

THE LOCAL STRUCTURE OF CONFORMALLY SYMMETRIC MANIFOLDS

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ABSTRACT. This is a final step in a local classification of pseudo-Riemannian manifolds with parallel Weyl tensor that are neither conformally flat, nor locally symmetric.

INTRODUCTION

The present paper provides a finishing touch in a local classification of essentially conformally symmetric pseudo-Riemannian metrics.

A pseudo-Riemannian manifold of dimension $n \geq 4$ is called *essentially conformally symmetric* if it is *conformally symmetric* [2] (in the sense that its Weyl conformal tensor is parallel) without being conformally flat or locally symmetric.

The metric of an essentially conformally symmetric manifold is always indefinite [4, Theorem 2]. *Compact* essentially conformally symmetric manifolds are known to exist in all dimensions $n \geq 5$ with $n \equiv 5 \pmod{3}$, where they represent all indefinite metric signatures [8], while examples of essentially conformally symmetric pseudo-Riemannian metrics on open manifolds of all dimensions $n \geq 4$ were first constructed in [16].

On every conformally symmetric manifold there is a naturally distinguished parallel distribution \mathcal{D} , of some dimension d , which we call the *Olszak distribution*. As shown by Olszak [13], for an essentially conformally symmetric manifold $d \in \{1, 2\}$.

In [7] we described the local structure of all conformally symmetric manifolds with $d = 2$. See also Section 3. This paper establishes an analogous result (Theorem 4.1) for the case $d = 1$.

In both cases, some of the metrics in question are locally symmetric. In Remark 4.2 we explain why a similar classification result cannot be valid just for *essentially* conformally symmetric manifolds.

Essentially conformally symmetric manifolds with $d = 1$ are all *Ricci-recurrent*, in the sense that, for every tangent vector field v , the Ricci tensor ρ and the covariant derivative $\nabla_v \rho$ are linearly dependent at each point. The local structure of essentially conformally symmetric Ricci-recurrent manifolds at points with $\rho \otimes \nabla \rho \neq 0$ has already been determined by the second author [16]. Our new contribution

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settles the one case still left open in the local classification problem, namely, that of essentially conformally symmetric manifolds with $d = 1$ at points where $\rho \otimes \nabla \rho = 0$.

The literature dealing with conformally symmetric manifolds includes, among others, [9, 10, 12, 15, 17, 18] and the papers cited above. A local classification of *homogeneous* essentially conformally symmetric manifolds can be found in [3].

1. PRELIMINARIES

Throughout this paper, all manifolds and bundles, along with sections and connections, are assumed to be of class C^∞ . A manifold is, by definition, connected. Unless stated otherwise, a mapping is always a C^∞ mapping between manifolds.

Given a connection ∇ in a vector bundle \mathcal{E} over a manifold M , a section ψ of \mathcal{E} , and vector fields u, v tangent to M , we use the sign convention

$$(1) \quad R(u, v)\psi = \nabla_v \nabla_u \psi - \nabla_u \nabla_v \psi + \nabla_{[u, v]}\psi$$

for the curvature tensor $R = R^\nabla$.

The Levi-Civita connection of a given pseudo-Riemannian manifold (M, g) is always denoted by ∇ . We also use the symbol ∇ for connections induced by ∇ , in various ∇ -parallel subbundles of TM and their quotients.

The Schouten tensor σ and Weyl conformal tensor W of a pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ are given by $\sigma = \rho - (2n - 2)^{-1} s g$, with ρ denoting the Ricci tensor, $s = \text{tr}_g \rho$ standing for the scalar curvature, and

$$(2) \quad W = R - (n - 2)^{-1} g \wedge \sigma.$$

Here \wedge is the exterior multiplication of 1-forms valued in 1-forms, which uses the ordinary \wedge as the valewise multiplication; thus, $g \wedge \sigma$ is a 2-form valued in 2-forms.

Let $(t, s) \mapsto x(t, s)$ be a fixed *variation of curves* in a pseudo-Riemannian manifold (M, g) , that is, an M -valued C^∞ mapping from a rectangle (product of intervals) in the ts -plane. By a *vector field w along the variation* we mean, as usual, a section of the pullback of TM to the rectangle (so that $w(t, s) \in T_{x(t, s)}M$). Examples are x_s and x_t , which assign to (t, s) the velocity of the curve $t \mapsto x(t, s)$ or $s \mapsto x(t, s)$ at s or t . Further examples are provided by restrictions to the variation of vector fields on M . The partial covariant derivatives of a vector field w along the variation are the vector fields w_t, w_s along the variation, obtained by differentiating w covariantly along the curves $t \mapsto x(t, s)$ or $s \mapsto x(t, s)$. Skipping parentheses, we write w_{ts}, w_{stt} , etc., rather than $(w_t)_s, ((w_s)_t)_t$ for higher-order derivatives, as well as x_{ss}, x_{st} instead of $(x_s)_s, (x_s)_t$. One always has $w_{ts} = w_{st} + R(x_t, x_s)w$, cf. [11, formula (5.29) on p. 460], and, since the Levi-Civita connection ∇ is torsionfree, $x_{st} = x_{ts}$. Thus, whenever $(t, s) \mapsto x(t, s)$ is a variation of curves in M ,

$$(3) \quad x_{tss} = x_{sst} + R(x_t, x_s)x_s.$$

2. THE OLSZAK DISTRIBUTION

The *Olszak distribution* of a conformally symmetric manifold (M, g) is the parallel subbundle \mathcal{D} of TM , the sections of which are the vector fields u with the property that $\xi \wedge \Omega = 0$ for all vector fields v, v' and for the differential forms $\xi = g(u, \cdot)$ and $\Omega = W(v, v', \cdot, \cdot)$. The distribution \mathcal{D} was introduced, in a more general situation, by Olszak [13], who also proved the following lemma.

