Affine vector fields on compact pseudo-Kähler manifolds

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Dedicated to Paolo Piccione on the occasion of his 60th birthday

Abstract. It is known that a Killing field on a compact pseudo-Kähler manifold is necessarily (real) holomorphic, as long as the manifold satisfies some relatively mild additional conditions. We provide two further proofs of this fact and discuss the natural open question whether the same conclusion holds for affine – rather than Killing – vector fields. The question cannot be settled by invoking the Killing case: Boubel and Mounoud [Trans. Amer. Math. Soc. 368, 2016, 2223–2262] constructed examples of non-Killing affine vector fields on compact pseudo-Riemannian manifolds. We show that an affine vector field v is necessarily symplectic, and establish some algebraic and differential properties of the Lie derivative of the metric along v, such as its being parallel, antilinear and nilpotent as an endomorphism of the tangent bundle. As a consequence, the answer to the above question turns out to be 'yes' whenever the underlying manifold admits no nontrivial holomorphic quadratic differentials, which includes the case of compact almost homogeneous complex manifolds with nonzero Euler characteristic.

Introduction

A *pseudo-Kähler manifold* is a pseudo-Riemannian manifold endowed with a parallel almost-complex structure J, making the metric Hermitian. This is well known, cf. [**3**, p. 6], to imply integrability of J.

It is also well known [**1**, pp. 60–61] that Killing fields on compact *Riemannian* Kähler manifolds are necessarily (real) holomorphic, compactness being essential (as illustrated by flat manifolds). This remains valid, under some additional assumptions, in the pseudo-Kähler case [**8, 3**].

A vector field v on a manifold endowed with a connection ∇ is said to be *affine* if its local flow preserves ∇. When ∇ is the Levi-Civita connection of

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a Riemannian metric, such v are usually Killing fields, with very few – always noncompact – exceptions [**5**, Ch. IV]. However, Boubel and Mounoud [**2**] provided examples of compact pseudo-Riemannian manifolds admitting non-Killing affine vector fields. Their examples are not of the pseudo-Kähler type, which raises a question: are the affine-to-Killing and affine-to-holomorphic implications true for compact pseudo-Kähler manifolds?

This question remains open. The present paper establishes some properties of the Lie derivative $\pounds_v g$ for an affine vector field v on compact pseudo-Kähler manifold (M,g) with the $\partial\bar\partial$ property (Theorem 5.1): in addition to being obviously parallel, $\pounds_v g$ is also antilinear and nilpotent as an endomorphism of TM, while v is a Killing field if and only if it is real holomorphic. Also, $\pounds_v\omega = 0$ for the Kähler form ω (Corollary 3.2). Finally, as a partial answer to the above question, we show that v must be real holomorphic when M admits no nontrivial holomorphic quadratic differentials (Theorem 6.2) and hence, in particular, when M is almost homogeneous and has nonzero Euler characteristic (Corollary 6.3).

1. Preliminaries

Manifolds and mappings are assumed smooth, the former also connected.

Let (M,g) be a pseudo-Riemannian manifold. We write $\beta \sim B$ when tensor fields β and B of types $(0,2)$ and $(1,1)$ are related by $\beta = g(B\cdot\,,\,\cdot)$. Thus, $g \sim \mathrm{Id}$. On a pseudo-Kähler manifold

(1.1) $\omega \sim J$ for the Kähler form ω and the complex-structure tensor J.

If $\beta \sim B$, as defined above, and we set

$$A \,=\, \nabla v \tag{1.2}$$

for a fixed vector field v, one easily sees that

$$\pounds_v\beta \,\sim\, \nabla_{\!v}B \,+\, BA \,+\, A^*B, \tag{1.3}$$

A^* being the pointwise g-adjoint of A. Two obvious special cases are

$$\text{a)}\ \ \pounds_v g \,\sim\, A + A^*, \qquad \text{b)}\ \ \pounds_v\omega \,\sim\, JA + A^*J, \tag{1.4}$$

the latter in the pseudo-Kähler case. Assuming (1.2), we obviously get

$$d[g(v,\,\cdot)] \,\sim\, A \,-\, A^*. \tag{1.5}$$

If $\beta \sim B$ for a real differential 2-form on a complex manifold,

(1.6) β is a $(1,1)$-form if and only if $[J,B]=0$, as both conditions are clearly equivalent to $\beta(J\cdot\,,J\cdot) \,=\, \beta$.

Given a pseudo-Riemannian manifold (M,g), using J,

(1.7) we treat TM as a complex vector bundle.

