\mathrm{Ad}\,x)(\mathrm{Ad}\,y)(\mathrm{Ad}\,z)] = C(x,y,z)$, which in component notation reads

$$2C_{ir}{}^p C_{jq}{}^r C_{kp}{}^q \,=\, C_{ijk}. \tag{2.5}$$

To prove (2.5), we note that the equalities $C^k{}_p{}^q = C_p{}^{qk}$ and $C_i{}^{rp} = -C_i{}^{pr}$ (obvious from (2.4.ii)) along with (2.4.ii), (2.4.iii) and (2.4.i–ii) give $2C_{ir}{}^pC_{jq}{}^rC^k{}_p{}^q = 2C_i{}^{rp}C_{jqr}C_p{}^{qk} = C_i{}^{rp}(C_{jqr}C_p{}^{qk} - C_{jqp}C_r{}^{qk}) = C_i{}^{rp}(C_{jr}{}^qC_{qp}{}^k + C_{pj}{}^qC_{qr}{}^k) = -C_i{}^{rp}C_{rp}{}^qC_{qj}{}^k = \delta_i^qC_{qj}{}^k = C_{ij}{}^k$. Now (2.5) follows if one lowers the index k. Next, we introduce the linear operator

$$T:[\mathfrak{g}^*]^{\otimes 2}\to[\mathfrak{g}^*]^{\otimes 2} \;\text{ with }\; (T\sigma)_{ij} = T_{ij}^{kl}\sigma_{kl}, \;\text{ where }\; T_{ij}^{kl} = 2C_{ip}{}^kC_j{}^{lp}. \tag{2.6}$$

Lemma 2.1 *For T and the operator $\Omega:[\mathfrak{g}^*]^{\odot 2}\to[\mathfrak{g}^*]^{\odot 2}$ given by (1.2),*

(a) *T leaves the subspaces $[\mathfrak{g}^*]^{\odot 2}$ and $[\mathfrak{g}^*]^{\wedge 2}$ invariant,*
(b) *Ω coincides with the restriction of T to $[\mathfrak{g}^*]^{\odot 2}$,*
(c) *the restriction of T to $[\mathfrak{g}^*]^{\wedge 2}$ is diagonalizable, with the eigenvalues 0 and 1,*
(d) *the eigenspace $[\mathfrak{g}^*]^{\wedge 2}\cap\mathrm{Ker}\,(\Omega-\mathrm{Id})$ equals $\{C(x,\cdot,\cdot): x\in\mathfrak{g}\}$, for C given by (1.8).*

Proof Assertions (a) – (b) are obvious from (2.6) and the fact that, by (2.6), $T\sigma$ is the same as $\Omega\sigma$ in (1.2), except that now $\sigma : \mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$ need not be symmetric. Next, $(T\sigma)_{ij} = -C_{ij}{}^{q}C_{q}{}^{kl}\sigma_{kl}$ for $\sigma\in[\mathfrak{g}^*]^{\wedge 2}$, as one sees raising the index k in (2.4.iii), then transvecting with σ_{kl}, and using (2.4.ii). Hence, if $\sigma = C(x,\cdot,\cdot)$ lies in $C(\mathfrak{g}) = \{C(x,\cdot,\cdot) : x\in\mathfrak{g}\}$ (or, $\sigma\in[\mathfrak{g}^*]^{\wedge 2}$ is β-orthogonal to $C(\mathfrak{g})$), so that $\sigma_{kl} = x^pC_{pkl}$ (or, respectively, $C_p{}^{kl}\sigma_{kl} = 0$), then, by (2.4.i–ii), $T\sigma = \sigma$ (or, respectively, $T\sigma = 0$). As $C(\mathfrak{g})$ and its β-orthogonal complement must now span $[\mathfrak{g}^*]^{\wedge 2}$ for dimensional reasons, (c) and (d) follow. □

The next result is a direct consequence of Meyberg's theorem. See the Appendix, the last three lines of which justify assertions (c), (d) and (e).

Theorem 2.2 *For any complex simple Lie algebra* $\mathfrak{g}$ *and* $\Omega : [\mathfrak{g}^*]^{\odot 2}\to[\mathfrak{g}^*]^{\odot 2}$ *with* (1.2),

(a) Ω *is diagonalizable,*
(b) 2 *is an eigenvalue of* Ω *with multiplicity* 1,
(c) 0 *is not an eigenvalue of* Ω,
(d) Ω *has the eigenvalue* 1 *if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{sl}(n,\mathbb{C})$ *for some* $n\ge 3$,
(e) $\dim\operatorname{Ker}(\Omega+\mathrm{Id})$ *equals* 5 *when* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{sl}(2,\mathbb{C})$*, and* 0 *otherwise.*

Remark 2.3 The isomorphism types of all complex simple Lie algebras are: $\mathfrak{sl}_n$, for $n\ge 2$, $\mathfrak{sp}_n$ (even $n\ge 4$), $\mathfrak{so}_n$ with $n\ge 7$, as well as $\mathfrak{g}_2,\mathfrak{f}_4,\mathfrak{e}_6,\mathfrak{e}_7$ and $\mathfrak{e}_8$. See [9, pp. 8 and 77].

Remark 2.4 One has $\operatorname{Ker}[\Theta(\Theta+a\,\mathrm{Id})] \subset \operatorname{Ker}\Theta\oplus\operatorname{Ker}(\Theta+a\,\mathrm{Id})$ for any scalar $a\ne 0$ and linear endomorphism Θ of a vector space. In fact, the required decomposition of any $\sigma\in\operatorname{Ker}[\Theta(\Theta+a\,\mathrm{Id})]$ is given by $a\sigma = (\Theta+a\,\mathrm{Id})\sigma - \Theta\sigma$.