LEMMA 2.1. *The following conclusions hold for the dimension d of the Olszak distribution \mathcal{D} in any conformally symmetric manifold (M, g) with $\dim M = n \geq 4$.*

- (i) $d \in \{0, 1, 2, n\}$, and $d = n$ if and only if (M, g) is conformally flat.
- (ii) $d \in \{1, 2\}$ if (M, g) is essentially conformally symmetric.
- (iii) $d = 2$ if and only if $\text{rank } W = 1$, in the sense that W , as an operator acting on exterior 2-forms, has rank 1 at each point.
- (iv) If $d = 2$, the distribution \mathcal{D} is spanned by all vector fields of the form $W(u, v)v'$ for arbitrary vector fields u, v, v' on M .

PROOF. See Appendix I. □

In the next lemma, parts (a) and (d) are due to Olszak [13, 2° and 3° on p. 214].

LEMMA 2.2. *If $d \in \{1, 2\}$, where d is the dimension of the Olszak distribution \mathcal{D} of a given conformally symmetric manifold (M, g) with $\dim M = n \geq 4$, then*

- (a) \mathcal{D} is a null parallel distribution,
- (b) at any $x \in M$ the space \mathcal{D}_x contains the image of the Ricci tensor ρ_x treated, with the aid of g_x , as an endomorphism of $T_x M$,
- (c) the scalar curvature is identically zero and $R = W + (n - 2)^{-1}g \wedge \rho$,
- (d) $W(u, \cdot, \cdot, \cdot) = 0$ whenever u is a section of \mathcal{D} ,
- (e) $R(v, v', \cdot, \cdot) = W(v, v', \cdot, \cdot) = 0$ for any sections v and v' of \mathcal{D}^\perp ,
- (f) of the connections in \mathcal{D} and $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$, induced by the Levi-Civita connection of g , the latter is always flat, and the former is flat if $d = 1$.

PROOF. Assertion (e) for W is immediate from the definition of \mathcal{D} . Namely, at any point $x \in M$, every 2-form Ω_x in the image of W_x (for W_x acting on 2-forms at x) is \wedge -divisible by $\xi = g_x(u, \cdot)$ for each $u \in \mathcal{D}_x \setminus \{0\}$, and so $\Omega_x(v, v') = 0$ if $v, v' \in \mathcal{D}_x^\perp$.

We now proceed to prove (a), (b), (c) and (d).

First, let $d = 2$. By Lemma 2.1(iii), this amounts to the condition $\text{rank } W = 1$, so that (a), (b) and (c) follow from Lemma 2.1(iv) combined with [7, Lemma 17.1(ii) and Lemma 17.2]. Also, for a nonzero 2-form Ω_x chosen as in the last paragraph, \mathcal{D}_x is the image of Ω_x , that is, Ω_x equals the exterior product of two vectors in \mathcal{D}_x (treated as 1-forms, with the aid of g_x). Now (d) follows since, by (a), $\Omega_x(u_x, \cdot) = 0$ if u is a section of \mathcal{D} .

Next, suppose that $d = 1$. Replacing M by a neighborhood of any given point, we may assume that \mathcal{D} is spanned by a vector field u . If u were not null, we would have $W(u, v, u, v') = 0$ for any sections v, v' of \mathcal{D}^\perp , as one sees contracting the twice-covariant tensor field $W(\cdot, v, \cdot, v') = 0$, at any point x , in an orthogonal basis containing the vector u_x . (We have already established (e) for W .) Combined with (e) for W and the symmetries of W , the relation $W(u, v, u, v') = 0$ for v, v' in \mathcal{D}^\perp would then give $W = 0$, contrary to the assumption that $d = 1$. Thus, u is null, which yields (a). Now

(4) we choose, locally, a null vector field u' with $g(u, u') = 1$.

For any section v of \mathcal{D}^\perp one sees that $W(u, \cdot, u', v) = 0$ by contracting the tensor field $W(\cdot, \cdot, \cdot, v) = 0$ in the first and third arguments, at any point x , in

(5) a basis of $T_x M$ formed by u_x, u'_x and $n - 2$ vectors orthogonal to them,

and using (e) for W , along with the inclusion $\mathcal{D} \subset \mathcal{D}^\perp$, cf. (a). Since u' and \mathcal{D}^\perp span TM , assertion (e) for W thus implies (d).

To obtain (b) and (c) when $d = 1$, we distinguish two cases: (M, g) is either essentially conformally symmetric, or locally symmetric. For (c), it suffices to establish vanishing of the scalar curvature s (cf. (2)). Now, in the former case, $s = 0$ according to [5, Theorem 7], while (b) follows since, as shown in [6, Theorem 7 on p. 18], for arbitrary vector fields v, v' and v'' on an essentially conformally symmetric pseudo-Riemannian manifold, $\xi \wedge \Omega = 0$, where $\xi = \rho(v, \cdot)$ and $\Omega = W(v', v'', \cdot, \cdot)$. In the case where g is locally symmetric, (b) and (c) are proved in Appendix II.

