

Almost-product Kähler surfaces

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ABSTRACT. We study the case where the tangent bundle of a Kähler surface splits orthogonally into two integrable complex-line subbundles. This amounts to the presence of a unit-length closed anti-self-dual 2-form, leading in turn to a curvature condition. Therefore, such a decomposition need not exist, even locally, in contrast with general Riemannian four-manifolds which, under the assumption of real-analyticity, were shown by Grant and Vickers [11] to always admit, locally, two mutually orthogonal integrable rank-two distributions. We exhibit local coordinates naturally adapted to a decomposition as above in a Kähler surface, which may be used for simple constructions of examples. We also provide a characterization of the Kähler case within the wider class of almost-Kähler surfaces arising from the coordinates just mentioned. The characterization involves a system of four first-order quasilinear partial differential equations imposed on four unknown functions of four variables and, using Cartan's test, we prove the system's local solvability. Finally, we show that, for such a decomposition in a Kähler surface, one of the summands is holomorphic if and only if the other one has totally geodesic leaves, and describe a general construction of examples with this last property.

1. Introduction

Grant and Vickers [11] showed that every real-analytic Riemannian four-manifold admits, locally, a decomposition of its tangent bundle into an orthogonal direct sum of two integrable real-plane subbundles. As we point out in Proposition B, the analogous assertion fails to hold, in general, for real-analytic Kähler surfaces, if one in addition requires the two summands to be J -invariant (with J denoting, here and below, the almost-complex structure).

The goal of this paper is to provide a complete local description – see Theorem C below – of Kähler surfaces (M, g) that do admit such a decomposition: the case where there exist $\mathcal{H}$ and $\mathcal{V}$ such that

$$(1.1) \quad TM = \mathcal{H} \oplus \mathcal{V} \text{ is the } g\text{-orthogonal direct sum of} \\ \text{two integrable complex-line subbundles } \mathcal{H} \text{ and } \mathcal{V}.$$

In the following two propositions, proved in Sect. 3, M as well as the leaves of $\mathcal{H}$ and $\mathcal{V}$, being almost-complex manifolds, are canonically oriented.

2020 *Mathematics Subject Classification.* 53B35.

Key words and phrases. Kähler surface; almost-product structure; holomorphic distribution; totally geodesic foliation; exterior differential system.

Proposition A. *On an almost-Kähler surface (M, g) the decompositions (1.1) stand in a one-to-one correspondence with unit-length closed anti-self-dual 2-forms. The correspondence associates with a given decomposition (1.1) the 2-form ζ such that $\mathcal{H}$ and $\mathcal{V}$ are ζ -orthogonal, while ζ restricted to each leaf of $\mathcal{H}$, or $\mathcal{V}$, equals $-1/\sqrt{2}$ or, respectively, $1/\sqrt{2}$ times the area form of the leaf.*

In a distant analogy to the results of Tod [15], cf. also [10], (1.1) implies a curvature condition: a pointwise inequality relating the lowest eigenvalue of the Weyl tensor acting on anti-self-dual bivectors to the scalar curvature s .

Proposition B. *At points of a Kähler surface (M, g) where, locally, a decomposition (1.1) exists, one necessarily has*

$$(1.2) \quad \min \text{spec } W^- \leq s/6 \text{ and, consequently, } |W^-| \geq -s/6.$$

As a result, (1.1) does not hold, even locally, in a self-dual Kähler surface with negative scalar curvature, an example of which is provided by the complex hyperbolic plane with the standard metric. For more examples, see Remark 3.1.

The most trivial examples of (1.1) for Kähler surfaces (M, g) arise when

$$(1.3) \quad \begin{array}{l} \mathcal{H} \text{ and } \mathcal{V} \text{ are, locally, the factor distributions} \\ \text{of a Riemannian-product decomposition of } g. \end{array}$$

Due to closedness of the Ricci form, (1.1) is satisfied by the Ricci eigendistributions $\mathcal{H}$ and $\mathcal{V}$, in any Kähler surface (M, g) in which

$$(1.4) \quad \text{the Ricci tensor Ric of } g \text{ has two distinct constant eigenvalues.}$$

Several authors studied consequences of (1.4) under additional assumptions.

Apostolov, Drăghici and Moroianu [1, Theorem 1.2] showed that, in the *compact* case, (1.4) implies (1.3) unless the Ricci eigenvalues are both negative (and, if they are, M must be a minimal surface of general type with even positive signature and ample canonical bundle). Apostolov *et al.* also pointed out that, without the compactness assumption, (1.4) does not imply (1.3), with counterexamples having one Ricci eigenvalue equal to zero.

Nihonyanagi, Oguro and Sekigawa [13], assuming *homogeneity* of (M, g) , proved that (1.4) leads to (1.3) unless Ric has a zero eigenvalue. The latter case does occur, as shown by Kowalski [12, Theorem VI.3], while Drăghici [9] explained why Kowalski's example is essentially unique in this regard.

Here and below we often use local coordinates

$$(1.5) \quad \begin{array}{l} x^1, \dots, x^4 \text{ on a coordinate domain } U \subseteq M \text{ with the index} \\ \text{ranges } j, k, l \in \{1, 2, 3, 4\}, \quad \lambda, \mu, \nu \in \{1, 2\}, \quad p, q, r \in \{3, 4\}. \end{array}$$

Any smooth functions $\alpha, \beta, \gamma, \delta$ of $x^1, \dots, x^4$ give rise to a metric g with

$$(1.6) \quad \begin{array}{l} (g_{11}, g_{12}, g_{22}) = (e^{-\beta} \sec \alpha, \tan \alpha, e^{\beta} \sec \alpha), \\ (g_{33}, g_{34}, g_{44}) = (e^{-\delta} \sec \gamma, \tan \gamma, e^{\delta} \sec \gamma), \\ (g_{13}, g_{14}, g_{23}, g_{24}) = (0, 0, 0, 0), \text{ where} \\ \alpha, \gamma : U \rightarrow (-\pi/2, \pi/2) \text{ and } \beta, \delta : U \rightarrow \mathbb{R}. \end{array}$$

The $(1, 1)$ tensor field defined, using (1.6), by

$$(1.7) \quad \begin{bmatrix} J_1^1 & J_2^1 \\ J_1^2 & J_2^2 \end{bmatrix} = \begin{bmatrix} -g_{12} & -g_{22} \\ g_{11} & g_{12} \end{bmatrix}, \quad \begin{bmatrix} J_3^3 & J_4^3 \\ J_3^4 & J_4^4 \end{bmatrix} = \begin{bmatrix} -g_{34} & -g_{44} \\ g_{33} & g_{34} \end{bmatrix}, \quad J_\lambda^p = J_p^\lambda = 0,$$

is an almost-complex structure, since (1.6) gives

$$(1.8) \quad \begin{aligned} &\text{i)} \quad \det[g_{\lambda\mu}] = \det[g_{pq}] = 1, \text{ with this obvious consequence:} \\ &\text{ii)} \quad (g^{11}, g^{12}, g^{22}, g^{33}, g^{34}, g^{44}) = (g_{22}, -g_{12}, g_{11}, g_{44}, -g_{34}, g_{33}). \end{aligned}$$

Theorem C. *If the functions $\alpha, \beta, \gamma, \delta$ in (1.6) satisfy the system*

$$(1.9) \quad \begin{aligned} &\text{a)} \quad (-1)^p \beta_p = \alpha_q e^{(-1)^p \delta} \sec \gamma - \alpha_p \tan \gamma \quad \text{whenever } \{p, q\} = \{3, 4\}, \\ &\text{b)} \quad (-1)^\lambda \delta_\lambda = \gamma_\mu e^{(-1)^\lambda \beta} \sec \alpha - \gamma_\lambda \tan \alpha \quad \text{whenever } \{\lambda, \mu\} = \{1, 2\} \end{aligned}$$

of four first-order partial differential equations, where the subscripts denote partial derivatives, then g in (1.6) is a Kähler metric on U with the complex-structure tensor J given by (1.7), and (1.1) holds, on U rather than M , for $\mathcal{H}$ and $\mathcal{V}$ spanned by the first two and last two coordinate vector fields.