Cartan's homotopy formula $\pounds_v = \imath_v d + d\imath_v$ for $\pounds_v$ acting on differential forms [**6**, Thm. 14.35, p. 372] and the Leibniz rule $\pounds_v[\nabla\Theta] = [\pounds_v\nabla]\Theta + \nabla[\pounds_v\Theta]$ imply that, for any vector field v on a manifold,

$$\pounds_v\omega \,=\, d[\omega(v,\,\cdot\,)] \text{ if } \omega \text{ is a closed differential form,} \tag{1.8}$$

while, whenever v happens to be affine relative to a connection ∇,

$$\nabla[\pounds_v\Theta] = 0 \quad \text{if } \Theta \text{ is a tensor field with } \nabla\Theta = 0. \tag{1.9}$$

Remark 1.1. Any constant-rank twice-covariant symmetric tensor field β on a manifold has the same algebraic type at all points: its positive and negative indices, being lower semicontinuous, with a constant sum, must be locally constant.

Remark 1.2. Due to the Leibniz rule, for any vector field v on a pseudoKähler manifold, $\pounds_v J = [J, A]$, where $A = \nabla v$, so that v is real holomorphic if and only if ∇v commutes with J. On the other hand, holomorphic complex-valued functions ϕ on a complex manifold M are characterized by the Cauchy-Riemann condition $(d\phi)J = i d\phi$, where $(d\phi)J$ denotes the composite bundle morphism $TM \to TM \to M \times \mathbb{C}$.

2. The $\partial\bar\partial$ property

Every compact complex manifold admitting a Riemannian Kähler metric has the following *$\partial\bar\partial$ property*, also referred to as *the $\partial\bar\partial$ lemma* [**9**, Prop. 6.17 on p. 144]: given integers $p, q \ge 0$, any closed ∂-exact or $\bar\partial$-exact (p,q)-form equals $\partial\bar\partial\lambda$ for some $(p-1, q-1)$-form λ. Then, since exactness of a $(p,0)$-form amounts to its ∂-exactness, and implies its closedness.

$$M \text{ admits no nonzero exact } (p,0)\text{- or } (0,p)\text{-forms.} \tag{2.1}$$

As a special case, on a compact complex $\partial\bar\partial$ manifold M,

$$\text{every exact real } (1,1)\text{-form } \alpha \text{ equals } i\partial\bar\partial\phi \text{ for some } \phi : M \to \mathrm{I\!R}. \tag{2.2}$$

Namely, writing $\alpha = d\xi = \partial\xi + \bar\partial\xi$ we see that $\partial\xi$ and $\bar\partial\xi$ are both closed: $d\partial\xi = \bar\partial\alpha = 0 = \partial\alpha = d\bar\partial\xi$. Note that, for any $\phi : M \to \mathrm{I\!R}$,

$$2i\partial\bar\partial\phi \,=\, -d[(d\phi)J], \tag{2.3}$$

$(d\phi)J$ being the composite bundle morphism $TM \to TM \to M \times \mathrm{I\!R}$. Consequently,

$$-2\alpha(J\,\cdot\,,\,\cdot\,) \,=\, \theta(J\,\cdot\,, J\,\cdot\,) \,+\, \theta(\,\cdot\,,\,\cdot\,) \text{ if } \alpha = i\partial\bar\partial\phi \text{ and } \theta = \nabla d\phi. \tag{2.4}$$

where ∇ is any torsion-free connection on M with $\nabla J = 0$. Whether or not such ∇ exists, at any critical point z of a function $\phi : M \to \mathrm{I\!R}$, by (2.3),

$$-2\alpha(J\,\cdot\,,\,\cdot\,) \,=\, \theta(J\,\cdot\,, J\,\cdot\,) \,+\, \theta(\,\cdot\,,\,\cdot\,) \text{ if } \alpha = i\partial\bar\partial\phi \text{ and } \theta = \mathrm{Hess}_z\phi. \tag{2.5}$$

Lemma 2.1. *A compact complex manifold M satisfying the special case* (2.2) *of the $\partial\bar\partial$ condition admits no nonzero constant-rank real-valued exact* $(1,1)$*-forms.*

PROOF. Applying (2.2) and (2.5) to a constant-rank real-valued exact $(1,1)$-form α, at a point $z \in M$ where ϕ assumes its maximum (or, minimum) value, we see that the symmetric 2-tensor field $\alpha(J\cdot\,,\,\cdot)$ is positive (or, negative) semidefinite at z. Remark 1.1 applied to $\beta = \alpha(J\cdot\,,\,\cdot)$, which has constant rank (equal to the rank of α), implies both positive and negative semidefiniteness of β at all points, so that $\beta = 0$ everywhere. □

3. Consequences of the Hodge decomposition

For each cohomology space $H^p(M,\mathbb{C})$ of a compact complex manifold M with the $\partial\bar\partial$ property, denoting by $H^{r,s}M$ the space of cohomology classes of closed complex-valued (r,s)-forms, one has the Hodge decomposition

$$H^p(M,\mathbb{C}) = H^{p,0}M \oplus H^{p-1,1}M \oplus \ldots \oplus H^{1,p-1}M \oplus H^{0,p}M. \tag{3.1}$$

See, e.g., [**4**, p. 296, subsect. (5.21)].