Remark 2.5 The curvature operator of a (pseudo)Riemannian metric γ on a manifold, acting on symmetric 2-tensors, has been studied by various authors [5], [3], [2, pp. 51–52]. It is given by $\sigma\mapsto\tau$, where $2\tau_{ij} = \gamma^{pq}R_{ipj}{}^{k}\sigma_{qk}$ in terms of components relative to a basis of the tangent space at any point, the sign convention about the curvature tensor R being that a Euclidean tangent plane with an orthonormal basis x,y has the sectional curvature $\gamma_{pq}R_{ijk}{}^{p}x^iy^jx^ky^q$. When γ is the Killing form β of a semisimple Lie group G, treated as a left-invariant metric (see the lines following (1.1)), this operator equals $-\Omega/16$, for Ω with (1.2). In fact, the description of the Levi-Civita connection D of β in the Introduction gives $4R(x,y)z = [[x,y],z]$ for left-invariant vector fields x,y,z, that is, $4R_{ijk}{}^{l} = C_{ij}{}^{p}C_{pk}{}^{l}$. Lemma 2.1(b) now implies our claim, as $T_{ij}^{kl} = -8\beta^{kp}R_{jpi}{}^{l}$ due to (2.4.ii) and (2.6).

3 Proof of Theorem A

We use the notation of Section 2. For $C_{ij}{}^{k}$ as in (2.4), relations (1.4) and (1.6) give

$$\begin{aligned}&(\Lambda\sigma)_{ijkl} = \Lambda_{ijkl}{}^{rs}\sigma_{rs} \quad\text{where}\quad \Lambda_{ijkl}{}^{rs} = C_{ij}{}^{r}C_{kl}{}^{s} + C_{jk}{}^{r}C_{il}{}^{s} + C_{ki}{}^{r}C_{jl}{}^{s},\\ &(\Pi\zeta)_{pq} = C^{ij}{}_{p}C^{kl}{}_{q}\zeta_{ijkl}, \quad\text{whenever}\quad \sigma\in[\mathfrak{g}^*]^{\odot 2} \text{ and } \zeta\in[\mathfrak{g}^*]^{\wedge 4},\end{aligned} \tag{3.1}$$

in any real/complex semisimple Lie algebra $\mathfrak{g}$. Next, with T_{ij}^{kl} defined by (2.6),

$$2C^{ij}{}_{p}C^{kl}{}_{q}(C_{ij}{}^{r}C_{kl}{}^{s} + C_{jk}{}^{r}C_{il}{}^{s} + C_{ki}{}^{r}C_{jl}{}^{s}) = 2\delta_p^r\delta_q^s + T_{pq}^{rs} - T_{pq}^{ik}T_{ik}^{rs}. \tag{3.2}$$

In fact, the first of the three terms naturally arising on the left-hand side of (3.2) equals $2\delta_p^r\delta_q^s$ since, by (2.4.i–ii), $C^{ij}{}_pC_{ij}{}^r = -\delta_p^r$ and $C^{kl}{}_qC_{kl}{}^s = -\delta_q^s$. The other two terms coincide (as skew-symmetry of $C^{ij}{}_p$ in i,j gives $C^{ij}{}_pC_{jk}{}^rC_{il}{}^s = -C^{ij}{}_pC_{ik}{}^rC_{jl}{}^s = C^{ij}{}_pC_{ki}{}^rC_{jl}{}^s$), and so they add up to $4C^{ij}{}_pC^{kl}{}_qC_{ki}{}^rC_{jl}{}^s$, that is, $4C^{kl}{}_qC_{jl}{}^sC_p{}^{ji}C_{ik}{}^r = 4C^{kl}{}_qC^j{}_l{}^sC_{pj}{}^iC_{ik}{}^r = -4C^{kl}{}_qC^j{}_l{}^s(C_{jk}{}^iC_{ip}{}^r + C_{kp}{}^iC_{ij}{}^r)$; the rightmost equality is due to the Jacobi identity (2.4.iii). The last expression consists of the first term, $-4C^{kl}{}_qC^j{}_l{}^sC_{jk}{}^iC_{ip}{}^r = -4C_{ip}{}^r(C^i{}_k{}^jC_{ql}{}^kC^s{}_j{}^l) = -2C_{ip}{}^rC^i{}_q{}^s$, cf. (2.5), equal, by (2.4.ii) and (2.6), to $2C_{pi}{}^rC_q{}^{si} = T_{pq}^{rs}$, and the second term, $-(2C_{kp}{}^iC^{kl}{}_q)(2C_{ij}{}^rC^j{}_l{}^s)$, the two parenthesized factors of which are, for the same reasons, nothing else than T_{pq}^{il} and T_{il}^{rs}. This proves (3.2).

Theorem A is now an obvious consequence of (3.1) – (3.2) and Lemma 2.1(b).

4 The spectrum of Ω in real simple Lie algebras

Theorem 4.1 *Let Ω denote the operator with (1.2) corresponding to a fixed real simple Lie algebra $\mathfrak{g}$, and $\Omega^{\mathfrak{h}}$ its analog for $\mathfrak{h}$, chosen so that $\mathfrak{g}$ and $\mathfrak{h}$ satisfy (1.3).*

(i) *Ω is always diagonalizable.*
(ii) *In case (1.3.a), Ω has the same spectrum as $\Omega^{\mathfrak{h}}$, including the multiplicities.*
(iii) *In case (1.3.b), the spectrum of Ω arises from that of $\Omega^{\mathfrak{h}}$ by doubling the original multiplicities and then including 0 as an additional eigenvalue with the required complementary multiplicity. Note that, by Theorem 2.2(c), 0 is not an eigenvalue of $\Omega^{\mathfrak{h}}$.*
(iv) *The eigenspace $\mathrm{Ker}\,(\Omega - 2\,\mathrm{Id})$ is spanned in case (1.3.a) by β, and in case (1.3.b) by $\mathrm{Re}\,\beta^{\mathfrak{h}}$ and $\mathrm{Im}\,\beta^{\mathfrak{h}}$, for the Killing forms β of $\mathfrak{g}$ and $\beta^{\mathfrak{h}}$ of $\mathfrak{h}$.*

