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THE LOCAL STRUCTURE OF ESSENTIALLY CONFORMALLY SYMMETRIC MANIFOLDS WITH CONSTANT FUNDAMENTAL FUNCTION

II. THE HYPERBOLIC CASE

BY

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1. Introduction. By a *conformally symmetric* manifold [1] we mean an n-dimensional ($n \geqslant 4$) manifold M with a (not necessarily definite) Riemannian metric g, the Weyl conformal curvature tensor of which is parallel, i.e., $C_{hijk,l} = 0$, where

$$C_{hijk} = R_{hijk} - (n-2)^{-1}(g_{ij}R_{hk} + g_{hk}R_{ij} - g_{hj}R_{ik} - g_{ik}R_{hj}) + \\ + R(n-1)^{-1}(n-2)^{-1}(g_{ij}g_{hk} - g_{hj}g_{ik}),$$

R_{hijk}, R_{ij} and R being the curvature tensor, Ricci tensor and scalar curvature of M (more precisely, of (M, g)), respectively. Such a manifold is said to be *essentially conformally symmetric* (shortly, e.c.s.) if it is neither *conformally flat* ($C_{hijk} = 0$) nor *locally symmetric* ($R_{hijk,l} = 0$). Examples of e.c.s. manifolds can be found in [8], [2], [7] and [3]. All e.c.s. metrics are indefinite ([5], Theorem 2).

By Theorem 3 of [7], every e.c.s. manifold M satisfies the relation

$$R_{ij}R_{hk} - R_{hj}R_{ik} = FC_{hijk} \tag{1}$$

for a certain function F, called the *fundamental function* of M. According to Lemma 1 of [3], any e.c.s. manifold with *constant* fundamental function F is either *elliptic* ($F \neq 0$, R_{ij} semidefinite everywhere) or *hyperbolic* ($F \neq 0$, R_{ij} semidefinite nowhere), or *parabolic* ($F = 0$). The present paper is devoted to hyperbolic e.c.s. manifolds with $F = \text{const}$ (for elliptic and parabolic ones, see [3] and [4]). Our main results (Theorems 2 and 3) consist in describing their local structure. We also deliver an example of a homogeneous hyperbolic e.c.s. manifold (Theorem 1).

By a *manifold* we shall mean in the sequel a connected, paracompact manifold of class C^∞ or analytic. We shall follow strictly the methods of [3].

2. Preliminaries. The following theorem establishes the existence of *homogeneous* hyperbolic e.c.s. manifolds (with fundamental function constant in view of homogeneity).

THEOREM 1. *Let G denote the open subset of $\boldsymbol{R}^4$ given by*

$$G = \{(u^1, u^2, u^3, u^4) \mid (u^1)^2+(u^2)^2 > 0\}.$$

We can endow the set G with a Lie group structure by identifying $(u^1, u^2, u^3, u^4) \in G$ with the matrix

$$\begin{bmatrix} u^1 & u^2 & 0 & 0 \\ -u^2 & u^1+u^2 & 0 & 0 \\ u^3 & u^4 & 1 & 0 \\ u^4 & -u^3-u^4 & 0 & 1 \end{bmatrix},$$

so that G becomes isomorphic to a closed subgroup of $GL(4, \boldsymbol{R})$. Let g be the left-invariant metric on G determined at the unit element $e = (1, 0, 0, 0) \in G$ by

$$g_e(d_1, d_1) = 2t, \quad g_e(d_1, d_2) = -1, \quad g_e(d_1, d_3) = 2,$$
$$g_e(d_1, d_4) = g_e(d_2, d_3) = g_e(d_2, d_4) = 1,$$
$$g_e(d_2, d_2) = g_e(d_3, d_3) = g_e(d_3, d_4) = g_e(d_4, d_4) = 0,$$

where t is an arbitrary real number and $d_1, \ldots, d_4$ are the vectors of the canonical frame of $\boldsymbol{R}^4$ at e.

Then (G, g) is a hyperbolic e.c.s. manifold with fundamental function $F = -4$.