Assertion (e) for R is now obvious from (e) for W and (c), since, by (b), $\rho(v, \cdot) = 0$ for any section v of \mathcal{D}^\perp . The claim about \mathcal{E} in (f) is in turn immediate from (1) and (e) for R , which states that $R(w, w')v$, for arbitrary vector fields w, w' and any section v of \mathcal{D}^\perp , is orthogonal to all sections of \mathcal{D}^\perp (and hence must be a section of \mathcal{D}). Finally, to prove (f) for \mathcal{D} , with $d = 1$, let us fix a section u of \mathcal{D} , a vector field v , and define a differential 2-form ζ by $\zeta(w, w') = (n - 2)R(w, w', u, v)$ for any vector fields w, w' . By (c) and (e), $\zeta = g(u, \cdot) \wedge \rho(v, \cdot)$, as $\mathcal{D} \subset \mathcal{D}^\perp$ (cf. (a)), and so $\rho(u, \cdot) = 0$ in view of (b) and symmetry of ρ . However, by (b), both $g(u, \cdot)$ and $\rho(v, \cdot)$ are sections of the subbundle of T^*M corresponding to \mathcal{D} under the bundle isomorphism $TM \rightarrow T^*M$ induced by g , so that $\zeta = 0$ since the distribution \mathcal{D} is one-dimensional. \square

3. THE CASE $d = 2$

For more details of the construction described below, we refer the reader to [7].

Let there be given a surface Σ , a projectively flat torsionfree connection D on Σ with a D -parallel area form α , an integer $n \geq 4$, a sign factor $\varepsilon = \pm 1$, a real vector space V with $\dim V = n - 4$, and a pseudo-Euclidean inner product $\langle \cdot, \cdot \rangle$ on V .

We also assume the existence of a twice-contravariant symmetric tensor field T on Σ with $\operatorname{div}^D(\operatorname{div}^D T) + (\rho^D, T) = \varepsilon$ (in coordinates: $T^{jk}{}_{,jk} + T^{jk}R_{jk} = \varepsilon$). Here div^D denotes the D-divergence, ρ^D is the Ricci tensor of D, and $(,)$ stands for the obvious pairing. Such T always exists locally in Σ . In fact, according to [7, Theorem 10.2(i)] combined with [7, Lemma 11.2], T exists whenever Σ is simply connected and noncompact.

For T chosen as above, we define a twice-covariant symmetric tensor field τ on Σ , that is, a section of $[T^*\Sigma]^{\odot 2}$, by requiring τ to correspond to the section T of $[T\Sigma]^{\odot 2}$ under the vector-bundle isomorphism $T\Sigma \rightarrow T^*\Sigma$ which acts on vector fields v by $v \mapsto \alpha(v, \cdot)$. In coordinates, $\tau_{jk} = \alpha_{jl}\alpha_{km}T^{lm}$.

Next, we denote by h^D the *Patterson-Walker Riemann extension metric* [14] on the total space $T^*\Sigma$, obtained by requiring that all vertical and all D-horizontal vectors be h^D -null, while $h_x^D(\zeta, w) = \zeta(d\pi_x w)$ for $x \in T^*\Sigma$, any vector $w \in T_x T^*\Sigma$, any vertical vector $\zeta \in \operatorname{Ker} d\pi_x = T_{\pi(x)}^* \Sigma$, and the bundle projection $\pi : T^*\Sigma \rightarrow \Sigma$.

Finally, let γ and θ be the constant pseudo-Riemannian metric on V corresponding to the inner product \langle, \rangle , and the function $V \rightarrow \mathbf{R}$ with $\theta(v) = \langle v, v \rangle$.

Our $\Sigma, D, \alpha, n, \varepsilon, V, \langle, \rangle$ now give rise to the pseudo-Riemannian manifold

$$(6) \quad (T^*\Sigma \times V, h^D - 2\tau + \gamma - \theta\rho^D),$$

of dimension n , with the metric $h^D - 2\tau + \gamma - \theta\rho^D$, where the function θ and covariant tensor fields $\tau, \rho^D, h^D, \gamma$ on $\Sigma, T^*\Sigma$ or V are identified with their pull-backs to $T^*\Sigma \times V$. (Thus, for instance, $h^D - 2\tau + \gamma$ is a product metric.)

We have the following local classification result, in which d stands for the dimension of Olszak distribution \mathcal{D} .

THEOREM 3.1. *The pseudo-Riemannian manifold (6) obtained as above from any data $\Sigma, D, \alpha, n, \varepsilon, V, \langle, \rangle$ with the stated properties is conformally symmetric and has $d = 2$. Conversely, in any conformally symmetric pseudo-Riemannian manifold such that $d = 2$, every point has a connected neighborhood isometric to an open subset of a manifold (6) constructed above from some data $\Sigma, D, \alpha, n, \varepsilon, V, \langle, \rangle$.*

The manifold (6) is never conformally flat, and it is locally symmetric if and only if the Ricci tensor ρ^D is D-parallel.