Proof By [6, Lemma 14.3(ii) and formulae (14.5) – (14.7)], if $\mathfrak{g}$ is of type (1.3.a), the complexification of $[\mathfrak{g}^*]^{\odot 2}$ may be naturally identified with its (complex) counterpart $[\mathfrak{h}^*]^{\odot 2}$ for $\mathfrak{h}$, in such a way that $\Omega^{\mathfrak{h}}$ and the Killing form $\beta^{\mathfrak{h}}$ become the unique $\mathbb{C}$-linear extensions of Ω and β. Now Theorem 2.2(a)–(b) and (1.5.ii) yield (i), (ii) and (iv) in case (1.3.a).

For $\mathfrak{g}$ of type (1.3.b), Lemma 13.1 of [6] states the following. First, $[\mathfrak{g}^*]^{\odot 2}$ is the direct sum of two Ω-invariant subspaces: one formed by the real parts of $\mathbb{C}$-bilinear symmetric functions $\sigma : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$, the other by the real parts of functions $\sigma : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$ which are antilinear and Hermitian. Secondly, Ω vanishes on the "Hermitian" summand, and its action on the "symmetric" summand is equivalent, via the isomorphism $\sigma \mapsto \mathrm{Re}\,\sigma$, to the action of $\Omega^{\mathfrak{h}}$ on $\mathbb{C}$-bilinear symmetric functions σ. With diagonalizability of $\Omega^{\mathfrak{h}}$ again provided by Theorem 2.2(a), this proves our remaining claims. (The multiplicities are doubled since the original complex eigenspaces are viewed as real, while the eigenspace $\Omega^{\mathfrak{h}}$ for the eigenvalue 2 consists, by Theorem 2.2(b) and (1.5.ii), of complex multiples of $\beta^{\mathfrak{h}}$, the real parts of which are precisely the real linear combinations of $\mathrm{Re}\,\beta^{\mathfrak{h}}$ and $\mathrm{Im}\,\beta^{\mathfrak{h}}$.) □

Remark 4.2 It is well known [9, p. 30] that, up to isomorphisms, $\mathfrak{sl}(n,\mathbb{R})$ as well as $\mathfrak{su}(p,q)$ with $p+q=n$ and, if n is even, $\mathfrak{sl}(n/2,\mathbb{H})$, are the only real forms of $\mathfrak{sl}(n,\mathbb{C})$.

Lemma 4.3 *The only complex, or real, simple Lie algebras of dimensions less than 7 are, up to isomorphisms, $\mathfrak{sl}(2,\mathbb{C})$ or, respectively, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2,\mathbb{C})$, the last one being both complex three-dimensional and real six-dimensional. Consequently,*

(i) *a complex simple Lie algebra cannot be six-dimensional,*

(ii) *there is just one isomorphism type of a complex or, respectively, real simple Lie algebra of dimension* 3 *or, respectively,* 6*, both represented by* $\mathfrak{sl}(2,\mathbb{C})$*,*

(iii) $\dim\mathfrak{g}\notin\{1,2,4,5\}$ *for every real or complex simple Lie algebra* $\mathfrak{g}$*.*

Proof According to Remark 2.3, in the complex case, only $\mathfrak{sl}(2,\mathbb{C})$ is possible. For real Lie algebras, one can use Remark 4.2 and (1.3). □

Remark 4.4 We can now justify the claim, made in [6, Theorem 12.3], that 1 is not an eigenvalue of Ω in any real or complex simple Lie algebra except the ones isomorphic to $\mathfrak{sl}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{C}), \mathfrak{su}(p,q)$ or, for even n only, $\mathfrak{sl}(n/2,\mathbb{H})$, where $n=p+q\geq 3$.

In fact, by Theorem 2.2 and parts (ii) – (iii) of Theorem 4.1, the only real or complex simple Lie algebras in which Ω has the eigenvalue 1 are, up to isomorphisms, $\mathfrak{sl}(n,\mathbb{C})$ for $n\geq 3$ and their real forms. According to Remark 4.2, these are all listed in the last paragraph.

Remark 4.5 For any real/complex simple Lie algebra $\mathfrak{g}$, Theorems 2.2 and 4.1(ii)-(iii) give $3\dim\mathrm{Ker}\,(\Omega+\mathrm{Id})=5\dim\mathfrak{g}$ if $\dim\mathfrak{g}\in\{3,6\}$, and $\mathrm{Ker}\,(\Omega+\mathrm{Id})=\{0\}$ otherwise.

5 Proofs of (1.7) and Theorem B

Let $\sigma\in[\mathfrak{g}^*]^{\odot 2}$ and $\Lambda\sigma=0$. Consequently, by (1.4), $\sigma([x,y],[z,z'])+\sigma([y,z],[x,z'])+\sigma([z,x],[y,z'])=0$ for all x,y,z,z' in $\mathfrak{g}$. Thus, $\sigma([x,y],[z,z'])=0$ whenever $x,y\in\mathfrak{h}_i$ and $z,z'\in\mathfrak{h}_j$ with $j\neq i$. The summands $\mathfrak{h}_i$ and $\mathfrak{h}_j$, being simple, are spanned by such brackets $[x,y]$ and $[z,z']$, and so $\mathfrak{h}_i$ is σ-orthogonal to $\mathfrak{h}_j$. As this is the case for any two summands, we obtain Theorem B(i), the right-to-left inclusion being obvious. Theorem B(ii) is also immediate, since $[\mathfrak{g}^*]^{\wedge 4}=\{0\}$ when $\dim\mathfrak{g}=3$.