LEMMA 3.1. *Let ζ be an exact ∇-parallel complex-valued p-form on a compact complex $\partial\bar\partial$ manifold M with a torsion-free connection ∇ such that $\nabla J = 0$.*

(i) *ζ has zero $(p,0)$ and $(0,p)$ components.*
(ii) *The (r,s) components of ζ, $r+s=p$, are all exact and ∇-parallel.*
(iii) *$\zeta = 0$ when $p = 2$.*

PROOF. The decomposition of the bundle of complex-valued exterior p-forms on M into its (r,s) summands, with $r+s=p$, is invariant under parallel transports, since J uniquely determines the decomposition and $\nabla J = 0$. The components $\zeta^{r,s}$ of the decomposition of ζ are thus all ∇-parallel, and hence closed. The resulting cohomology relation $\sum_{r,s}[\zeta^{r,s}] = [\zeta] = 0$ gives, by (3.1), $[\zeta^{r,s}] = 0$ whenever $r+s=p$, proving (ii), while (i) follows from (ii) and (2.1). Lemma 2.1 and (i) – (ii) now easily yield (iii). □

We have an obvious consequence of (1.8) – (1.9) and Lemma 2.1(iii).

COROLLARY 3.2. *Let v be an affine vector field on a compact pseudo-Kähler $\partial\bar\partial$ manifold (M,g) with the Kähler form $\omega = g(J\cdot\,,\,\cdot)$. Then $\pounds_v\omega = 0$.*

4. The case of Killing fields

The paper [**3**] provides two different proofs of the fact that, *on a compact pseudo-Kähler $\partial\bar\partial$ manifold, all Killing fields are real holomorphic.*

Our preceding discussion gives rise to two more simple proofs of this fact. As

$$\pounds_v[g(J\cdot\,,\,\cdot)] \;=\; g(\pounds_v J\cdot\,,\,\cdot) \;\text{ when }\; \pounds_v g = 0, \tag{4.1}$$

the first proof comes directly from Corollary 3.2.

For the other proof, note that $\zeta = \pounds_v\omega$, being exact and parallel by (1.8) and (1.9), must – due to Lemma 2.1(i) – be a $(1,1)$-form. Since (4.1) gives $\pounds_v\omega \sim \pounds_v J$, (1.6) implies that J and $C = \pounds_v J$ commute. However, they also anticommute: $0 = -\pounds_v \mathrm{Id} = \pounds_v J^2 = CJ + JC$.

5. The four components of the covariant derivative

On a pseudo-Kähler manifold (M, g), the operation $A \mapsto JAJ$ applied to bundle morphisms $A : TM \to TM$ obviously commutes with $A \mapsto A^*$ and, as both are involutions, every A is decomposed into four components (complex-linear self-adjoint, complex-linear skew-adjoint, antilinear self-adjoint, antilinear skew-adjoint). In the case where $A = \nabla v$ for an affine vector field v on a compact pseudoKähler $\partial\bar\partial$ manifold (M, g), two of the four components – the first and last ones – are absent, according to the assertions (5.1-b) and (5.1-c) below, while the third one has rather special algebraic and differential properties, cf. (5.1-d).

Theorem 5.1. *Let v be an affine vector field on a compact pseudo-Kähler $\partial\bar\partial$ manifold (M, g). Then, for $A = \nabla v$,*

(5.1)
- a) $A^* = JAJ$,
- b) $A - A^*$ commutes with J,
- c) $A + A^*$ anticommutes with J,
- d) $A + A^*$ is parallel, and nilpotent at every point,
- e) $\mathrm{div}\, v = 0$,
- f) v is a Killing field if and only if it is real-holomorphic.