Proof. Denoting by $\bar{d}_i$ the left-invariant vector fields on G whose values at e are d_i, $i = 1, \ldots, 4$, we easily obtain

$$[\bar{d}_1, \bar{d}_2] = 0, \quad [\bar{d}_1, \bar{d}_3] = -\bar{d}_3, \quad [\bar{d}_1, \bar{d}_4] = -\bar{d}_4,$$
$$[\bar{d}_2, \bar{d}_3] = -\bar{d}_4, \quad [\bar{d}_2, \bar{d}_4] = \bar{d}_3-\bar{d}_4, \quad [\bar{d}_3, \bar{d}_4] = 0.$$

Define the left-invariant frame field $e_1, \ldots, e_4$ on G by

$$e_1 = \bar{d}_4, \quad e_2 = \bar{d}_3-\bar{d}_4, \quad e_3 = \bar{d}_2, \quad e_4 = \bar{d}_1-\bar{d}_2-(1+t)\bar{d}_3+(2+t)\bar{d}_4,$$

so that we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_2, \quad [e_1, e_4] = e_1+e_2,$$
$$[e_2, e_3] = e_1+e_2, \quad [e_2, e_4] = -e_1, \quad [e_3, e_4] = (1+t)e_1+(2+t)e_2.$$

Denoting now by g_{ij}, Γ_{ij}^k, R_{hijk}, etc. the components of the metric, the Riemannian connection, the curvature tensor, etc., respectively, in the frame field $e_1, \ldots, e_4$, it is easy to see that all g_{ij} vanish except for $g_{13} = g_{31} = g_{24} = g_{42} = 1$. Using the above-given relations and the well-known formula

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{ks}(g_{ir}c_{sj}^{r} + g_{jr}c_{si}^{r}) + \frac{1}{2} c_{ij}^{k}$$

(cf., e.g., [3], the proof of Theorem 1), where c_{ij}^k are the structure constants given by $[e_i, e_j] = c_{ij}^k e_k$, we obtain the following covariant derivative expressions:

$$D_{e_1}e_1 = D_{e_1}e_2 = D_{e_2}e_1 = D_{e_2}e_2 = 0, \quad D_{e_1}e_3 = -e_2, \quad D_{e_1}e_4 = e_1,$$
$$D_{e_2}e_3 = e_2, \quad D_{e_2}e_4 = -e_1,$$
$$D_{e_3}e_1 = 0, \quad D_{e_3}e_2 = -e_1, \quad D_{e_3}e_3 = -(1+t)e_2+e_4, \quad D_{e_3}e_4 = (1+t)e_1,$$
$$D_{e_4}e_1 = -e_2, \quad D_{e_4}e_2 = 0, \quad D_{e_4}e_3 = -(2+t)e_2, \quad D_{e_4}e_4 = (2+t)e_1+e_3.$$

These equalities, together with

$$R_{hijk} = g(e_k, D_{e_h}D_{e_i}e_j - D_{e_i}D_{e_h}e_j - D_{[e_h,e_i]}e_j), \tag{2}$$

show that the only non-zero components of the curvature tensor, the Ricci tensor and the Weyl conformal tensor are those related to $R_{1334} = 1$, $R_{2434} = -1$, $R_{3434} = -1$, $R_{34} = -2$ and $C_{3434} = -1$. Making use of the relation

$$C_{hijk,l} = -\Gamma_{lh}^{r}C_{rijk} - \Gamma_{li}^{r}C_{hrjk} - \Gamma_{lj}^{r}C_{hirk} - \Gamma_{lk}^{r}C_{hijr},$$

and similarly computing $R_{33,3}$, we can easily verify that (G, g) is e.c.s. Hyperbolicity is obvious. Using (1) we now obtain $F = -4$, which completes the proof.

A Riemannian manifold is said to be *Ricci-recurrent* if $R_{hi}R_{jk,l} = R_{jk}R_{hi,l}$. Examples show that e.c.s. manifolds may be Ricci-recurrent ([8], Theorem 3) or not ([2] and [7], Theorem 6). Ricci-recurrency implies vanishing of the fundamental function ([7], Theorem 5).

LEMMA 1 ([7], Theorem 4). *If M is a non-Ricci-recurrent e.c.s. manifold, then $C_{hijk} = \delta\omega_{hi}\omega_{jk}$, where $|\delta| = 1$ and ω is a parallel, absolute (i.e. determined at each point up to a sign) exterior 2-form on M such that* $\operatorname{rank}\omega = 2$ *and* $\omega_{ir}\omega^{r}{}_{j} = 0$.