From now on, $\mathfrak{g}$ is assumed to be simple. The first of the following two inclusions is then clear from Theorem 4.1(iv), (1.5.i) and (2.1.iii) (applied to $\sigma=a\beta^{\mathfrak{h}}$, with $a\in\mathbb{C}$), the second one – from Theorem A and Remark 2.4 (for $\Theta=\Omega-2\,\mathrm{Id}$ and $a=3$):

$$\mathrm{Ker}\,(\Omega-2\,\mathrm{Id})\subset\mathrm{Ker}\,\Lambda\subset\mathrm{Ker}\,(\Omega-2\,\mathrm{Id})\oplus\mathrm{Ker}\,(\Omega+\mathrm{Id}). \tag{5.1}$$

If $\dim\mathfrak{g}\notin\{3,6\}$, Remark 4.5 gives $\mathrm{Ker}\,(\Omega+\mathrm{Id})=\{0\}$. The inclusions in (5.1) thus are equalities, which both proves (1.7) in this case and, combined with Theorem 4.1(iv), implies Theorem B(iv). When $\dim\mathfrak{g}=3$, (1.7) follows as the second inclusion in (5.1) is an equality: both spaces are 6-dimensional by Theorems B(ii), 4.1(iv) and Remark 4.5.

Finally, suppose that $\dim\mathfrak{g}=6$. According to Lemma 4.3(i)-(ii), $\mathfrak{g}$ is then real and isomorphic to the underlying real algebra of $\mathfrak{h}=\mathfrak{sl}(2,\mathbb{C})$. From (2.1.iii), with $\Lambda^{\mathfrak{h}}\sigma=0$ by Theorem B(ii), we thus get $\mathscr{F}\subset\mathrm{Ker}\,\Lambda$ for $\mathscr{F}=\{\mathrm{Re}\,\sigma:\sigma\in[\mathfrak{h}^*]^{\odot 2}\}$, where $[\mathfrak{h}^*]^{\odot 2}$ denotes the space of all symmetric $\mathbb{C}$-bilinear forms $\sigma:\mathfrak{h}\times\mathfrak{h}\to\mathbb{C}$. As the operator $\sigma\mapsto\mathrm{Re}\,\sigma$ is injective, that is, any such σ is uniquely determined by $\mathrm{Re}\,\sigma$, one must have $\dim_{\mathbb{R}}\mathscr{F}=12$. The second inclusion in (5.1) is therefore an equality, and $\mathscr{F}=\mathrm{Ker}\,\Lambda$, for dimensional reasons: $\mathrm{Ker}\,\Lambda$ contains the subspace $\mathscr{F}$ of real dimension 12, equal, in view of Theorem 4.1(iv) and Remark 4.5, to the real dimension of $\mathrm{Ker}\,(\Omega-2\,\mathrm{Id})\oplus\mathrm{Ker}\,(\Omega+\mathrm{Id})$. This yields (1.7) in the remaining case $\dim\mathfrak{g}=6$ while, due to the definition of $\mathscr{F}$, the relations $\dim_{\mathbb{R}}\mathscr{F}=12$ and $\mathscr{F}=\mathrm{Ker}\,\Lambda$ prove assertion (iii) of Theorem B.

6 Some facts needed from linear algebra

In this section $\mathfrak{g}$ is the underlying real space of a finite-dimensional complex vector space $\mathfrak{h}$ and $J:\mathfrak{g}\to\mathfrak{g}$ is the operator of multiplication by i, also referred to as the *complex structure*. We denote by $\beta^{\mathfrak{h}}$ a fixed nondegenerate $\mathbb{C}$-bilinear symmetric form on $\mathfrak{h}$, so that the $\mathbb{R}$-bilinear symmetric form $\beta = 2\,\mathrm{Re}\,\beta^{\mathfrak{h}}$ on $\mathfrak{g}$ is nondegenerate as well. The same applies to any nonzero complex multiple of $\beta^{\mathfrak{h}}$. Thus, β and $\gamma = 2\,\mathrm{Im}\,\beta^{\mathfrak{h}}$ constitute a basis of a real vector space $\mathscr{P}$ of $\mathbb{R}$-bilinear symmetric forms on $\mathfrak{g}$. All nonzero elements of $\mathscr{P}$ are nondegenerate. As $\beta^{\mathfrak{h}}$ is $\mathbb{C}$-bilinear, $\gamma(x,y) = -\beta(x,Jy)$ for all $x,y\in\mathfrak{g}$. We use components relative to a basis of $\mathfrak{g}$, as in Section 2.

Lemma 6.1 *The real spaces $\mathfrak{g}$ and $\mathscr{P}$ uniquely determine the pair $(J,\beta^{\mathfrak{h}})$ up to its replacement by $(J,a\beta^{\mathfrak{h}})$ or $(-J,\overline{a\beta^{\mathfrak{h}}})$, with any $a\in\mathbb{C}\smallsetminus\{0\}$.*

Proof For any basis κ,λ of $\mathscr{P}$, replacing $\beta^{\mathfrak{h}}$ by a complex multiple, which leaves $\mathscr{P}$ unchanged, we assume that $\kappa=\beta$. Thus, $\lambda = u\beta+v\gamma$, where $u,v\in\mathbb{R}$ and $v\neq 0$. Writing the equality $\gamma = -\beta(\,\cdot\,,J\,\cdot\,)$ as $\gamma_{rq} = -\beta_{rs}J_q^s$, and then using the reciprocal components $\kappa^{pr}=\beta^{pr}$, we obtain $\kappa^{pr}\lambda_{rq} = \beta^{pr}(u\beta_{rq} - v\beta_{rs}J_q^s) = u\delta_q^p - vJ_q^p$. Now $\pm J$ may be defined by declaring the matrix J_q^p to be the traceless part of $\kappa^{pr}\lambda_{rq}$, normalized so that $J^2=-\mathrm{Id}$.