PROOF. See [7, Section 22]. Note that, in view of Lemma 2.1(iii), the condition $\operatorname{rank} W = 1$ used in [7] is equivalent to $d = 2$. \square

The objects $\Sigma, D, \alpha, n, \varepsilon, V, \langle, \rangle$ are treated as parameters of the above construction, while T is merely assumed to exist, even though the metric g in (6) clearly depends on τ (and hence on T). This is justified by the fact that, with fixed $\Sigma, D, \alpha, n, \varepsilon, V, \langle, \rangle$, the metrics corresponding to two choices of T are, locally, isometric to each other, cf. [7, Remark 22.1].

The metric signature of (6) is clearly given by $-- \dots ++$, with the dots standing for the sign pattern of \langle, \rangle .

4. THE CASE $d = 1$

Let there be given an open interval I , a C^∞ function $f : I \rightarrow \mathbf{R}$, an integer $n \geq 4$, a real vector space V of dimension $n - 2$ with a pseudo-Euclidean inner product \langle, \rangle , and a nonzero traceless linear operator $A : V \rightarrow V$, self-adjoint relative to \langle, \rangle . As in [16], we then define an n -dimensional pseudo-Riemannian manifold

$$(7) \quad (I \times \mathbf{R} \times V, \kappa dt^2 + dt ds + \gamma),$$

where products of differentials represent symmetric products, t, s denote the Cartesian coordinates on the $I \times \mathbf{R}$ factor, γ stands for the pullback to $I \times \mathbf{R} \times V$ of the flat pseudo-Riemannian metric on V that corresponds to the inner product \langle, \rangle , and the function $\kappa : I \times \mathbf{R} \times V \rightarrow \mathbf{R}$ is given by $\kappa(t, s, \psi) = f(t)\langle\psi, \psi\rangle + \langle A\psi, \psi\rangle$.

The manifolds (7) are characterized by the following local classification result, analogous to Theorem 3.1. As before, d is the dimension of the Olszak distribution.

THEOREM 4.1. *For any $I, f, n, V, \langle, \rangle, A$ as above, the pseudo-Riemannian manifold (7) is conformally symmetric and has $d = 1$. Conversely, in any conformally symmetric pseudo-Riemannian manifold such that $d = 1$, every point has a connected neighborhood isometric to an open subset of a manifold (7) constructed from some such $I, f, n, V, \langle, \rangle, A$.*

The manifold (7) is never conformally flat, and it is locally symmetric if and only if f is constant.

A proof of Theorem 4.1 is given at the end of the next section.

Obviously, the metric $\kappa dt^2 + dt ds + \gamma$ in (7) has the sign pattern $-\dots +$, where the dots stand for the sign pattern of \langle, \rangle .

REMARK 4.2. A classification result of the same format as Theorem 4.1 cannot be true just for *essentially* conformally symmetric manifolds with $d = 1$. Namely, such manifolds do not satisfy a principle of unique continuation: formula (7) with f which is nonconstant on I , but constant on some nonempty open subinterval I' of I , defines an essentially conformally symmetric manifold with a locally symmetric open submanifold $U = I' \times \mathbf{R} \times V$. At points of U , the local structure of (7) does not, therefore, arise from a construction that, locally, produces all essentially conformally symmetric manifolds and nothing else.

As explained in [7, Section 24], an analogous situation arises when $d = 2$.

5. PROOF OF THEOREM 4.1

The following assumptions will be used in Lemma 5.1.

- (a) (M, g) is a conformally symmetric manifold of dimension $n \geq 4$ and $y \in M$.

- (b) The Olszak distribution \mathcal{D} of (M, g) is one-dimensional.
- (c) u is a global parallel vector field spanning \mathcal{D} .
- (d) $t : M \rightarrow \mathbf{R}$ is a C^∞ function with $g(u, \cdot) = dt$ and $t(y) = 0$.
- (e) $\dim V = n - 2$ for the space V of all parallel sections of $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$.
- (f) $\rho = (2 - n)f(t) dt \otimes dt$ for some C^∞ function $f : I' \rightarrow \mathbf{R}$ on an open interval I' , where ρ is the Ricci tensor and $f(t)$ denotes the composite $f \circ t$.

For local considerations, only (a) and (b) are essential. In fact, condition (e) (in which ‘parallel’ refers to the connection in \mathcal{E} induced by the Levi-Civita connection of g), as well (c) and (d) for some u and t , follow from (a) – (b) if M is simply connected. See Lemma 2.2(f). On the other hand, (c) – (d), Lemma 2.2(b) and symmetry of ρ give $\nabla dt = 0$ and $\rho = \chi dt \otimes dt$ for some function $\chi : M \rightarrow \mathbf{R}$, so that $\nabla \rho = d\chi \otimes dt \otimes dt$. However, $\nabla \rho$ is totally symmetric (that is, ρ satisfies the Codazzi equation): our assumption $\nabla W = 0$ implies the condition $\operatorname{div} W = 0$, well known [11, formula (5.29) on p. 460] to be equivalent to the Codazzi equation for the Schouten tensor σ , while $\sigma = \rho$ by Lemma 2.2(c). Thus, $d\chi$ equals a function times dt , and so χ is, locally, a function of t , which (locally) yields (f).