LEMMA 2 ([3], Lemma 3). *Let M be a non-Ricci-recurrent e.c.s. manifold (e.g., a hyperbolic one with constant fundamental function) and let ω be the absolute 2-form defined in Lemma 1. Then*

(i) *The image* $\operatorname{im}\omega$ *of ω, i.e. the set of all vectors u of the type $u_i = \pm\omega_{ij}v^j$, is a parallel field of totally isotropic (2-dimensional) planes, which contains all vectors u of the form $u_i = R_{ij}v^j$.*

(ii) *The orthogonal complement of* $\operatorname{im}\omega$ *coincides with the kernel* $\ker\omega$ *of ω (the set of all vectors v with $\omega_{ij}v^j = 0$) and each $v \in \ker\omega$ satisfies $R_{ij}v^j = 0$.*

(iii) *Both* $\operatorname{im}\omega$ *and* $\ker\omega$ *are integrable. The tensor fields* R_{ij} *and* $R_{ij,k}$ *are parallel along the integral manifolds of* $\ker\omega$.

LEMMA 3. *Suppose that we are given a manifold M with a symmetric* $(0,2)$ *tensor field P on M such that* $\operatorname{rank}P = 2$ *and P is not semidefinite at any point. Then, for any $p \in M$, there exists a covariant 2-frame field a, b (i.e. a pair of C^∞ covariant vector fields, linearly independent at each point) in a neighbourhood U of p such that* $P_{ij} = a_i b_j + a_j b_i$ *in U.*

Proof. Choose covariant vector fields $\bar{a}$, $\bar{b}$ spanning the image $\operatorname{im}P$ of P in a neighbourhood U of p, so that

$$P_{ij} = X\bar{a}_i\bar{a}_j + Y(\bar{a}_i\bar{b}_j + \bar{b}_i\bar{a}_j) + Z\bar{b}_i\bar{b}_j$$

for some functions X, Y, Z. It is clear that there exists a pair c, d of C^∞ contravariant vector fields, defined in U and such that $c^i a_i = d^i b_i = 1$ and $c^i b_i = d^i a_i = 0$. Since c, d and $\ker P$ span the tangent bundle of U, the expressions

$$P_{ij}(sc^i + td^i)(sc^j + td^j), \tag{3}$$

where s and t run over real numbers, must assume both positive and negative values (for, otherwise, P would be semidefinite). Note that (3) is equal to $s^2X + 2stY + t^2Z$. Hence, the matrix

$$\begin{bmatrix} X & Y \\ Y & Z \end{bmatrix}$$

defines (for any fixed point of U) a hyperbolic quadratic form in $\boldsymbol{R}^2$, which implies $XZ - Y^2 < 0$. Therefore, the quadratic equation

$$[Q(q)]^2 - Y(q)Q(q) + \frac{1}{4}X(q)Z(q) = 0$$

has a non-zero solution $Q(q)$ (at any point $q \in U$) and the solution function $Q\colon q \mapsto Q(q)$ can be chosen to be C^∞. Setting now $a = Q\bar{a} + \frac{1}{2}Z\bar{b}$ and $b = \bar{b} + \frac{1}{2}XQ^{-1}\bar{a}$, we obtain our assertion.

Given an e.c.s. manifold M, we shall say that it is *singular* if

$$R_{hi,j}R_{kl,m} = R_{hi,m}R_{kl,j}. \tag{4}$$

M will be called *regular* if (4) fails at some point.

In the sequel, we shall often assume the following hypotheses:

(5) (M, g) is an n-dimensional ($n \geqslant 4$) *singular* hyperbolic e.c.s. Riemannian manifold with fundamental function $F = \text{const} \neq 0$, and p is a point of M such that $R_{ij,k}(p) \neq 0$.

(6) (M, g) is an n-dimensional ($n \geqslant 4$) *regular* hyperbolic e.c.s. manifold with fundamental function $F = \text{const} \neq 0$, and p is a point of M such that $R_{hi,j}(p)R_{kl,m}(p) \neq R_{hi,m}(p)R_{kl,j}(p)$ (cf. (4)).