At the same time, fixing any $\kappa\in\mathscr{P}\smallsetminus\{0\}$ we may assume, as before, that $\kappa=\beta$. Then κ and $\gamma=-\kappa(\,\cdot\,,J\,\cdot\,)$, determine $2\beta^{\mathfrak{h}}$, being its real and imaginary parts. Combined with the last sentence of the preceding paragraph, this completes the proof. ☐

The next fact concerns two mappings, $\mathrm{rec}:\mathscr{P}\smallsetminus\{0\}\to\mathfrak{g}^{\odot 2}$ and $\mathfrak{g}^{\odot 2}\ni\mu\mapsto\mu_\flat\in\mathrm{End}\,\mathfrak{g}$. The former sends every nonzero element of $\mathscr{P}$ (which, as we know, is nondegenerate) to its reciprocal. The latter is the operator of index-lowering via β, and takes values in the space of $\mathbb{R}$-linear endomorphisms of $\mathfrak{g}$, which include complex multiples of Id. We then have

$$\{[\mathrm{rec}(\sigma)]_\flat : \sigma\in\mathscr{P}\smallsetminus\{0\}\} = \{a\mathrm{Id} : a\in\mathbb{C}\smallsetminus\{0\}\}. \tag{6.1}$$

Namely, under index raising with the aid of β, the operators $A=a\mathrm{Id}$, for $a\in\mathbb{C}\smallsetminus\{0\}$, correspond to elements μ of $\mathfrak{g}^{\odot 2}$ characterized by $\mu^{pq}=\beta^{pr}A_r^q$. Every such μ is in turn the reciprocal of $\sigma\in[\mathfrak{g}^*]^{\odot 2}$ defined by $\sigma_{pq}=H_p^k\beta_{kq}$, where $H=A^{-1}$ (as $\sigma_{pq}\mu^{sq}=H_p^k\beta_{ks}\beta^{sr}A_r^q = H_p^rA_r^q=\delta_p^q$). Symmetry of μ, and hence σ, is obvious from β-self-adjointness of A. The inverses H of our operators $A=a\mathrm{Id}$ range over nonzero complex multiples of Id as well, and so the resulting symmetric forms σ act on $x,y\in\mathfrak{g}$ by $\sigma(x,y)=\beta(ux+vJx,y)$, where (u,v) range over $\mathbb{R}\smallsetminus\{0\}$. Therefore $\sigma=u\beta-v\gamma$, as required.

Remark 6.2 The relation $\gamma=-\beta(\,\cdot\,,J\,\cdot\,)$ for $\beta=2\,\mathrm{Re}\,\beta^{\mathfrak{h}}$ and $\gamma=2\,\mathrm{Im}\,\beta^{\mathfrak{h}}$ shows that, once J is fixed, $\mathrm{Re}\,\beta^{\mathfrak{h}}$ uniquely determines $\beta^{\mathfrak{h}}$. Similarly, $\mathrm{Re}\,C^{\mathfrak{h}}$ and J determine the Cartan three-form $C^{\mathfrak{h}}$ of a complex Lie algebra $\mathfrak{h}$, cf. (1.8). In fact, $\mathrm{Im}\,C^{\mathfrak{h}} = -\mathrm{Re}\,C^{\mathfrak{h}}(\,\cdot\,,\,\cdot\,,J\,\cdot\,)$.

Remark 6.3 The bracket $[\,,]$ of a real/complex semisimple Lie algebra is uniquely determined by C and β via (1.8). Knowing C and the set of nonzero scalar multiples of β, rather than β itself, makes $[\,,]$ unique up to multiplications by cubic roots of 1. Such factors must be allowed as multiplying $[\,,]$ by a scalar r replaces β and C with $r^2\beta$ and r^3C.

Remark 6.4 In the first sentence of Remark 6.3, treating C and β formally, we see that in the complex case $\bar{C}$ and $\bar{\beta}$ determine, via (1.8), the same bracket $[\,,]$ as C and β.

Lemma 6.5 *The Lie algebra* $\mathfrak{a}$ *of infinitesimal automorphisms of the Cartan three-form* C *of a simple real/complex Lie algebra* $\mathfrak{g}$ *has the vector-space decomposition* $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, *where* $\mathfrak{a}_+$ *is the space of all* Σ *related as in* (1.2.b) *to elements* σ *of* $\operatorname{Ker}(\Omega + \mathrm{Id})$, *and* $\mathfrak{a}_-$ *consists of all derivations of* $\mathfrak{g}$. *The operators forming* $\mathfrak{a}_+$ *are all* β*-self-adjoint, those in* $\mathfrak{a}_-$ *are* β*-skew-adjoint, and* $\mathfrak{a}_-$ *coincides with* $\operatorname{Ad}(\mathfrak{g}) = \{\operatorname{Ad} x : x \in \mathfrak{g}\}$.