Conversely, for any Kähler surface (M, g) with (1.1), $g, J, \mathcal{H}$ and $\mathcal{V}$ arise, locally, in suitable local coordinates $y^1, \dots, y^4$, from the above construction applied to some functions $\alpha, \beta, \gamma, \delta$ satisfying (1.9).

We also derive Theorem C, in Sect. 4, from our next result, which addresses the meaning of the system (1.9).

Theorem D. *Formulae (1.6) – (1.7) always define an almost-Kähler metric, in the usual sense of skew-symmetry and closedness of $\omega = g(J\cdot, \cdot)$. The five conditions listed below are mutually equivalent.*

- (i) *Equations (1.9).*
- (ii) *$\nabla \omega = 0$ for $\omega = g(J\cdot, \cdot)$ and the Levi-Civita connection ∇ of g .*
- (iii) *$\nabla J = 0$, that is, the Kähler property of g .*
- (iv) *Integrability of the almost-complex structure J .*
- (v) *$\mathcal{V}$ -holomorphicity of $\alpha + i\beta$ and $\mathcal{H}$ -holomorphicity of $\gamma + i\delta$.*

Furthermore, as we show in Sect. 4,

- (vi) *the first part of (v) is equivalent to (1.9-a), the second to (1.9-b).*

We say here that a function $\phi : M \rightarrow \mathbb{C}$ on an almost-complex manifold M is $\mathcal{V}$ -holomorphic, for an integrable complex-line subbundle $\mathcal{V}$ of TM , if the restriction of ϕ to every leaf of $\mathcal{V}$ is a holomorphic function on the leaf. (Note that the leaves of $\mathcal{V}$ are complex manifolds.)

Remark E. The assumptions in (1.6) – (1.7) and, consequently, their conclusions, remain unaffected when one switches $(1, 2, \alpha, \beta)$ with $(3, 4, \gamma, \delta)$.

An obvious question arises as to when $\mathcal{H}$ or $\mathcal{V}$ in (1.1) is holomorphic. Without loss of generality, we may consider this to be the case for $\mathcal{V}$. Note that holomorphicity implies integrability (the complex dimension being 1), and $\mathcal{H}, \mathcal{V}$ must both be holomorphic whenever they are, locally, the factor distributions of a Riemannian-product decomposition of g .

Theorem F. *For any Kähler surface (M, g) with (1.1), $\mathcal{V}$ in (1.1) is holomorphic as a complex-line subbundle of TM if and only if*

- (*) *$\mathcal{H}$ has totally geodesic leaves.*

In the case of $g, \mathcal{H}, \mathcal{V}$ arising from the construction of Theorem C, () means that α and β or, equivalently, g_{11}, g_{12}, g_{22} , depend only on x^1 and x^2 .*

All these conclusions remain valid when $\mathcal{H}$ and $\mathcal{V}$ are switched.

Corollary G. *Whenever (1.1) holds on a Kähler surface (M, g) , holomorphicity of both $\mathcal{H}$ and $\mathcal{V}$ is equivalent to their being parallel, that is, to (1.3).*

We prove Theorem F and Corollary G in Section 5, where we also point out (see Remark 5.1) that the situation of Theorem F has a local description in terms of Theorem C, much simpler than the general case of (1.1). More precisely, Remark 5.1 provides a *general local construction of Kähler surfaces (M, g) which constitute holomorphic bundles over real surfaces such that the horizontal distribution is integrable and the bundle projection is a Riemannian submersion.*

The following result, proved in Appendix II, establishes the existence of numerous solutions to the system (1.9).

Proposition H. *Real-analytic solutions $(\alpha, \beta, \gamma, \delta)$ to (1.9) exist on a neighborhood of any $z \in \mathbb{R}^4$ and realize all first-order initial data satisfying (1.9) at z .*

2. Preliminaries

All manifolds and mappings, including tensor fields, are assumed to be smooth, and all manifolds – to be connected.

In local coordinates $y^1, \dots, y^4$, for functions ϕ and ψ , obviously,

$$(2.1) \quad \begin{aligned} & \text{the differential 2-form } \phi dy^1 \wedge dy^2 + \psi dy^3 \wedge dy^4 \text{ is} \\ & \text{closed if and only if } \partial_3 \phi = \partial_4 \phi = \partial_1 \psi = \partial_2 \psi = 0, \end{aligned}$$

that is, if and only if ϕ and ψ depend only on y^1, y^2 and, respectively, y^3, y^4 .

The Nijenhuis tensor N of an almost-complex structure J sends vector fields v, w to the vector field $N(v, w) = [v, w] + J[Jv, w] + J[v, Jw] - [Jv, Jw]$, easily seen to coincide with $[J\nabla_v J - \nabla_{Jv} J]w + [\nabla_{Jw} J - J\nabla_w J]v$ for any torsion-free connection ∇ . The Newlander-Nirenberg theorem now yields this well-known observation:

$$(2.2) \quad \begin{aligned} & \text{an almost-complex structure parallel with re-} \\ & \text{spect to a torsion-free connection is integrable.} \end{aligned}$$

For a 2-form ζ on a manifold, $\Theta = d\zeta$, and

$$(2.3) \quad \Theta(v, w, w') \text{ equals } d_v[\zeta(w, w')] - \zeta([v, w], w') + \text{cycl},$$

+ cycl meaning here *summed cyclically over v, w, w'* . See, e.g., [2, p. 21].

We will use Cartan's lemma [14, p. 18]: in a vector space, given any vectors $w_1, \dots, w_k$ and linearly independent vectors $v_1, \dots, v_k$, one has

$$(2.4) \quad \begin{aligned} & v_1 \wedge w_1 + \dots + v_k \wedge w_k = 0 \text{ if and only if } w_i = h_{i1}v_1 + \dots + h_{ik}v_k \text{ for} \\ & \text{all } i, \text{ with some } h_{ij} \text{ forming a symmetric } k \times k \text{ matrix: } h_{ij} = h_{ji}. \end{aligned}$$

We also need the following well-known fact – see, e.g., [8, formula (11.1)]:

$$(2.5) \quad \begin{aligned} & \text{in dimension } n \text{ any differential } n\text{-form without zeros equals, lo-} \\ & \text{cally, } dx^1 \wedge dx^2 \wedge \dots \wedge dx^n \text{ for some local coordinates } x^1, \dots, x^n. \end{aligned}$$

Remark 2.1. Given a rank-two differential 2-form $\zeta = \xi^1 \wedge \xi^2$ on a Riemannian manifold (M, g) of dimension $n \geq 2$, the distribution $\text{Ker } \zeta$ is integrable if and only if $d\zeta = \eta \wedge \zeta$ for some 1-form η , and such η becomes unique if one assumes that $\eta = g(v, \cdot)$ with a vector field v having $\zeta(v, \cdot) = 0$. In fact, the ‘if’ part is obvious from (2.3). Assuming integrability of $\text{Ker } \zeta$ and using (2.3), for $\Theta = d\zeta$, we get $\Theta(v, v', \cdot) = 0$ whenever $\zeta(v, \cdot) = \zeta(v', \cdot) = 0$. Extending $\xi^1 \wedge \xi^2$ locally to a local trivialization $\xi^1, \dots, \xi^n$ of T^*M dual to a local frame $e_1, \dots, e_n$, and expressing Θ as a functional combination of linearly independent threefold exterior

products of the 1-forms ξ^i , we now see that the only possibly nonzero coefficients occur when $i > 2$ for at most one i present in the product, which proves our claim.