LEMMA 4. (i) *Under the hypothesis* (5), *there exists a unique* 2*-frame* a, b *at* p *such that*

$$R_{ij} = a_i b_j + b_i a_j, \tag{7}$$

$$R_{ij,k} = a_i a_j a_k. \tag{8}$$

Moreover, a, b *can be extended to a unique* C^∞*-field of* 2*-frames in a neighbourhood of* p*, satisfying* (7) *and* (8) *at each point. This field is parallel along* $\ker\omega$ (cf. Lemma 2).

(ii) *Under the hypothesis* (6), *there exists a* 2*-frame* a, b *at* p *having property* (7) *and such that there exist* $S = S(p) > 0$ *and* $\varepsilon = \pm 1$*, satisfying*

$$R_{ij,k} = 2\varepsilon S(a_i a_j a_k + \varepsilon b_i b_j b_k). \tag{9}$$

Such frames are exactly two (*the other one being just* $\varepsilon b, \varepsilon a$). *Either of them has a unique* C^∞ *local extension satisfying* (7) *and* (9) (*where* $S > 0$ *may vary*) *at any point near* p. *Both extensions are parallel along* $\ker\omega$.

Proof. In virtue of (iii) of Lemma 2 we infer that a, b are parallel along $\ker\omega$, since they are algebraically determined by R_{ij} and $R_{ij,k}$ with the aid of (7)-(9). By Lemma 3, there exists a C^∞-field $\bar{a}, \bar{b}$ of 2-frames in a neighbourhood of p such that $R_{ij} = \bar{a}_i\bar{b}_j + \bar{b}_i\bar{a}_j$. It is clear that we can find C^∞ vector fields c, d, defined near p and dual to $\bar{a}, \bar{b}$ in the sense that $c^i\bar{a}_i = d^i\bar{b}_i = 1$ and $c^i\bar{b}_i = d^i\bar{a}_i = 0$. By Lemma 5 of [3], we have

$$R_{ij,k} = A\bar{a}_i\bar{a}_j\bar{a}_k + B\bar{b}_i\bar{b}_j\bar{b}_k + C(\bar{a}_i\bar{b}_j\bar{b}_k + \bar{b}_i\bar{a}_j\bar{b}_k + \bar{b}_i\bar{b}_j\bar{a}_k) + \\ + D(\bar{a}_i\bar{a}_j\bar{b}_k + \bar{a}_i\bar{b}_j\bar{a}_k + \bar{b}_i\bar{a}_j\bar{a}_k)$$

for some functions A, B, C, D. Differentiating (1) covariantly, we obtain

$$R_{ij,m}R_{hk} + R_{ij}R_{hk,m} = R_{hj,m}R_{ik} + R_{hj}R_{ik,m}.$$

Transvecting this with $d^id^jc^hc^kc^m$ and with $c^ic^jd^hd^kd^m$ we obtain $C = D = 0$, i.e.

$$R_{ij,k} = A\bar{a}_i\bar{a}_j\bar{a}_k + B\bar{b}_i\bar{b}_j\bar{b}_k.$$

Clearly, $A^2 + B^2 > 0$ in a neighbourhood of p, for $R_{ij,k}(p) \neq 0$. We have now two cases:

(i) Assume (5). It is then easy to see that $AB = 0$. Interchanging $\bar{a}$ and $\bar{b}$ if necessary, we may assume $A \neq 0$ and $B = 0$ near p. Setting now $a = A^{1/3}\bar{a}$ and $b = A^{-1/3}\bar{b}$, we obtain (7) and (8), so that (i) is immediate.

(ii) Suppose that (6) holds. Then $AB \neq 0$ in a neighbourhood of p and, putting

$$a = \varphi \bar{a}, \quad b = \varphi^{-1}\bar{b}, \quad \text{where } \varphi = \operatorname{sign}(B)|A|^{1/6}|B|^{-1/6},$$

we obtain (7) and (9) with $\varepsilon = \operatorname{sign}(AB)$, $S = \frac{1}{2}|AB|^{1/2}$. Since (7) determines the *unordered* pair of lines $\{\mathbf{R}a, \mathbf{R}b\}$ completely, it is clear that (7) and (9) hold only for the pairs a, b and $\varepsilon b, \varepsilon a$, which completes the proof.