Proof We have the obvious inclusions $\operatorname{Ad}(\mathfrak{g}) \subset \mathfrak{a}_- \subset \mathfrak{a}$. For any fixed $\Sigma \in \mathfrak{a}$, define σ by (1.2.b). Transvecting the equality $\sigma_{iq} C_{jk}{}^q + \sigma_{jq} C_{ki}{}^q + \sigma_{kq} C_{ij}{}^q = 0$ with $C^{jk}{}_p$, we see that, by (2.4.i–ii), $\sigma_{ip} = 2C^{jk}{}_p C_{ki}{}^q \sigma_{jq}$. Hence (2.6) and (2.4.ii) give $T\tau = -\sigma$, where $\tau = \sigma^*$ is the 2-tensor with $\tau_{ij} = \sigma_{ji}$. As $(T\tau)^* = T\tau^*$, cf. (2.6), $\sigma \pm \sigma^*$ is an eigenvector of T for the eigenvalue ∓ 1. Lemma 2.1(b),(d) thus shows that the self-adjoint and skew-adjoint parts of any $\Sigma \in \mathfrak{a}$ lie in $\mathfrak{a}_+$ and, respectively, in $\operatorname{Ad}(\mathfrak{g}) \subset \mathfrak{a}_- \subset \mathfrak{a}$. Consequently, noting that Lie-algebra automorphisms of $\mathfrak{g}$ leave β invariant, and so all derivations of $\mathfrak{g}$ must be β-skew-adjoint, one obtains $\mathfrak{a}_- = \operatorname{Ad}(\mathfrak{g})$ and $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{a}_-$ for the space $\mathfrak{b}$ of all $\Sigma \in \mathfrak{a}_+$ which are at the same time infinitesimal automorphisms of C.

It now suffices to show that $\mathfrak{b} = \mathfrak{a}_+$. If $\dim \mathfrak{g} \notin \{3, 6\}$, this is clear from Remark 4.5, which gives $\mathfrak{b} = \mathfrak{a}_+ = \{0\}$. When $\dim \mathfrak{g} = 3$, the inclusion $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{a}_- \subset \mathfrak{a}_+ \oplus \mathfrak{a}_-$ is an equality, as $8 = \dim \mathfrak{a} \le \dim \mathfrak{a}_+ + \dim \mathfrak{a}_- = 5 + \dim \operatorname{Ad}(\mathfrak{g}) \le 5 + \dim \mathfrak{g} = 8$. (Here $\dim \mathfrak{a}_+ = 5$ by Remark 4.5, and $\dim \mathfrak{a} = 8$, since C is a volume form in the 3-space $\mathfrak{g}$.) Finally, let $\dim \mathfrak{g} = 6$ and $\Sigma \in \mathfrak{a}_+$. Thus, $\mathfrak{g}$ is real and isomorphic to the underlying real algebra of $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C})$ (see Lemma 4.3(i)-(ii)). As σ with (1.2.b) lies in $\operatorname{Ker}(\Omega + \mathrm{Id})$, and so, by (1.7), in $\operatorname{Ker} \Lambda$, Theorem B(iii) gives $\sigma = 2\operatorname{Re} \sigma^{\mathfrak{h}}$, where $\sigma^{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ is $\mathbb{C}$-bilinear and symmetric. Clearly, $\sigma^{\mathfrak{h}}(x, y) = \beta^{\mathfrak{h}}(\Sigma^{\mathfrak{h}} x, y)$ for all $x, y \in \mathfrak{h}$, the Killing form $\beta^{\mathfrak{h}}$ of $\mathfrak{h}$, and some complex-linear operator $\Sigma^{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$. Taking $2\operatorname{Re}$ of both sides, we see that (2.1.i) yields (1.2.b) with Σ replaced by $\Sigma^{\mathfrak{h}}$. Consequently, $\Sigma^{\mathfrak{h}} = \Sigma$, and $\Sigma : \mathfrak{h} \to \mathfrak{h}$ is complex-linear. At the same time, (2.6) and Lemma 2.1(b) easily imply that Ω is self-adjoint. The two summands in (1.7) are therefore β-orthogonal, and so, by Theorem 4.1(iv), σ is orthogonal to $\beta = 2\operatorname{Re} \beta^{\mathfrak{h}}$ and $\gamma = 2\operatorname{Im} \beta^{\mathfrak{h}}$. Since σ, β and γ correspond as in (1.2.b) to Σ, Id and $-J$, cf. Remark 6.2, these orthogonality relations read $\operatorname{tr}_{\mathbb{R}} \Sigma = \operatorname{tr}_{\mathbb{R}} J\Sigma = 0$, that is, $\operatorname{tr}_{\mathbb{C}} \Sigma = 0$. On the other hand, the Cartan three-form $C^{\mathfrak{h}}$ of $\mathfrak{h}$ is a volume form in its underlying complex 3-space. Being traceless, Σ is thus an infinitesimal automorphism of both $C^{\mathfrak{h}}$ and $C = 2\operatorname{Re} C^{\mathfrak{h}}$ (see (2.1.ii)), as required. □

The first paragraph of the above proof obviously remains valid if $\mathfrak{g}$ is only assumed to be semisimple, and so it constitutes a direct argument showing that, for any semisimple real or complex Lie algebra $\mathfrak{g}$, all derivations of $\mathfrak{g}$ lie in $\operatorname{Ad}(\mathfrak{g})$.