Remark 2.2. For any holomorphic function $f : M \rightarrow \mathbb{C}$ without critical points on a complex manifold M of any complex dimension $n \geq 2$, the distribution $\mathcal{H}$ on M tangent to the level hypersurfaces of f is holomorphic as a vector subbundle of TM . In fact, given holomorphic vector fields w_j , $j = 1, \dots, n$, on an open set $U \subseteq M$, trivializing TU , and rearranged so that $f_1 \neq 0$ everywhere in U , with f_j denoting the directional derivative of f along w_j , the restriction of $\mathcal{H}$ to U has the holomorphic local trivialization v_j , $j = 2, \dots, n$, given by $v_j = f_1 w_j - f_j w_1$.

Remark 2.3. It is well known – see, for instance, [7, Lemma 7.1] – that the spaces $\Lambda^\pm$ of self-dual and anti-self-dual bivectors in an oriented Euclidean four-space $\mathcal{T}$ are themselves naturally oriented, and their length $\sqrt{2}$ positive orthogonal bases are precisely the same as all the triples

$$(2.6) \quad \pm e_1 \wedge e_2 + e_3 \wedge e_4, \quad \pm e_1 \wedge e_3 + e_4 \wedge e_2, \quad \pm e_1 \wedge e_4 + e_2 \wedge e_3,$$

where e_1, e_2, e_3, e_4 is a positive orthonormal basis of $\mathcal{T}$. In addition, e_1, e_2, e_3, e_4 is uniquely determined by (2.6) up to an overall sign change. Since, as pointed out in [7, formula (7.7)], the mapping sending (e_1, e_2, e_3, e_4) to (2.6) is equivariant relative to a two-fold covering homomorphism $\mathrm{SO}(4) \rightarrow \mathrm{SO}(3) \times \mathrm{SO}(3)$,

$$(2.7) \quad \pm(e_1, e_2, e_3, e_4) \text{ depends real-analytically on (2.6).}$$

Remark 2.4. Given a subspace V of a vector space W , the set $\mathcal{Y}$ of all subspaces H of W with $W = H \oplus V$ is well known – see, e.g., [3, p. 109] – to carry a natural structure of an affine space. To be specific, $\mathcal{Y}$ stands in a one-to-one correspondence with the preimage of the point $\mathrm{Id}_{W/V}$ under the surjective linear operator $\mathrm{Hom}(W/V, W) \rightarrow \mathrm{End}(W/V)$ sending any Φ to the composition $\Pi\Phi$, where $\Pi : W \rightarrow W/V$ is the quotient projection. The correspondence associates with $H \in \mathcal{Y}$ the inverse of the restriction $\Pi : H \rightarrow W/V$. More explicitly, the vector space $\mathcal{X} = \mathrm{Hom}(W/V, V)$ acts on $\mathcal{Y}$ simply transitively, so that $H + \Psi$, for $(H, \Psi) \in \mathcal{Y} \times \mathcal{X}$, equals the image of H under $\mathrm{Id}_W + \Psi\Pi$.

3. Proofs of Propositions A and B

Proof of Proposition A. We use the metric to identify vectors with 1-forms, and bivectors with both 2-forms and skew-adjoint $(1, 1)$ tensors. The “purely algebraic” version of Proposition A is also true. It arises when one deletes the words *integrable*, in (1.1), and *closed*, and uses the area forms of $\mathcal{H}$ and $\mathcal{V}$ rather than referring to “leaves” (which now need not exist). In fact, for ζ described in the final clause of the proposition, $\sqrt{2}\zeta^\pm$ are the initial entries in (2.6), where ζ^- stands for ζ , and $\sqrt{2}\zeta^+$ for the Kähler form $\omega = g(J\cdot, \cdot)$, while e_1, e_2 and e_3, e_4 are fixed positive orthonormal local trivializations of $\mathcal{H}$ and $\mathcal{V}$. Due to Remark 2.3, $*\zeta^\pm = \pm\zeta^\pm$ and $|\zeta^\pm| = 1$, while $\zeta^\pm$ uniquely determine $\mathcal{H}$ and $\mathcal{V}$ since, obviously,

$$(3.1) \quad \mathcal{H} = \mathrm{Ker}(\zeta^+ + \zeta^-) \text{ and } \mathcal{V} = \mathrm{Ker}(\zeta^+ - \zeta^-),$$

showing that the assignment sending (1.1) to ζ is injective.

To verify its surjectivity, we start with ζ as in the first part of the proposition, let $\zeta^- = \zeta$ and $\sqrt{2}\zeta^+ = \omega$ (the Kähler form), and then extend $\sqrt{2}\zeta^\pm$ to length $\sqrt{2}$ positive orthogonal local trivializations of the oriented bundles of self-dual and

anti-self-dual 2-forms, cf. Remark 2.3. Next, we again invoke Remark 2.3, including (2.7), to express the latter local trivializations as (2.6) for a smooth local orthonormal frame (e_1, e_2, e_3, e_4) . Finally, we set $\mathcal{H} = \text{span}(e_1, e_2)$ and $\mathcal{V} = \text{span}(e_3, e_4)$, so that surjectivity follows.

Proving Proposition A is thus reduced to showing that closedness of ζ implies integrability of both distributions $\mathcal{H}, \mathcal{V}$, and vice versa. The first implication is straightforward: due to closedness of $\omega = \sqrt{2}\zeta^+$, if $\zeta = \zeta^-$ is closed, so are both $\zeta^+ \pm \zeta^-$, and, consequently, $\mathcal{H}$ and $\mathcal{V}$ are integrable in view of (3.1) and (2.3).

For the converse implication, use Remark 2.1 and (3.1) to write

$$d[\zeta^+ \pm \zeta^-] = \eta^\pm \wedge [\zeta^+ \pm \zeta^-] = \eta \wedge [\zeta^+ \pm \zeta^-]$$

for $\zeta^\pm$ and e_1, e_2, e_3, e_4 as above, where $\eta = \eta^+ + \eta^-$, and the 1-form η^+ , or η^- , treated as a vector field, is a functional combination of e_1, e_2 or, respectively, e_3, e_4 . Adding/subtracting, we get $d\omega = \eta \wedge \omega$ and $d\zeta = \eta \wedge \zeta$. As ω is closed and nondegenerate, $\eta = 0$, which completes the proof. $\square$