For any section v of \mathcal{D}^\perp , we denote by \underline{v} the image of v under the quotient-projection morphism $\mathcal{D}^\perp \rightarrow \mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$.

The data for the construction in Section 4 consist of I, f, n, V appearing in (a) – (f), of the pseudo-Euclidean inner product $\langle \cdot, \cdot \rangle$ in V , induced in an obvious way by g (cf. Lemma 2.2(f)), and of $A : V \rightarrow V$ with $\langle A\psi, \psi' \rangle = W(u', v, v', u')$, for $\psi, \psi' \in V$, where a vector field u' and sections v, v' of \mathcal{D}^\perp are chosen, locally, so that $g(u, u') = 1$, $\psi = \underline{v}$ and $\psi' = \underline{v}'$. (The bilinear form $(\psi, \psi') \mapsto \langle A\psi, \psi' \rangle$ on V then is well-defined, that is, unaffected by the choices of u', v or v' , in view of Lemma 2.2(d),(e), while the function $W(u', v, v', u')$ is in fact constant, by Lemma 2.2(d), as one sees differentiating it via the Leibniz rule and noting that, since \underline{v} and \underline{v}' are parallel, the covariant derivatives of v and v' in the direction of any vector field are sections of \mathcal{D} .) That A is traceless and self-adjoint is immediate from the symmetries of W . Finally, $A \neq 0$ since, otherwise, W would vanish. (Namely, in view of Lemma 2.2(d),(e), W would yield 0 when evaluated on any quadruple of vector fields, each of which is either u' or a section of \mathcal{D}^\perp .)

Under the assumptions (a) – (f), with $f = f(t)$, we then have

$$(8) \quad R(u', v)v' = [fg(v, v') + \langle A\underline{v}, \underline{v}' \rangle]g(u', u)u$$

for any sections v, v' of \mathcal{D}^\perp and any vector field u' . In fact, $\rho(v, \cdot) = \rho(v', \cdot) = 0$ from symmetry of ρ and Lemma 2.2(b), so that, by Lemma 2.2(c), $R(u', v)v' = W(u', v)v' - (n - 2)^{-1}g(v, v')\rho u'$, where $\rho u'$ denotes the unique vector field with $g(\rho u', \cdot) = \rho(u', \cdot)$. Now (8) follows: due to (d), (f) and the definition of A , both sides have the same g -inner product with u' , and are orthogonal to $u^\perp = \mathcal{D}^\perp$ (since $R(u', v)v'$ is orthogonal to \mathcal{D}^\perp as a consequence of Lemma 2.2(e)).

We fix an open subinterval I of I' , containing 0, and a null geodesic $I \ni t \mapsto x(t)$ in M with $x(0) = y$, parametrized by the function t (in the sense that the function t restricted to the geodesic coincides with the geodesic parameter). Namely, since $\nabla dt = 0$, the restriction of t to any geodesic is an affine function of the parameter; thus, by (d), it suffices to prescribe the initial data formed by $x(0) = y$ and a null vector $\dot{x}(0) \in T_y M$ with $g(\dot{x}(0), u_y) = 1$.

As $g(\dot{x}(0), u_y) = 1$, the plane P in $T_y M$, spanned by the null vectors $\dot{x}(0)$ and u_y (cf. Lemma 2.2(a)) is g_y -nondegenerate, and so $T_y M = P \oplus \tilde{V}$, for $\tilde{V} = P^\perp$. Let $\text{pr} : T_y M \rightarrow \tilde{V}$ be the orthogonal projection. Since $\text{pr}(\mathcal{D}_y) = \{0\}$, the restriction of pr to \mathcal{D}_y^\perp descends to an isomorphism $\mathcal{E}_y = \mathcal{D}_y^\perp / \mathcal{D}_y \rightarrow \tilde{V}$, also denoted by pr . Finally, for $\psi \in V$, we let $t \mapsto \tilde{\psi}(t) \in T_{x(t)} M$ be the parallel field with $\tilde{\psi}(0) = \text{pr} \psi_y$, and set $\kappa(t, s, \psi) = f(t)\langle \psi, \psi \rangle + \langle A\psi, \psi \rangle$, as in Section 4.

The formula $F(t, s, \psi) = \exp_{x(t)}(\tilde{\psi}(t) + su_{x(t)}/2)$ now defines a C^∞ mapping F from an open subset of $\mathbf{R}^2 \times V$ into M .