7 Proof of Theorem C

For a real/complex Lie algebra $\mathfrak{g}$, let the mapping $\Phi : [\mathfrak{g}^*]^{\wedge 3} \times \mathfrak{g}^{\odot 2} \to [\mathfrak{g}^*]^{\wedge 4}$ be defined by $[\Phi(C, \mu)](x, y, z, z') = \mu(C(x, y), C(z, z')) + \mu(C(y, z), C(x, z')) + \mu(C(z, x), C(y, z'))$, where $\mu \in \mathfrak{g}^{\odot 2}$ is treated as a symmetric real/complex-bilinear form on $\mathfrak{g}^*$, and $C(x, y)$ stands for the element $C(x, y, \cdot)$ of $\mathfrak{g}^*$. If $\mathfrak{g}$ is also semisimple, the isomorphic identification $\mathfrak{g} \approx \mathfrak{g}^*$ provided by the Killing form β induces an isomorphism $[\mathfrak{g}^*]^{\odot 2} \to \mathfrak{g}^{\odot 2}$, which we write as $\sigma \mapsto \sigma^\sharp$. Then, in view of (1.4) and (1.8),

$$\Phi(C, \sigma^\sharp) = \Lambda\sigma \quad \text{for any } \sigma \in \mathfrak{g}^{\odot 2} \text{ and the Cartan three-form } C. \tag{7.1}$$

Theorem C is a trivial consequence of the following result combined with Lemma 4.3(ii) and the fact that, by multiplying a Lie-algebra bracket operation $[\,,]$ by a nonzero scalar, one obtains a Lie-algebra structure isomorphic to the original one. Note that the final clause of Theorem C is immediate from Lemma 7.1(a) along with Lemma 6.5 and Remark 4.5.

Lemma 7.1 *In a real or complex semisimple Lie algebra* $\mathfrak{g}$*, the Cartan three-form and the vector-space structure of* $\mathfrak{g}$ *uniquely determine each of the following objects.*

(a) *The vector subspaces constituting the simple direct summand ideals of* $\mathfrak{g}$.
(b) *Up to a sign, in the real case, the complex structure, defined as in Section 6, of every summand ideal* $\mathfrak{g}'$ *with* $\dim_{\mathbb{R}}\mathfrak{g}' \neq 6$ *which is a complex Lie algebra, treated as real.*
(c) *Up to multiplications by cubic roots of* 1*, the restrictions of the Lie-algebra bracket of* $\mathfrak{g}$ *to all such summands of dimensions other than* 3 *or* 6.
(d) *The Lie algebra isomorphism types of all summand ideals* $\mathfrak{g}'$ *with* $\dim_{\mathbb{R}}\mathfrak{g}' \neq 3$.

Proof Let C be the Cartan three-form of $\mathfrak{g}$. By (7.1), $\operatorname{Ker}\Delta = \{\sigma^\sharp : \sigma \in \operatorname{Ker}\Lambda\}$ for the real/complex-linear operator $\Delta : \mathfrak{g}^{\odot 2} \to [\mathfrak{g}^*]^{\wedge 4}$ given by $\Delta\mu = \Phi(C,\mu)$. Then, if one views all $\mu \in \operatorname{Ker}\Delta \subset \mathfrak{g}^{\odot 2}$ as linear operators $\mu : \mathfrak{g}^* \to \mathfrak{g}$,

(e) *the simple direct summands of* $\mathfrak{g}$ *are precisely the minimal elements, in the sense of inclusion, of the set* $\mathbf{S} = \{\mu(\mathfrak{g}^*) : \mu \in \operatorname{Ker}\Delta,\ \text{and}\ \dim\mu(\mathfrak{g}^*) = 3 \text{ or } \dim\mu(\mathfrak{g}^*) \geq 6\}$.

In fact, $\mathbf{S}$ consists of the images of those linear endomorphisms $\Sigma : \mathfrak{g} \to \mathfrak{g}$ which correspond via (1.2.b) to elements σ of $\operatorname{Ker}\Lambda$, and have $\operatorname{rank}\Sigma \notin \{0,1,2,4,5\}$. To describe all such Σ, we use the four parts of Theorem B, referring to them as (i) – (iv). Specifically, by (i), our endomorphisms Σ are direct sums of linear endomorphisms Σ_i of the simple direct summands $\mathfrak{g}_i$ of $\mathfrak{g}$, while the endomorphisms Σ_i are themselves subject to just two restrictions: one due to the exclusion of ranks $0,1,2,4$ and 5, the other depending, in view of (ii) – (iv), on $d_i = \dim\mathfrak{g}_i$, as follows. If $d_i = 3$, (ii) states that Σ_i is only required to be β-self-adjoint (to reflect symmetry of σ_i related to Σ_i as in (1.2.b)). Similarly, it is clear from (iv) and (2.2) that, with a specific the scalar field $\mathbb{F}$,

$$\begin{array}{l}\Sigma_i \text{ is a nonzero } \mathbb{F}\text{-multiple of Id when } d_i \notin \{3,6\}, \text{ where } \mathbb{F} = \mathbb{C} \text{ if } \mathfrak{g}_i \text{ is} \\ \text{either complex or real of type (1.3.b), and } \mathbb{F} = \mathbb{R} \text{ for real } \mathfrak{g}_i \text{ of type (1.3.a).}\end{array} \tag{7.2}$$

In the remaining case, $d_i = 6$. Then, by (iii), Σ_i is complex-linear and β-self-adjoint, cf. (2.3) and (2.1.i), but otherwise arbitrary.

The image $\Sigma(\mathfrak{g})$ of any Σ as above is the direct sum of the images of its summands Σ_i, and so it can be minimal only if there exists just one i with $\Sigma_i \neq 0$. For this i, minimality of $\Sigma(\mathfrak{g}) = \Sigma_i(\mathfrak{g}_i)$ implies that $\Sigma(\mathfrak{g}) = \mathfrak{g}_i$. In fact, in view of the last paragraph, the cases $d_i = 3$ and $d_i \notin \{3,6\}$ are obvious (the former since $\operatorname{rank}\Sigma_i \geq 3$) while, if $d_i = 6$, complex-linearity of Σ_i precludes not just $0,1,2,4$ and 5, but also 3 from being its real rank.