In a Riemannian manifold of dimension $n \geq 4$ with the scalar curvature s , the action of the Weyl tensor on 2-forms ζ satisfies the following well-known formula [5, p. 409], [6, p. 458], easily derived from the Ricci identity:

$$(3.2) \quad W\zeta = \frac{1}{2} [\delta(\nabla\zeta - d\zeta) - d\delta\zeta] + \frac{n-4}{2(n-2)} \{r, \zeta\} + \frac{s}{(n-1)(n-2)} \zeta,$$

where $\{, \}$ is the anticommutator and r the Ricci tensor. In local coordinates,

$$(3.3) \quad \begin{aligned} W_{ijpq}\zeta^{pq} = & -\zeta_{pi,j}{}^p - \zeta_{jp,i}{}^p - \zeta_{pj,i}{}^p + \zeta_{pi,j}{}^p \\ & + \frac{n-4}{n-2} (r_j^p \zeta_{ip} + r_i^p \zeta_{pj}) + \frac{2s}{(n-1)(n-2)} \zeta_{ij}, \end{aligned}$$

the index range, in contrast with (1.5), being $i, j, p, q = 1, \dots, n$, and $\delta(\nabla\zeta - d\zeta)$ having the coordinate expression $-\zeta_{pi,j}{}^p - \zeta_{jp,i}{}^p = \zeta_{ij,p}{}^p - (\zeta_{ij,p} + \zeta_{pi,j} + \zeta_{jp,i})^p$. As $2|\zeta|^2 = \zeta^{ij}\zeta_{ij}$, the Laplacian of $|\zeta|^2$ equals $\zeta^{ij,p} \zeta_{ij,p} + \zeta^{ij} \zeta_{ij,p}{}^p = 2|\nabla\zeta|^2 + 2\langle\zeta, \delta\nabla\zeta\rangle$. When $|\zeta|$ is constant, $n = 4$, and $d\zeta = \delta\zeta = 0$, (3.2) with $\langle\zeta, \delta\nabla\zeta\rangle = -|\nabla\zeta|^2$ gives

$$(3.4) \quad \langle W\zeta, \zeta \rangle - s|\zeta|^2/6 = -|\nabla\zeta|^2/2.$$

Proof of Proposition B. The existence of a decomposition (1.1) gives rise, via Proposition A, to a unit-length closed anti-self-dual 2-form ζ which, due to its anti-self-duality, is also co-closed. Now the first part of (1.2) follows from (3.4) since $\min \text{spec } W^-$, at any point z , equals the minimum of $\langle W\zeta, \zeta \rangle$ over all unit-length anti-self-dual exterior 2-forms ζ at z . The second part of (1.2) is in turn obvious as tracelessness of W^- gives $|W^-| \geq -\min \text{spec } W^-$. $\square$

Remark 3.1. Due to Propositions A, and B, both (1.2) and (1.1), fail to hold, even locally, in self-dual Kähler surfaces of negative scalar curvature, such as the standard complex hyperbolic plane, some examples constructed in [4], and small deformations of the above metrics, arising, for instance, when one adds to the Kähler form $i\partial\bar{\partial}\varphi$ for a real-valued function φ which is C^4 -close to zero.

4. Proofs of Theorems D and C

Before proving the theorems, we derive some consequences of (1.6) – (1.7), which will be used in Sect. 5 as well. Without assuming (1.9), note that, by (1.6),

$$(4.1) \quad g_{\lambda\lambda} = e^{(-1)^\lambda \beta} \sec \alpha, \quad g_{12} = \tan \alpha, \quad g_{pp} = e^{(-1)^p \delta} \sec \gamma, \quad g_{34} = \tan \gamma$$

if $\lambda \in \{1, 2\}$ and $p \in \{3, 4\}$. Therefore, one has (1.9) if and only if

$$(4.2) \quad \text{a) } (-1)^p \beta_p = g_{pp} \alpha_q - g_{34} \alpha_p, \quad \text{b) } (-1)^\lambda \delta_\lambda = g_{\lambda\lambda} \gamma_\mu - g_{12} \gamma_\lambda$$

whenever $\{\lambda, \mu\} = \{1, 2\}$ and $\{p, q\} = \{3, 4\}$. From (1.6) – (1.7) we also get

$$(4.3) \quad J_p^q = (-1)^q e^{(-1)^p \delta} \sec \gamma, \quad J_p^p = (-1)^p \tan \gamma \quad (\text{no summing}) \text{ if } \{p, q\} = \{3, 4\}.$$

For g, J defined by (1.6) – (1.7) and $\omega = g(J \cdot, \cdot)$, one easily sees that

$$(4.4) \quad \begin{aligned} \omega_{12} = \omega_{34} = -\omega_{21} = -\omega_{43} = 1 \quad \text{and} \quad \omega_{jk} = 0 \quad \text{other-} \\ \text{wise, or, equivalently, } \omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4. \end{aligned}$$

Thus, in terms of the components Γ_{jk}^l of the Levi-Civita connection ∇ of the metric g , with k', l' defined by $(1', 2', 3', 4') = (2, 1, 4, 3)$, the components of $\nabla \omega$ are

$$(4.5) \quad \omega_{kl', j} = (-1)^l \Gamma_{jk}^l + (-1)^k \Gamma_{jl'}^{k'}.$$

On the other hand, due to skew-symmetry of ω , cf. (4.4),

$$(4.6) \quad \nabla \omega \text{ is completely determined by the components } \omega_{kl, j} \text{ with } k < l.$$

With the index ranges (1.5), $\Gamma_{j\lambda}^\lambda = \Gamma_{jp}^p = 0$ due to (1.8-i). Thus,

$$(4.7) \quad \omega_{12, j} = \omega_{34, j} = 0,$$

since $\omega_{12, j} = -\Gamma_{j\lambda}^\lambda$ and $\omega_{34, j} = -\Gamma_{jp}^p$ in view of (4.5).

Next, (4.5), (1.6), (1.8-ii) and the Christoffel-symbol formula yield the following equalities, valid whenever $\{\lambda, \mu\} = \{1, 2\}$ and $\{p, q\} = \{3, 4\}$, with repeated indices *not* summed over, and (4.8-d) arising as a trivial consequence of (4.1):

$$(4.8) \quad \begin{aligned} \text{a) } \omega_{\lambda p, \lambda} &= (-1)^q \Gamma_{\lambda\lambda}^q + (-1)^\lambda \Gamma_{\lambda p}^\mu, & \omega_{\lambda p, \mu} &= (-1)^q \Gamma_{12}^q + (-1)^\lambda \Gamma_{\mu p}^\mu, \\ \text{b) } 2\Gamma_{\lambda\lambda}^q &= g_{34} \partial_p g_{\lambda\lambda} - g_{pp} \partial_q g_{\lambda\lambda}, & 2\Gamma_{\lambda p}^\mu &= g_{\lambda\lambda} \partial_p g_{12} - g_{12} \partial_p g_{\lambda\lambda}, \\ \text{c) } 2\Gamma_{12}^q &= g_{34} \partial_p g_{12} - g_{pp} \partial_q g_{12}, & 2\Gamma_{\mu p}^\mu &= g_{\lambda\lambda} \partial_p g_{\mu\mu} - g_{12} \partial_p g_{12}, \\ \text{d) } \partial_p \log g_{\lambda\lambda} &= (-1)^\lambda \beta_p + g_{12} \alpha_p, & \partial_p g_{12} &= g_{11} g_{22} \alpha_p. \end{aligned}$$

If $\{\lambda, \mu\} = \{1, 2\}$ and $\{p, q\} = \{3, 4\}$, (4.4) and (4.8) give

$$(4.9) \quad \begin{aligned} 2(-1)^\lambda \omega_{\lambda p, \lambda} / g_{\lambda\lambda} &= [(-1)^{\lambda+p} g_{12} + g_{34}] [g_{pp} \alpha_q - g_{34} \alpha_p - (-1)^p \beta_p] \\ &\quad + g_{pp} [g_{qq} \alpha_p - g_{34} \alpha_q - (-1)^q \beta_q] + (g_{11} g_{22} - g_{12}^2 + g_{34}^2 - g_{33} g_{44}) \alpha_p, \\ &\quad \text{where, due to (1.8-i), } (g_{11} g_{22} - g_{12}^2 + g_{34}^2 - g_{33} g_{44}) \alpha_p = 0, \\ 2\omega_{\lambda p, \mu} &= (-1)^p g_{11} g_{22} [g_{pp} \alpha_q - g_{34} \alpha_p - (-1)^p \beta_p]. \end{aligned}$$