LEMMA 5.1. *Under the above hypotheses, $F^*g = \kappa dt^2 + dt ds + h$.*

PROOF. The F -images $w, w', F_*\psi$ of the constant vector fields $(1, 0, 0), (0, 1, 0)$ and $(0, 0, \psi)$ in $\mathbf{R}^2 \times V$, for $\psi \in V$, are vector fields tangent to M along F (sections of F^*TM). Since \mathcal{D}^\perp is parallel, its leaves are totally geodesic and, by Lemma 2.2(e), the Levi-Civita connection of g induces on each leaf a flat torsion-free connection. Thus, w' and each $F_*\psi$ are parallel along each leaf of \mathcal{D}^\perp , as well as tangent to the leaf, and parallel along the geodesic $t \mapsto x(t)$. Therefore, $w' = u/2$, while the functions $g(w', F_*\psi)$ and $g(F_*\psi, F_*\psi')$, for $\psi, \psi' \in V$, are constant, and hence equal to their values at y , that is, 0 and $\langle \psi, \psi' \rangle$. It now remains to be shown that $g(w, w) = \kappa \circ F$, $g(w, u/2) = 1/2$ and $g(w, F_*\psi) = 0$. To this end, we consider the variation $x(t, s) = F(t, sa, s\psi)$ of curves in M , with any fixed $a \in \mathbf{R}$ and $\psi \in V$. Clearly, $w = x_t$ along the variation (notation of Section 1). Next, $x_{ts} = x_{st}$ is tangent to \mathcal{D}^\perp , since so is x_s , while \mathcal{D}^\perp is parallel. Consequently, $[g(x_t, u)]_s = 0$, as u is parallel and tangent to \mathcal{D} . Thus, $g(w, u) = g(x_t, u) = 1$. (Note that $g(x_t, u) = 1$ at $s = 0$, due to (d), as the geodesic $t \mapsto x(t)$ is parametrized by the function t .) However, $x_{ss} = 0$ and x_s is tangent to \mathcal{D}^\perp , so that (3) and (8) now give $x_{tss} = [fg(x_s, x_s) + \langle Ax_s, x_s \rangle]u$, which is parallel in the s direction, while $x_{ts} = x_{st} = 0$ at $s = 0$. Hence $x_{ts} = s[fg(x_s, x_s) + \langle Ax_s, x_s \rangle]u$, and so $g(x_{ts}, x_{ts}) = 0$ (cf. (c) above and Lemma 2.2(a)). This further yields $[g(x_t, x_t)]_{ss}/2 = g(x_t, x_{tss}) = fg(x_s, x_s) + \langle Ax_s, x_s \rangle$. The last function is constant in the s direction, while $g(x_t, x_t) = [g(x_t, x_t)]_s = 0$ at $s = 0$, and so $g(w, w) = g(x_t, x_t) = s^2[fg(x_s, x_s) + \langle Ax_s, x_s \rangle] = \kappa$. Finally, being proportional to u at each point, x_{ts} is orthogonal to \mathcal{D}^\perp , and hence to $F_*\psi$, which implies that $[g(x_t, F_*\psi)]_s = 0$, and, as $g(w, F_*\psi) = g(x_t, F_*\psi) = 0$ at $s = 0$, we get $g(w, F_*\psi) = 0$ everywhere. \square

We are now in a position to prove Theorem 4.1. First, (7) is conformally symmetric and has $d = 1$, as one can verify by a direct calculation, cf. [16, Theorem 3]. Conversely, if conditions (a) and (b) above are satisfied, we may also assume (c) – (f). (See the comment following (f).) Our assertion is now immediate from Lemma 5.1.

APPENDIX I: PROOF OF LEMMA 2.1

We prove Lemma 2.1 here, since Olszak's paper [13] may be difficult to obtain.

The condition $d = n$ is equivalent to conformal flatness of (M, g) , since $n > 2$ and so $\Omega = 0$ is the only 2-form \wedge -divisible by all nonzero 1-forms ξ . At a fixed point x , the metric g_x allows us to treat the Ricci tensor ρ_x and any 2-form Ω_x as endomorphisms of T_xM , so that we may consider their images (which are subspaces of T_xM). If $W \neq 0$, fixing a nonzero 2-form Ω_x in the image of W_x acting on 2-forms at x we see that, for every $u \in \mathcal{D}_x$, our Ω_x is \wedge -divisible by $\xi = g_x(u, \cdot)$, and so the image of Ω_x contains \mathcal{D}_x . Thus, $d \leq 2$, and (i) follows. (Being nonzero and decomposable, Ω_x has rank 2.) As shown in [6, Theorem 7 on p. 18], if (M, g) is essentially conformally symmetric, the image of ρ_x is a subspace of \mathcal{D}_x , so that (i) yields (ii), since g in (ii) cannot be Ricci-flat. Next, if $d = 2$, the image of our Ω_x coincides with \mathcal{D}_x (as $\text{rank } \Omega_x = 2$). Every 2-form in the image of W_x thus is a multiple of Ω_x , being the exterior product of two vectors in \mathcal{D}_x , identified, via g_x , with 1-forms. Hence $\text{rank } W = 1$. Conversely, if $\text{rank } W = 1$, all nonzero 2-forms Ω_x in the image of W_x are of rank 2, as W_x , being self-adjoint, is a multiple of $\Omega_x \otimes \Omega_x$, and so the Bianchi identity for W gives $\Omega_x \wedge \Omega_x = 0$. All such Ω_x are therefore \wedge -divisible by $\xi = g_x(u, \cdot)$, for every nonzero vector u in the common 2-dimensional image of such Ω_x , which shows that $d = 2$. Finally, (iv) follows if one chooses $\Omega_x \neq 0$ equal to $W_x(v, v', \cdot, \cdot)$ for some $v, v' \in T_xM$.

APPENDIX II: LEMMA 2.2(b),(c) IN THE LOCALLY SYMMETRIC CASE

Parts (b) and (c) of Lemma 2.2 for locally symmetric manifolds with $d = 1$ could, in principle, be derived from Cahen and Parker's classification [1] of pseudo-Riemannian symmetric manifolds. We prove them here directly, for the reader's convenience. Our argument uses assertions (a), (d) in Lemma 2.2, along with (e) for W , which were established in the proof of Lemma 2.2 before Appendix II was mentioned.