Remark 1. The argument above implies that, given a hyperbolic e.c.s. manifold M with constant fundamental function, it must admit a field of unordered pairs of tangent isotropic lines. Using an arbitrary positive definite metric on M, we can construct a finite isometric covering $\bar{M}$ of M, admitting a C^∞-field a, b of 2-frames such that $a^i a_i = a^i b_i = b^i b_i = 0$. Therefore (see [3], Lemma 4), $\bar{M}$ admits a field of 4-frames.

Remark 2. From the uniqueness statement of (ii) it is clear that the number $S = S(p) > 0$ is a well-defined metric invariant (at any p where $R_{ij,k}(p) \neq 0$). Thus, by setting $S(p) = 0$ whenever $R_{ij,k}(p) = 0$, we assign a function $S: M \to [0, \infty)$ (called the *fundamental invariant*) to every regular hyperbolic e.c.s. manifold M with constant fundamental function. Clearly, S is of class C^∞ wherever $R_{ij,k} \neq 0$. Similarly, the number $\varepsilon = \pm 1$ occurring in (9) is an invariant, defined on the set where $R_{ij,k} \neq 0$ and locally constant. Putting $\varepsilon = 0$ wherever $R_{ij,k} = 0$, we define a function ε which will be called the *sign* of the regular hyperbolic manifold M (with constant fundamental function).

LEMMA 5. (i) *Under the notation and assumptions as in* (i) *of Lemma* 4, *the frame field* a, b *determined by* (7) *and* (8) *satisfies*

$$a_{i,j} = \mu a_i a_j, \quad b_{i,j} = \frac{1}{2} a_i a_j - \mu b_i a_j \tag{10}$$

for some function μ *defined in a neighbourhood of* p.

(ii) *In the conditions of* (ii) *of Lemma* 4, *we have*

$$a_{i,j} = \sigma a_i a_j + \lambda a_i b_j + S b_i b_j, \quad b_{i,j} = \varepsilon S a_i a_j - \sigma b_i a_j - \lambda b_i b_j,$$

σ *and* λ *being functions defined near* p, *while* ε *and* S *are the sign and the fundamental invariant of* M, *respectively* (see Remark 2 and (9)).

Proof. In both cases, a and b are parallel along $\ker\omega$ (Lemma 4). On the other hand, by (i) of Lemma 2, they span the parallel plane field $\operatorname{im}\omega$. Therefore, $a_{i,j}$ and $b_{i,j}$ are combinations of tensor squares and products of a and b. The corresponding coefficient functions can be determined by differentiating (7) covariantly and comparing the result with (8) (respectively, with (9)). Assertion (ii) is now immediate.

As for (i), we obtain

$$a_{i,j} = \mu a_i a_j - \tau a_i b_j, \qquad b_{i,j} = \frac{1}{2} a_i a_j - \mu b_i a_j + \tau b_i b_j$$

for some functions μ and τ. Thus, $a_{i,j} = a_i c_j$, where $c = \mu a - \tau b$. The expression $R_{ij,kl} = 3a_i a_j a_k c_l$ (cf. (8)) is symmetric in l, k by Theorem 9 of [6]. Hence a and c are collinear, which implies $\tau = 0$. This completes the proof.

Definition. Under the hypothesis (5) (respectively, (6)), a C^∞-field $c, d, e_3, \ldots, e_{n-2}, b, a$ of n-frames in a neighbourhood of p is called *S-special* (respectively, *R-special*) if a and b satisfy (7) and (8) (respectively, (7) and (9)) and

$$g(a, e_x) = g(b, e_x) = g(c, e_x) = g(d, e_x) = 0, \qquad g(e_x, e_y) = \varepsilon_x \delta_{xy},$$

$$g(a, d) = g(b, c) = g(c, c) = g(c, d) = g(d, d) = 0,$$

$$g(a, c) = g(b, d) = 1, \tag{11}$$

$$D_a e_x = D_b e_x = D_{e_y} e_x = 0,$$

$$D_a c = D_b c = D_{e_x} c = D_a d = D_b d = D_{e_x} d = 0,$$

where D denotes the Riemannian covariant derivative and $|\varepsilon_x| = 1$. Here and in the sequel we adopt the convention that the indices x, y, z range over the set $\{3, \ldots, n-2\}$ (empty if $n = 4$).