Proof. Corollary 3.2 and (1.4.b) yield (5.1-a), while (5.1-a) trivially implies (5.1-b) – (5.1-c). Next, (1.9) and (1.4.b) prove the first part of (5.1-d). Thus, $2\,\mathrm{div}\, v = 2\,\mathrm{tr}\, A = \mathrm{tr}\,(A + A^*)$ is constant, and has zero integral, which gives (5.1.e) and the equality $\mathrm{tr}\,(A + A^*)^k = 0$ for $k = 1$. The same equality for $k \geq 2$ now follows: setting $C = (A + A^*)^{k-1}$, we get $\mathrm{tr}\,(A + A^*)^k = \mathrm{tr}\,(CA + CA^*) = 2\,\mathrm{tr}\, CA$. (Note that $\mathrm{tr}\, CA^* = \mathrm{tr}\,(CA^*)^* = \mathrm{tr}\, AC = \mathrm{tr}\, CA$.) As C is parallel due to (5.1-d), $\mathrm{tr}\, CA = C_k^i v^k{}_{,i}$ integrated by parts yields zero. The zero-integral constant $\mathrm{tr}\,(A + A^*)^k$ thus equals 0, which proves the remainder of (5.1-d). □

6. Two holomorphic covariant tensors

Equation (6.1) in Theorem 6.1 is more than just a pseudo-Riemannian analog oa a Hodge decomposition for the 1-form $\xi = g(v, \cdot)$, with the exact part absent in accordance with (5.1-e); it also actually constitutes the Riemannian Hodge decomposition of ξ relative to any Riemannian Kähler metric h, as long as one exists on M. This is immediate since $(d\phi)J$ and ξ are, clearly, h-coexact and h-harmonic for every such metric h.

Theorem 6.1. *Given an affine vector field v on a compact pseudo-Kähler $\partial\bar\partial$ manifold (M, g), one has*

(6.1) $$g(v, \cdot) = (d\phi)J + \xi$$

for some $\phi : M \to \mathrm{I\!R}$ *and the real part* ξ *of a holomorphic* 1*-form* $\xi - i\xi J$.

Proof. The assertions (1.5), (5.1.b), and (1.6) applied to $B = A - A^*$, show that $d[g(v,\,\cdot\,)]$ is an exact $(1,1)$-form. Choosing ϕ as in (2.2) for $\alpha = -d[g(v,\,\cdot\,)]/2$, we now see that, by (2.3), the 1-form $\xi = g(v,\,\cdot\,) - (d\phi)J$ is closed. However, $\xi J = d\phi - g(Jv,\,\cdot\,) = d\phi - \omega(v,\,\cdot\,)$ is also closed, due to (1.8) and Corollary 3.2. The relations $d\xi = d(\xi J) = 0$ amount to holomorphicity of $\xi - i\xi J$. □

Theorem 6.1 does not seems to be relevant to the question stated in the Introduction: for instance, vanishing of the holomorphic 1-form $\xi - i\xi J$ (which follows if M is simply connected) does not lead to any immediate answer. This stands in marked contrast with the next result.

Theorem 6.2. *Suppose that* v *is an affine vector field on a compact pseudo-Kähler* $\partial\bar\partial$ *manifold* (M,g). *Then the* $(0,2)$ *tensor field* $\pounds_v g$ *is the real part of a holomorphic section* θ *of the second complex symmetric power of the complex dual of* TM, *with the convention* (1.7).

Consequently, $\pounds_v g = 0$ *if no such nonzero holomorphic section* θ *exists.*

Proof. By (1.4.a) and (5.1-c), $[\pounds_v g](J\cdot\,,J\cdot) = -\pounds_v g$, so that $\theta = \pounds_v g - i[\pounds_v g](J\cdot\,,\,\cdot\,)$ is complex-bilinear at every point. As it is parallel – by (1.9) – its holomorphicity follows: Remark 1.2 easily implies that, for any (local) real holomorphic vector fields w, w', the function $\theta(w,w')$ is holomorphic. □

A complex manifold M is called *almost homogeneous* [**7**] if, for some $x \in M$, every vector in T_xM is the value at x of some real holomorphic vector field.

Corollary 6.3. *If* g *is a pseudo-Kähler metric on a compact almost homogeneous complex* $\partial\bar\partial$ *manifold* M *with nonzero Euler characteristic* $\chi(M)$, *then every* g*-affine vector field* v *on* M *is real holomorphic.*

In fact, let θ be the holomorphic quadratic differential mentioned in Theorem 6.2. Thus, $\theta(w,w')$ is constant for any real holomorphic vector fields w, w'. If θ were nonzero – everywhere, due to its being parallel by (1.9) – choosing x as above and a real holomorphic vector field w with $\theta(w,w) \neq 0$ at x we would get $w \neq 0$ everywhere, and hence $\chi(M) = 0$.

Conflict of interest statement

The author states that there is no conflict of interest.

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