Suppose that $\nabla R = 0$ and $d = 1$. Replacing M by an open subset, we also assume that the Olszak distribution \mathcal{D} is spanned by a vector field u . By (1),

$$(9) \quad \text{i) } R(\cdot, \cdot)u = \Omega \otimes u \quad \text{or, in coordinates,} \quad \text{ii) } u^l R_{jkl}{}^s = \Omega_{jk} u^s,$$

for some differential 2-form Ω , which obviously does not depend on the choice of u . (It is also clear from (1) that Ω is the curvature form of the connection in the line bundle \mathcal{D} , induced by the Levi-Civita connection of g .) Being unique, Ω is

parallel, and so are ρ and W , which implies the Ricci identities $R \cdot \Omega = 0$, $R \cdot \rho = 0$, and $R \cdot W = 0$. Equivalently, $R_{mlj}{}^s \tau_{sk} + R_{mlk}{}^s \tau_{js} = 0$ for $\tau = \Omega$ or $\tau = \rho$, and

$$(10) \quad R_{qpj}{}^s W_{sklm} + R_{qpk}{}^s W_{jstl} + R_{qpl}{}^s W_{jksm} + R_{qpk}{}^s W_{jkl} = 0.$$

Summing $R_{mlj}{}^s \Omega_{sk} + R_{mlk}{}^s \Omega_{js} = 0$ against u^l , we obtain $\Omega \circ \Omega = 0$, where the metric g is used to treat Ω as a bundle morphism $TM \rightarrow TM$ that sends each vector field v to the vector field Ωv with $g(\Omega v, v') = \Omega(v, v')$ for all vector fields v' . Lemma 2.2(d) and (9.i) give $W(\cdot, \cdot, u, v) = R(\cdot, \cdot, u, v) = 0$ for our fixed vector field u , spanning \mathcal{D} , and any section v of \mathcal{D}^\perp . Hence, by (2), $g(u, \cdot) \wedge \sigma(v, \cdot) = g(v, \cdot) \wedge \sigma(u, \cdot)$. Thus, $\sigma u = cu$ for the Schouten tensor σ and some constant c , with σu defined analogously to Ωv . (Otherwise, choosing v such that $u, \sigma u$ and v are linearly independent at a given point x , we would obtain a contradiction with the equality between planes in $T_x M$, corresponding to the above equality between exterior products.) Consequently, $g(u, \cdot) \wedge (\sigma + cg)(v, \cdot) = 0$, and so $\sigma v + cv$ is a section of \mathcal{D} whenever v is a section of \mathcal{D}^\perp . Let us now fix u' as in (4). Symmetry of σ gives $g(\sigma u', u) = c$. In a suitably ordered basis with (5), at any point x , the endomorphism of $T_x M$ corresponding to σ_x thus has an upper triangular matrix with the diagonal entries $c, -c, \dots, -c, c$, so that $\text{tr}_g \sigma = (4 - n)c$. Consequently, $(n - 2)s = 2(n - 1)(4 - n)c$, for the scalar curvature s , and $(n - 2)\rho u = 2cu$. However, contracting (9.ii) in $k = s$, we get $\rho u = -\Omega u$, and so $(n - 2)\Omega u = -2cu$. The equality $\Omega \circ \Omega = 0$ that we derived from the Ricci identity $R \cdot \Omega = 0$ now gives $c = 0$. Hence $s = 0$ (which yields Lemma 2.2(c)), and $\rho u = 0$.

As $c = 0$ and $\sigma = \rho$, the assertion about $\sigma v + cv$ obtained above means that ρv is a section of \mathcal{D} whenever v is a section of \mathcal{D}^\perp . Let λ, μ, ξ be the 1-forms with $\lambda = g(u, \cdot)$, $\mu = g(u', \cdot)$, $\xi(u') = 0$, and $\rho v = \xi(v)u$ for sections v of \mathcal{D}^\perp . Transvecting (9.ii) with μ_s , we get $\Omega = R(\cdot, \cdot, u, u') = (n - 2)^{-1} \lambda \wedge \rho(u', \cdot)$ from Lemma 2.2(c) with $\rho u = 0$ and Lemma 2.2(d). However, evaluating $\rho(u', \cdot)$ on u', u and sections v of \mathcal{D}^\perp , we see that $\rho(u', \cdot) = h\lambda + \xi$, with $h = \rho(u', u')$. (Note that $\xi(u) = 0$ since $\rho u = 0$, while $\mathcal{D} \subset \mathcal{D}^\perp$ by Lemma 2.2(a).) Therefore,

$$(11) \quad \text{i) } (n - 2)\Omega = \lambda \wedge \xi, \quad \text{ii) } \rho = h\lambda \otimes \lambda + \lambda \otimes \xi + \xi \otimes \lambda.$$

In addition, if v' denotes the unique vector field with $g(v', \cdot) = \xi$, then u and v' are null and orthogonal, or, equivalently,

$$(12) \quad \text{the 1-forms } \lambda \text{ and } \xi \text{ are null and mutually orthogonal.}$$

In fact, $g(u, u) = 0$ by Lemma 2.2(a), $g(u, v') = 0$ as $\xi(u) = 0$, and v' is null since (11) yields $(n - 2)[\rho(\Omega u') - \Omega(\rho u')] = 2g(v', v')u$, while, transvecting the Ricci identity $R_{mlj}{}^s R_{sk} + R_{mlk}{}^s R_{js} = 0$ with u^l and using (9.ii), we see that ρ and Ω commute as bundle morphisms $TM \rightarrow TM$.