Proof of Theorem D. The almost-Kähler claim is obvious from (4.4), while (4.2), (4.6), (4.7) and (4.9) establish the equivalence between (i) and (ii). As $\omega = g(J \cdot, \cdot)$, (ii) is in turn equivalent to (iii). Next, (2.2) shows that (iii) implies (iv), while – see [2, Prop. 2.29] – (iv) and closedness of ω , cf. (4.4), give (iii). Finally, the $\mathcal{V}$ -holomorphicity of $\alpha + i\beta$ amounts to $(d\alpha + id\beta)J = i(d\alpha + id\beta)$, that is, $(d\alpha)J = -d\beta$ which, by (4.3), is nothing else than (1.9-a).

This proves the first assertion in (vi) and, combined with Remark E, yields the second one as well, completing the proof. $\square$

PROOF OF THEOREM C. The first claim is immediate from the fact that (iii) follows from (i) in Theorem D.

Let us now assume (1.1). The two subbundles are spanned, locally, by the first two and last two coordinate vector fields for some local coordinates $y^1, \dots, y^4$. Consequently, the Kähler form $\omega = g(J\cdot, \cdot)$ equals $\phi dy^1 \wedge dy^2 + \psi dy^3 \wedge dy^4$ for some functions ϕ, ψ and, according to the line following (2.1), the relation $d\omega = 0$ shows that ϕ and ψ are functions of y^1, y^2 and y^3, y^4 , respectively. Now (2.5) gives (4.4) with some new coordinates $x^1, \dots, x^4$. This yields (1.8-i) since, by (4.4), $dx^1 \wedge dx^2$ (or, $dx^3 \wedge dx^4$) restricted to each leaf of $\mathcal{H}$ (or, $\mathcal{V}$) in (1.1) equals the area form of the restriction of g to the leaf. Due to (1.8-i) and g -orthogonality in (1.1), some unique functions $\alpha, \beta, \gamma, \delta$ on the coordinate domain U satisfy (1.6). As (iii) implies (i) in Theorem D, the final clause of the theorem follows, which completes the proof. $\square$

5. Proofs of Theorem F and Corollary G

It is immediate from the Christoffel-symbol formula that the leaves of $\mathcal{H}$ are totally geodesic if and only if g_{11}, g_{12}, g_{22} depend only on x^1 and x^2 .

Now let $\mathcal{V}$ in Theorem F be holomorphic. We use the index ranges of (1.5):

$$(5.1) \quad \lambda, \mu, \nu \in \{1, 2\}, \quad p, q \in \{3, 4\}.$$

The final clause of Theorem C allows us to assume (1.6) – (1.9) for suitable local coordinates (1.5) and functions $\alpha, \beta, \gamma, \delta$. Thus, by (1.7), whenever $\{\lambda, \mu\} = \{1, 2\}$,

$$(5.2) \quad J_\lambda^\lambda = (-1)^\lambda g_{12}, \quad J_\mu^\lambda = (-1)^\lambda g_{\mu\mu}, \quad J_p^\mu = J_\lambda^p = 0 \text{ for } p \in \{3, 4\}.$$

With the index ranges (5.1), we then have

$$(5.3) \quad B_{\lambda p}^\mu = 0 \text{ for all } \lambda, \mu, p, \text{ where } B_{\lambda p}^\mu = J_\nu^\mu \Gamma_{\lambda p}^\nu - J_\lambda^\nu \Gamma_{\nu p}^\mu.$$

In fact, at each point x , some real-holomorphic vector field w on a neighborhood of x is tangent to $\mathcal{V}$ and has $w_x \neq 0$. Such w obviously realize all nonzero vectors tangent to $\mathcal{V}$ at all points. Holomorphicity of w amounts to vanishing of the $(1, 1)$ tensor field $S = [J, \nabla w]$. However, $S_\lambda^\mu = B_{\lambda p}^\mu w^p$, since w is tangent to $\mathcal{V}$ (and so $w^\lambda = 0$), which proves (5.3).

Next, replacing the Christoffel symbols and components of J with the expressions provided by (4.8-b) – (4.8-c) and (5.2), then combining two out of the four resulting terms with the aid of (1.8-i), while noting that the other two cancel each other, and, finally, using (4.8-d), we see that, if $\lambda \in \{1, 2\}$ and $p \in \{3, 4\}$, with no summation over the repeated index λ ,

$$(5.4) \quad B_{\lambda p}^\lambda = (-1)^\lambda g_{11} g_{22} \alpha_p.$$

Holomorphicity of $\mathcal{V}$ thus implies, via (5.3), (5.4), (1.9-a) and (1.6), that $\mathcal{H}$ has totally geodesic leaves or, in other words, g_{11}, g_{12}, g_{22} are functions of x^1 and x^2 .

Conversely, let g_{11}, g_{12}, g_{22} in (1.6) be functions of x^1, x^2 , so that they define a metric on an oriented surface Σ with the associated complex structure given by $J_\lambda^\lambda = (-1)^\lambda g_{12}$ and $J_\lambda^\mu = (-1)^\mu g_{\lambda\lambda}$ if $\{\lambda, \mu\} = \{1, 2\}$. Choosing, locally on this surface, a holomorphic function f without critical points, and extending it to our four-manifold M with local coordinates $x^1, \dots, x^4$, so as to make it independent of x^3 and x^4 , we obtain a holomorphic function on the complex surface M . Namely, since $J_\lambda^p = J_p^\lambda = 0$ in (1.7), and $\partial_3 f = \partial_4 f = 0$, the holomorphicity condition

$(df)J = idf$, assumed on Σ , remains valid on M . Remark 2.2 now implies that $\mathcal{V}$ is holomorphic. This completes the proof of Theorem F.

Corollary G is now obvious from the first paragraph of this section.

Remark 5.1. To construct examples illustrating Theorem F it is sufficient – as well as locally necessary, due to Theorem C and Theorem D(v) – to proceed as follows. One chooses functions $\alpha, \beta, \gamma, \delta$ of the local coordinates (1.5), so as to satisfy two requirements. First, α, β should depend only on (x^1, x^2) , and so the initial parts of (1.6) – (1.7) define both a metric on an oriented surface Σ with the local coordinates (x^1, x^2) , and the associated complex structure. Secondly, $\gamma + i\delta$ should form a family, parametrized by (x^3, x^4) , of holomorphic functions on Σ .

Appendix I: Cartan’s test

The terminology used in the appendices follows [3], and so does the notation, which makes the conventions adopted here inconsistent, at times, with those of the preceding sections. The resulting inconvenience is, in our view, minor compared to the benefit to the reader who wishes to look up our references to [3], for further details, and as a result is able to do that without the need of a “vocabulary.”