We thus obtain one of the inclusions claimed in (e): every minimal element of $\mathbf{S}$ equals some summand $\mathfrak{g}_i$. Conversely, any fixed summand $\mathfrak{g}_i$ is an element of $\mathbf{S}$, realized by Σ with $\Sigma_i = \mathrm{Id}$ and $\Sigma_j = 0$ for all $j \neq i$, cf. Lemma 4.3(iii). Minimality of $\mathfrak{g}_i$ is in turn obvious from (7.2) if $d_i \notin \{3,6\}$, while for $d_i = 3$ or $d_i = 6$ it follows from the restriction on $\operatorname{rank}\Sigma$ combined, in the latter case, with complex-linearity of Σ_i. This yields (e).

Now (a) is obvious from (e), as Δ and $\mathbf{S}$ depend only on C and the vector-space structure of $\mathfrak{g}$. To prove (b) – (c), we fix i with $d_i \notin \{3,6\}$. Elements μ of $\operatorname{Ker}\Delta$ having

$\mu(\mathfrak{g}^*) = \mathfrak{g}_i$ correspond, via (1.2.b) followed by the assignment $\sigma \mapsto \mu = \sigma^\sharp$, to endomorphisms Σ of $\mathfrak{g}$ which satisfy (7.2) and vanish on $\mathfrak{g}_j$ for $j \neq i$. Any such μ, now viewed as a bilinear form on $\mathfrak{g}^*$, is therefore obtained from a bilinear form μ_i on $\mathfrak{g}_i^*$ by the trivial extension to $\mathfrak{g}^*$, that is, pullback under the obvious restriction operator $\mathfrak{g}^* \to \mathfrak{g}_i^*$.

If $\mathbb{F} = \mathbb{R}$, it is immediate from (7.2) that the resulting forms μ_i are nonzero multiples of the reciprocal of the Killing form of $\mathfrak{g}_i$, and Remark 6.3 implies (c). Next, let $\mathbb{F} = \mathbb{C}$. We denote $\mathfrak{g}_i$ treated as a complex Lie algebra by $\mathfrak{h}$, and the Cartan three-form of $\mathfrak{h}$ by $C^\mathfrak{h}$. Formula (6.1) states that, in view of (7.2), the reciprocals of our μ_i are precisely the nonzero elements of the space $\mathscr{P}$ defined in Section 6. Thus, Lemma 6.1, (2.1.ii) and Remark 6.2 imply that C determines the triple $(J, \beta^\mathfrak{h}, C^\mathfrak{h})$ uniquely up to replacements by $(J, a\beta^\mathfrak{h}, aC^\mathfrak{h})$ or $(-J, a\overline{\beta^\mathfrak{h}}, a\overline{C^\mathfrak{h}})$, with $a \in \mathbb{C} \smallsetminus \{0\}$. This proves (b), while using Remarks 6.3 – 6.4 we obtain (c) for $\mathbb{F} = \mathbb{C}$ as well. Finally, (c) and Lemma 4.3(i)–(iii) easily yield (d). ☐

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Appendix: Meyberg's theorem

For any complex simple Lie algebra $\mathfrak{g}$, the operator Ω with (1.2) is diagonalizable. Its systems $\mathrm{Spec}[\mathfrak{g}]$ of eigenvalues and $\mathrm{Mult}[\mathfrak{g}]$ of the corresponding multiplicities are

$$\begin{array}{l}
\mathrm{Spec}[\mathfrak{sl}_n] = (2, 1, 2/n, -2/n) \ \text{ and} \\
\mathrm{Mult}[\mathfrak{sl}_n] = (1, n^2-1, n^2(n-3)(n+1)/4, n^2(n+3)(n-1)/4), \ \text{ if } n \geq 4. \\
\mathrm{Spec}[\mathfrak{sp}_n] = (2, (n+4)/(n+2), -4/(n+2), 2/(n+2)) \ \text{ for even } n \geq 4, \ \text{ and} \\
\mathrm{Mult}[\mathfrak{sp}_n] = (1, (n-2)(n+1)/2, n(n+1)(n+2)(n+3)/24, n(n-1)(n-2)(n+3)/12). \\
\mathrm{Spec}[\mathfrak{so}_n] = (2, (n-4)/(n-2), 4/(n-2), -2/(n-2)) \ \text{ if } n = 7 \text{ or } n \geq 9, \ \text{ while} \\
\mathrm{Mult}[\mathfrak{so}_n] = (1, (n+2)(n-1)/2, n(n-1)(n-2)(n-3)/24, n(n+1)(n+2)(n-3)/12)
\end{array}$$

and, if $\mathfrak{g}$ is one of the exceptional complex Lie algebras $\mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$,

$$\begin{array}{l}
\mathrm{Spec}[\mathfrak{g}] = (2, (1+w)/6, (1-w)/6), \ \text{ with } \mathrm{Mult}[\mathfrak{g}] \text{ equal to} \\
(1, 3d[(d+2)w-(d+32)]/[w(11-w)], 3d[(d+2)w+(d+32)]/[w(11+w)]),
\end{array} \tag{7.3}$$

with $d = \dim \mathfrak{g}$ and $w = [(d+242)/(d+2)]^{1/2}$. This is a result of Meyberg [8] who, rather than our Ω, studied the operator $T = \Omega/2$. (The formula for w in [8] misses the exponent $1/2$.) For $\mathfrak{sl}_2$, the resulting "eigenvalue" $4/3$ of multiplicity 0 should be disregarded. All isomorphism types of complex simple Lie algebras are listed above, cf. Remark 2.3.

The dimensions d of $\mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ are $3, 8, 14, 28, 52, 78, 133, 248$ [1, pp. 32, 37]. The eigenvalues $0, -1, 1$ in (7.3) would correspond to $w = 1, 7, 5$, of which only the latter two occur, for $d = 3, 8$ and $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$, in agreement with Theorem 2.2.

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