Furthermore, transvecting with $\mu^k \mu^m$ the coordinate form $R_{mlj}{}^s \tau_{sk} + R_{mlk}{}^s \tau_{js} = 0$ of the Ricci identity $R \cdot \tau = 0$ for the parallel tensor field $\tau = (n - 2)\Omega + \rho =$

$h\lambda \otimes \lambda + 2\lambda \otimes \xi$ (cf. (11)), we get $2\lambda_j b_{ls} \xi^s = 0$, where $b = W(u', \cdot, u', \cdot)$. Namely, $R = W + (n-2)^{-1}g \wedge \rho$ by Lemma 2.2(c), $W_{mlj}{}^s \tau_{sk} = 0$ in view of Lemma 2.2(d), $\mu^k \mu^m W_{mlk}{}^s \tau_{js} = 2\lambda_j b_{ls} \xi^s$ since $b(u, \cdot) = 0$ (again from Lemma 2.2(d)), and the remaining terms, related to $g \wedge \rho$, add up to 0 as a consequence of (12), (11.ii) and the formula for τ . (Note that (12) gives $R_j{}^s \tau_{sk} = R_j{}^s \tau_{ks} = 0$, and so four out of the eight remaining terms vanish individually.) However, $u \neq 0$, and so $\lambda \neq 0$, which gives $b(\cdot, v') = 0$, where v' is the vector field with $g(v', \cdot) = \xi$. Thus, $W(u', \cdot, u', v') = 0$. As a result, the 3-tensor $W(\cdot, \cdot, \cdot, v')$ must vanish: it yields the value 0 whenever each of the three arguments is either u' or a section of \mathcal{D}^\perp . (Lemma 2.2(e) for W is already established.)

The relation $W(\cdot, \cdot, \cdot, v') = 0$ implies in turn that $W(\cdot, \cdot, \cdot, \rho v) = 0$ (in coordinates: $W_{jkl}{}^s R_{sp} = 0$). In fact, by (11.ii), the image of ρ is spanned by u and v' , while $W(\cdot, \cdot, \cdot, u) = 0$ according to Lemma 2.2(d).

As in [13, 1° on p. 214], we have $W = (\lambda \otimes \lambda) \wedge b$ (notation of (2)), where, again, $b = W(u', \cdot, u', \cdot)$. Namely, by Lemma 2.2(e) for W , both sides agree on any quadruple of vector fields, each of which is either u' or a section of \mathcal{D}^\perp .

Finally, transvecting (10) with $\mu^k \mu^m$ and replacing R by $W + (n-2)^{-1}g \wedge \rho$, we obtain two contributions, one from W and one from $g \wedge \rho$, the sum of which is zero. Since $W = (\lambda \otimes \lambda) \wedge b$, the W contribution vanishes: its first two terms add up to 0, and so do its other two terms. (As we saw, $b(u, \cdot) = 0$, while, obviously, $b(u', \cdot) = 0$.) Out of the sixteen terms forming the $g \wedge \rho$ contribution, eight are separately equal to zero since $W_{jkl}{}^s R_{sp} = 0$, and so, in view of (11.ii) and the relation $W = (\lambda \otimes \lambda) \wedge b$, vanishing of the $g \wedge \rho$ contribution gives $\lambda_p S_{jlq} = \lambda_q S_{jlp}$, for $S_{jlq} = 2b_{jl} \xi_q - b_{ql} \xi_j - b_{qj} \xi_l$. Thus, $S_{jlq} = \eta_{jl} \lambda_q$ for some twice-covariant symmetric tensor field η , which, summed cyclically over j, l, q , yields 0 (due to the definition of S_{jlq} and symmetry of b). As $\lambda \neq 0$ and the symmetric product has no zero divisors, we get $\eta = 0$ and $S_{jlq} = 0$. The expression $b_{jl} \xi_q - b_{ql} \xi_j$ is, therefore, skew-symmetric in j, l . As it is also, clearly, skew-symmetric in j, q , it must be totally skew-symmetric and hence equal to one-third of its cyclic sum over j, l, q . That cyclic sum, however, is 0 in view of symmetry of b , so that $b_{jl} \xi_q = b_{ql} \xi_j$. Thus, $\xi = 0$, for otherwise the last equality would yield $b = \varphi \xi \otimes \xi$ for some function φ , and hence $W = (\lambda \otimes \lambda) \wedge b = \varphi (\lambda \otimes \lambda) \wedge (\xi \otimes \xi)$, which would clearly imply that the vector field v' with $g(v', \cdot) = \xi$ is a section of the Olszak distribution \mathcal{D} , not equal to a function times u (as $\xi(u') = 0$, while $g(u, u') = 1$), contradicting one-dimensionality of \mathcal{D} . Therefore, $\rho = h\lambda \otimes \lambda$ by (11.ii) with $\xi = 0$, which proves assertion (b) of Lemma 2.2 in our case.

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