An *exterior differential system* on a manifold M is an ideal $\mathcal{I}$ in the graded algebra Ω^*M of smooth differential forms on M , closed under exterior differentiation. The *integral elements* (or, *integral manifolds*) of $\mathcal{I}$ are the subspaces of tangent spaces of M (or, the submanifolds of M) on which all forms in $\mathcal{I}$ vanish [3, pp. 16, 65]. If $E \subseteq T_z M$ is a p -dimensional integral element of $\mathcal{I}$, one sets

$$(I.1) \quad H(E) = \{v \in T_z M : \zeta(v, e_1, \dots, e_p) = 0 \text{ for all } \zeta \in \mathcal{I} \cap \Omega^{p+1}M\}.$$

See [3, pp. 67–68]. The definition uses a basis $e_1, \dots, e_p$ of E , but does not depend on it, since $H(E)$ is clearly the vector subspace of $T_z M$ given by

$$(I.2) \quad H(E) = \{v \in T_z M : \text{span}(v, E) \text{ is an integral element of } \mathcal{I}\}.$$

The set $V_n(\mathcal{I})$ of all n -dimensional integral elements of $\mathcal{I}$ is contained in the total space of the Grassmannian bundle $G_n(TM)$. Slightly paraphrasing a part of [3, p. 74, Theorem 1.11], we may state it as follows.

Cartan’s test. *Let an exterior differential system $\mathcal{I}$ on a manifold M contain no nonzero forms of degree 0. If $\{0\}_z = E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots \subseteq E_n \subseteq T_z M$ is a flag of integral elements of $\mathcal{I}$, each E_k has the dimension k and $H(E_k)$ the codimension c_k in $T_z M$, while for some neighborhood U of E_n in $G_n(TM)$ the intersection $V_n(\mathcal{I}) \cap U$ is a smooth manifold of codimension $c_0 + \dots + c_{n-1}$ in U , then E_n is ordinary in the sense of [3, p. 73, Defn. 1.9] and, consequently, according to [3, p. 76, Cor. 2.3], when $\mathcal{I}$ is also assumed real-analytic, there exists an integral manifold of $\mathcal{I}$ passing through z and having at z the tangent space E_n .*

Remark I.1. In our application of Cartan’s test (see the next appendix) M will be a bundle over some base manifold X of dimension n . The *vertical distribution* on M is then tangent to the fibres (that is, levels of the bundle projection $\pi : M \rightarrow X$), and we will only be interested in integral elements or integral manifolds of $\mathcal{I}$ which are *horizontal* in the sense of having a trivial intersection with the vertical space at the point in question (or, respectively, of this being the case for their tangent spaces at all points). Horizontality is a special case of what [3, p. 103] calls an *independence condition*.

Horizontal integral manifolds of $\mathcal{I}$ are, locally, the same as graphs of sections $X \rightarrow M$ of the bundle M over X , and so their existence amounts to local solvability of a specific system of partial differential equations.

Appendix II: Solvability of the system (1.9)

To derive Proposition H from Cartan's test, we let M be the set of all

$$(II.1) \quad (x, y, u, v, \alpha, \beta, \gamma, \delta, A, B, C, D, E, F, G, H, K, L, P, Q)$$

in $\mathbb{R}^{20}$ having $\alpha, \gamma \in (-\pi/2, \pi/2)$, and treat M as a bundle over $\mathbb{R}^4$ with the projection sending (II.1) to (x, y, u, v) . On M one has the exterior differential system $\mathcal{I}$ generated by the following four 1-forms and four 2-forms:

$$(II.2) \quad \begin{aligned} & d\alpha - A dx - B dy - C du - D dv, \\ & d\beta - K dx - L dy + (De^{-\delta} \sec \gamma - C \tan \gamma) du \\ & \quad + (D \tan \gamma - Ce^{\delta} \sec \gamma) dv, \\ & d\gamma - E dx - F dy - G du - H dv, \\ & d\delta + (Fe^{-\beta} \sec \alpha - E \tan \alpha) dx - P du - Q dv \\ & \quad + (F \tan \alpha - Ee^{\beta} \sec \alpha) dy, \\ & dA \wedge dx + dB \wedge dy + dC \wedge du + dD \wedge dv, \\ & dK \wedge dx + dL \wedge dy - (e^{-\delta} \sec \gamma dD - \tan \gamma dC) \wedge du \\ & \quad - (\tan \gamma dD - e^{\delta} \sec \gamma dC) \wedge dv + \dots, \\ & dE \wedge dx + dF \wedge dy + dG \wedge du + dH \wedge dv, \\ & dP \wedge du + dQ \wedge dv + (\tan \alpha dE - e^{-\beta} \sec \alpha dF) \wedge dx \\ & \quad - (\tan \alpha dF - e^{\beta} \sec \alpha dE) \wedge dy + \dots \end{aligned}$$

Here $\dots$ stand for two specific functional combinations of the six exterior products $dx \wedge dy, dx \wedge du, dx \wedge dv, dy \wedge du, dy \wedge dv, du \wedge dv$, and the 2-forms in (II.2) are modified versions of the opposites of the exterior derivatives of the 1-forms. The modifications are based on a simple rule: any functional multiple of $d\alpha, d\beta, d\gamma$ or $d\delta$ occurring in an exterior derivative is replaced by what it would become if the corresponding 1-form in (II.2) were declared equal to zero. The ideal $\mathcal{I}$ thus coincides with the one generated by the four 1-forms in (II.2) and their exterior derivatives, as each of the above replacements consists in adding to a generator of the latter ideal a functional multiple of another generator.

It is clear that, locally, four-dimensional integral manifolds of $\mathcal{I}$ which are horizontal in the sense of Remark I.1 are nothing else than graphs of solutions $(\alpha, \beta, \gamma, \delta)$ to (1.9), with $(x^1, x^2, x^3, x^4) = (x, y, u, v)$.

We will apply Cartan's test, phrased as in Appendix I, by selecting, at a fixed point $z \in M$ with the projection $\mathbf{p} = (x, y, u, v) \in \mathbb{R}^4$, a horizontal integral flag E_k , $k = 0, 1, 2, 3, 4$, cf. Remark I.1, with each E_k of dimension k and $H(E_k)$ having the codimensions

$$(II.3) \quad (c_0, c_1, c_2, c_3) = (4, 8, 12, 16), \text{ with } c_0 + c_1 + c_2 + c_3 = 40.$$

The simultaneous kernel of the four linearly independent 1-forms in (II.2) is a co-dimension-four distribution $\mathcal{D}$ on M . Clearly, $E_0 = \{0\}$, and $H(E_0) = \mathcal{D}_z$ has the required codimension $c_0 = 4$, while E_k must all be contained in $\mathcal{D}_z$. Using the dotted versions of the coordinates (II.1) to represent vectors tangent to M at a

point (II.1), we will from now on restrict our consideration to vectors having

$$(II.4) \quad \begin{aligned} \dot{\alpha} &= A\dot{x} + B\dot{y} + C\dot{u} + D\dot{v}, \\ \dot{\beta} &= K\dot{x} + L\dot{y} - (De^{-\delta}\sec\gamma - C\tan\gamma)\dot{u} - (D\tan\gamma - Ce^{\delta}\sec\gamma)\dot{v}, \\ \dot{\gamma} &= E\dot{x} + F\dot{y} + G\dot{u} + H\dot{v}, \\ \dot{\delta} &= P\dot{u} + Q\dot{v} - (Fe^{-\beta}\sec\alpha - E\tan\alpha)\dot{x} - (F\tan\alpha - Ee^{\beta}\sec\alpha)\dot{y}, \end{aligned}$$

which amounts to requiring that they lie within the distribution $\mathcal{D}$. With any given $i, j \in \{1, 2, 3, 4\}$, for a pair $(\dot{x}_i, \dot{y}_i, \dots, \dot{P}_i, \dot{Q}_i), (\dot{x}_j, \dot{y}_j, \dots, \dot{P}_j, \dot{Q}_j)$ of vectors with (II.4), being annihilated by the 2-forms in (II.2) means precisely that

$$(II.5) \quad \begin{aligned} \dot{x}_j\dot{A}_i - \dot{x}_i\dot{A}_j + \dot{y}_j\dot{B}_i - \dot{y}_i\dot{B}_j + \dot{u}_j\dot{C}_i - \dot{u}_i\dot{C}_j + \dot{v}_j\dot{D}_i - \dot{v}_i\dot{D}_j &= 0, \\ \dot{x}_j\dot{K}_i - \dot{x}_i\dot{K}_j + \dot{y}_j\dot{L}_i - \dot{y}_i\dot{L}_j + (\tan\gamma)(\dot{u}_j\dot{C}_i - \dot{u}_i\dot{C}_j) - (\tan\gamma)(\dot{v}_j\dot{D}_i - \dot{v}_i\dot{D}_j) \\ &\quad + e^{\delta}(\sec\gamma)(\dot{v}_j\dot{C}_i - \dot{v}_i\dot{C}_j) - e^{-\delta}(\sec\gamma)(\dot{u}_j\dot{D}_i - \dot{u}_i\dot{D}_j) = \dots, \\ \dot{x}_j\dot{E}_i - \dot{x}_i\dot{E}_j + \dot{y}_j\dot{F}_i - \dot{y}_i\dot{F}_j + \dot{u}_j\dot{G}_i - \dot{u}_i\dot{G}_j + \dot{v}_j\dot{H}_i - \dot{v}_i\dot{H}_j &= 0, \\ \dot{u}_j\dot{P}_i - \dot{u}_i\dot{P}_j + \dot{v}_j\dot{Q}_i - \dot{v}_i\dot{Q}_j + (\tan\alpha)(\dot{x}_j\dot{E}_i - \dot{x}_i\dot{E}_j) - (\tan\alpha)(\dot{y}_j\dot{F}_i - \dot{y}_i\dot{F}_j) \\ &\quad + e^{\beta}(\sec\alpha)(\dot{y}_j\dot{E}_i - \dot{y}_i\dot{E}_j) - e^{-\beta}(\sec\alpha)(\dot{x}_j\dot{F}_i - \dot{x}_i\dot{F}_j) = \dots, \end{aligned}$$

where $\dots$ now denotes two specific functional combinations of the six expressions $\dot{x}_i\dot{y}_j - \dot{x}_j\dot{y}_i, \dots, \dot{u}_i\dot{v}_j - \dot{u}_j\dot{v}_i$. On a purely formal level, (II.5) is the result of replacing, in (II.2), $dA \wedge dx$ with $\dot{A}_i\dot{x}_j - \dot{A}_j\dot{x}_i$ (and analogously for the remaining eleven capital letters $B, C, \dots, P, Q$ and three lower-case letters y, u, v), with a similar replacement applied to the right-hand sides. For fixed $\dot{x}_i, \dot{x}_j, \dot{y}_i, \dot{y}_j, \dots, \dot{u}_i, \dot{u}_j, \dot{v}_i, \dot{v}_j$, with all choices of i, j , (II.5) are linear homogeneous in the forty-eight dotted-and-subscripted versions of our twelve capital letters.

To determine the codimensions c_k of $H(E_k)$, for suitably selected horizontal integral elements E_k of dimensions $k = 1, 2, 3$, we use a unified approach: fixing a basis of E_k , assumed to have some “generic” property, and finding out how many independent conditions are involved in extending this basis by one vector, so that the span will still be an integral element. The basis always consists of vectors $(\dot{x}_j, \dot{y}_j, \dots, \dot{P}_j, \dot{Q}_j)$ with (II.4), $1 \leq j \leq k$, with the projections $(\dot{x}_j, \dot{y}_j, \dot{u}_j, \dot{v}_j)$ linearly independent in $\mathbb{R}^4$ (due to horizontality), while equations (II.5) hold for all i, j with $1 \leq i < j \leq k$, since E_k is an integral element. A new vector $(\dot{x}_i, \dot{y}_i, \dots, \dot{P}_i, \dot{Q}_i)$ added to this basis so as to span an integral element is thus subject to precisely $4k$ equations (II.5), where j ranges from 1 to k and $i > k$ is fixed. Using the notation $\dot{\mathbf{x}}_j = (\dot{x}_j, \dot{y}_j)$ and $\dot{\mathbf{u}}_j = (\dot{u}_j, \dot{v}_j)$, we now represent each group of four out of the $4k$ linear homogeneous equations, with a single value of j and the sixteen unknowns $\dot{A}_i, \dot{B}_i, \dot{C}_i, \dot{D}_i, \dot{E}_i, \dot{F}_i, \dot{G}_i, \dot{H}_i, \dot{K}_i, \dot{L}_i, \dot{P}_i, \dot{Q}_i, \dot{x}_i, \dot{y}_i, \dot{u}_i, \dot{v}_i$, by the 4×16 matrix

$$(II.6) \quad \begin{bmatrix} \dot{\mathbf{x}}_j & \dot{\mathbf{u}}_j & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & * & * \\ \mathbf{0} & \Phi\dot{\mathbf{u}}_j & \mathbf{0} & \mathbf{0} & \dot{\mathbf{x}}_j & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & \dot{\mathbf{x}}_j & \dot{\mathbf{u}}_j & \mathbf{0} & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & \Psi\dot{\mathbf{x}}_j & \mathbf{0} & \mathbf{0} & \dot{\mathbf{u}}_j & * & * \end{bmatrix},$$

where every boldface vector entry stands for a row-like pair of scalar entries, including $\mathbf{0} = (0, 0)$, while the linear endomorphisms Φ, Ψ of $\mathbb{R}^2$ are given by

$$(II.7) \quad \begin{aligned} \Phi(u, v) &= (u \tan \gamma + ve^{\delta} \sec \gamma, -ue^{-\delta} \sec \gamma - v \tan \gamma) \quad \text{and} \\ \Psi(x, y) &= (x \tan \alpha + ye^{\beta} \sec \alpha, -xe^{-\beta} \sec \alpha - y \tan \alpha), \quad \text{so} \\ &\text{that } \Phi, \Psi \text{ are linear automorphisms with no real eigenvalues.} \end{aligned}$$

In fact, Φ and Ψ are complex structures, cf. (1.6) – (1.7). The eight asterisks in (II.6) are row-like pairs of entries, which we choose not to identify, as they have no bearing on our discussion.

Stacking k , that is, one, or two, or three, matrices (II.6) on top of one another, for k different values of j , we obtain a $4k \times 16$ matrix, which,

(II.8) with a generic choice of E_k , will have linearly independent rows.

Here are the details. For $k = 1$ our genericity requirement reads $\dot{\mathbf{x}}_j \neq \mathbf{0} \neq \dot{\mathbf{u}}_j$. In other words, we want the line $d\pi_z(E_1) \subseteq \mathbb{R}^4$ not to be contained in either of two specific coordinate planes. If a linear combination of the rows of (II.6) vanishes, the coefficient of the first (or second, or third, or fourth) row equals zero, as shown by the first, fifth, fourth, and sixth of the eight “columns” of (II.6).

If $k = 2$, we impose the following genericity condition: both pairs $\dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2$ and $\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2$ should be linearly independent in $\mathbb{R}^2$. Equivalently, the plane $d\pi_z(E_2)$ in $\mathbb{R}^4$ projects surjectively onto two specific coordinate planes (that is, intersects both of them trivially). Placing the matrix (II.6) with $j = 1$ on top of its version for $j = 2$, we obtain a matrix with eight linearly independent rows. In fact, as in the preceding paragraph, a vanishing linear combination of the rows must have zero coefficients of the rows 1 and 5, or 2 and 6, or 3 and 7, or 4 and 8, as one sees looking, again, at the first, fifth, fourth, and sixth “columns.”

Finally, let $k = 3$. This time we require both triples $\dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dot{\mathbf{x}}_3$ and $\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2, \dot{\mathbf{u}}_3$ to be in general position, in the sense that any two vectors from the triple are linearly independent in $\mathbb{R}^2$. Triples $(q_1, q_2, q_3) \in \mathbb{R}^3$ with $q_1\dot{\mathbf{x}}_1 + q_2\dot{\mathbf{x}}_2 + q_3\dot{\mathbf{x}}_3 = \mathbf{0}$ thus form a *one-dimensional subspace* of $\mathbb{R}^3$, and similarly for the vectors $\dot{\mathbf{u}}_j$. A further generic-type assumption needed here is that this

(II.9) one-dimensional subspace of $\mathbb{R}^3$ associated with $\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2, \dot{\mathbf{u}}_3$ is different from the one for $\dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dot{\mathbf{x}}_3$.

Stacking together, as before, three copies of (II.6), with different values of j , suppose now that we have a vanishing linear combination of the resulting twelve rows, with some coefficients $q_1, r_1, s_1, t_1, q_2, r_2, s_2, t_2, q_3, r_3, s_3, t_3$. Looking at the first, fifth, fourth and sixth “columns” we see, writing $\mathbf{q}$ for (q_1, q_2, q_3) , etc., that both pairs $\mathbf{q}, \mathbf{r}$ and $\mathbf{s}, \mathbf{t}$ are linearly dependent in $\mathbb{R}^3$.

Next, $\mathbf{q} = \mathbf{0}$ both when $\mathbf{r} = \mathbf{0}$ and when $\mathbf{r}$ is nonzero: we use the first two “columns” and – in the former case – invoke (II.9) while, in the latter, write $\mathbf{q} = \lambda\mathbf{r}$ with a scalar λ , so that the coefficients r_1, r_2, r_3 produce vanishing linear combinations of the vectors $\lambda\dot{\mathbf{x}}_j$, as well as of $\Phi\dot{\mathbf{u}}_j + \lambda\dot{\mathbf{u}}_j$ (and hence, due to the last line of (II.7), also of $\dot{\mathbf{u}}_j$). Then (II.9) gives $\lambda = 0$.

Very similarly, $\mathbf{s} = \mathbf{0}$, whether or not $\mathbf{t} = \mathbf{0}$, as one sees using the third and fourth “columns” instead of the first two. With $\mathbf{q} = \mathbf{s} = \mathbf{0}$, the second and fifth (or, third and sixth) “columns” show that, with the coefficients r_1, r_2, r_3 (or, t_1, t_2, t_3), the resulting linear combinations of the vectors $\Phi\dot{\mathbf{u}}_j$ and $\dot{\mathbf{x}}_j$ (or, $\Psi\dot{\mathbf{x}}_j$ and $\dot{\mathbf{u}}_j$) are both equal to zero. Now (II.9), combined with the last line of (II.7), implies that $\mathbf{r} = \mathbf{t} = \mathbf{0}$, proving our claim (II.8) about linear independence of the rows.

To choose E_k of dimensions $k = 1, 2, 3$, each of them generic and contained in the next one, we fix E_3 with a generic basis as described above, and declare E_k , for $k = 1, 2$, to be the span of the first k vectors of this basis, which obviously makes them generic. For each $k \in \{1, 2, 3\}$, (II.8) guarantees that the solution space,

forming the kernel of a rank $4k$ operator $\mathbb{R}^{16} \rightarrow \mathbb{R}^{4k}$, is of dimension $16 - 4k$, and hence of the codimension $c_k = 4(k + 1)$ in $\mathbb{R}^{20}$, as required in (II.3).

As a final step we will now show that, within the affine space of all four-dimensional horizontal subspaces of $\mathbb{R}^{20}$, cf. Remark 2.4, the integral elements E_4 of $\mathcal{I}$ at z form a codimension-forty affine subspace. Due to (II.3), this provides the correct codimension of $V_4(\mathcal{I}) \cap U$ in Cartan's test (Appendix I), with the manifold structure of an affine bundle (the fibres having the same dimension 24 at all points).

Every horizontal four-dimensional subspace E_4 of $\mathcal{D}_z \subseteq \mathbb{R}^{20}$ has a unique basis of vectors $(\dot{x}_j, \dot{y}_j, \dots, \dot{P}_j, \dot{Q}_j)$ with (II.4), $1 \leq j \leq 4$, such that the projections $(\dot{x}_j, \dot{y}_j, \dot{u}_j, \dot{v}_j)$ form the standard basis of $\mathbb{R}^4$. Requiring E_4 to be an integral element of $\mathcal{I}$ amounts to imposing, on the forty-eight unknowns

$$\dot{A}_j, \dot{B}_j, \dot{C}_j, \dot{D}_j, \dot{E}_j, \dot{F}_j, \dot{G}_j, \dot{H}_j, \dot{K}_j, \dot{L}_j, \dot{P}_j, \dot{Q}_j, \quad j = 1, 2, 3, 4,$$

the system of twenty-four (generally nonhomogeneous) linear equations (II.5), for the six pairs (i, j) with $1 \leq i < j \leq 4$.

The 24×48 matrix of this system is of rank twenty-four. Namely, replacing all the right-hand sides by zero, we obtain a system stating that the four 2-forms in (II.2) (with the dots $\dots$ treated as zero) vanish on E_4 . According to Cartan's lemma (2.4) this means precisely two things.

First, the matrix with the four rows $(\dot{A}_j, \dot{B}_j, \dot{C}_j, \dot{D}_j)$, $j = 1, 2, 3, 4$, is symmetric, and so is the one with the rows $(\dot{E}_j, \dot{F}_j, \dot{G}_j, \dot{H}_j)$, resulting in $10 + 10 = 20$ parameters, for now free (but subjected to two constraints below).

Secondly, two more symmetric 4×4 matrices similarly arise from the second and, respectively, fourth 2-forms in (II.2). However the last two (or, respectively, first two) of their columns are already expressed in terms of the two symmetric matrices mentioned earlier. The two new symmetry conditions thus amount to introducing six new free parameters: $\dot{K}_1, \dot{L}_2, \dot{K}_2 = \dot{L}_1, \dot{P}_3, \dot{Q}_4, \dot{P}_4 = \dot{Q}_3$, while also subjecting the previous ones to two constraints:

$$(e^\beta \dot{E}_1 + e^{-\beta} \dot{F}_2) \sec \alpha - 2\dot{E}_2 \tan \alpha = (e^\delta \dot{C}_3 + e^{-\delta} \dot{D}_4) \sec \gamma - 2\dot{C}_4 \tan \gamma = 0,$$

leading, in the end, to $20 + 6 - 2 = 24$ free parameters.

Since Cartan's lemma (2.4) has the form of an equivalence, the solution space of the corresponding homogeneous system is of dimension twenty-four, which proves the italicized claim made at the beginning of one of the earlier paragraphs.

The horizontal integral elements E_4 thus form an affine space of dimension twenty-four, and hence of the correct codimension forty required by Cartan's test.

Funding. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (RS-2024-00334956).

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