Note that, in view of (7) and (i) of Lemma 2, any S-special (respectively, R-special) frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ satisfies

$$a^i a_i = a^i b_i = b^i b_i = 0. \tag{12}$$

3. The local structure of singular hyperbolic manifolds.

LEMMA 6. *Under the hypothesis* (5), *there exists an S-special frame field* $c, d, e_3, \ldots, e_{n-2}, b, a$ *in a neighbourhood of* p. *Any such field satisfies the covariant derivative relations*

$$D_c c = \xi b - \mu c - \frac{1}{2} d - \sum_x \varepsilon_x A_x e_x, \qquad D_c d = -\xi a + \mu d - \sum_x \varepsilon_x B_x e_x,$$

$$D_c e_x = A_x a + B_x b + \sum_y C_{xy} e_y, \qquad C_{yx} = -\varepsilon_x \varepsilon_y C_{xy},$$

$$D_c b = \frac{1}{2} a - \mu b, \qquad D_c a = \mu a, \tag{13}$$

$$D_d c = \psi b - \sum_x \varepsilon_x E_x e_x, \qquad D_d d = -\psi a - \sum_x \varepsilon_x F_x e_x,$$

$$D_d e_x = E_x a + F_x b + \sum_y G_{xy} e_y, \quad G_{yx} = -\varepsilon_x \varepsilon_y G_{xy},$$

$$D_d b = 0, \quad D_d a = 0, \quad D_{e_x} \ldots = D_b \ldots = D_a \ldots = 0,$$

where ... stands for any frame vector, $x, y = 3, \ldots, n-2$, *and* ξ, ψ, A_x, B_x, E_x, F_x, C_{xy}, G_{xy} *are certain* C^∞*-functions in a neighbourhood of* p, *while* μ *is the function determined in* (i) *of Lemma* 5. *Moreover, these functions satisfy the following equations:*

$$(14)\quad \begin{cases} D_a\mu = D_a\psi = D_a A_x = D_a B_x = D_a E_x = D_a F_x = D_a C_{xy} \\ \qquad = D_a G_{xy} = 0, \\ D_a\xi = -(n-2)^{-1}, \end{cases}$$

$$(15)\quad \begin{cases} D_b\mu = D_b\xi = D_b A_x = D_b B_x = D_b E_x = D_b F_x = D_b C_{xy} \\ \qquad = D_b G_{xy} = 0, \\ D_b\psi = (n-2)^{-1}, \end{cases}$$

$$(16)\quad \begin{cases} D_{e_x}\mu = D_{e_x}\xi = D_{e_x}\psi = D_{e_y} A_x = D_{e_y} F_x = D_{e_x} C_{yz} = D_{e_x} G_{yz} = 0, \\ D_{e_x} B_y = D_{e_x} E_y = -(n-2)^{-1}\varepsilon_x \delta_{xy}, \end{cases}$$

$$(17)\quad D_d\mu = (n-2)^{-1},$$

$$(18)\quad D_c\psi - D_d\xi + \sum_x \varepsilon_x (A_x F_x - E_x B_x) - \mu\psi - F^{-1} = 0,$$

$$(19)\quad D_d A_x - D_c E_x + \sum_y (C_{xy} E_y - G_{xy} A_y) - \frac{1}{2} F_x = 0,$$

$$(20)\quad D_d B_x - D_c F_x + \sum_y (C_{xy} F_y - G_{xy} B_y) + 2\mu F_x = 0,$$

$$(21)\quad D_c G_{xy} - D_d C_{xy} + \sum_z (G_{xz} C_{zy} - C_{xz} G_{zy}) - \mu G_{xy} = 0.$$

Proof. The existence statement is an immediate consequence of Lemma 6 of [3] and Lemma 4. Using (11), (12) and (i) of Lemma 5 together with the Leibniz rule, it is easy to obtain (13). Since the scalar curvature of any e.c.s. manifold vanishes ([6], Theorem 7), (1) implies (provided $F \neq 0$) that

$$R_{hijk} = F^{-1}(R_{ij}R_{hk} - R_{hj}R_{ik}) + (n-2)^{-1}(g_{ij}R_{hk} + g_{hk}R_{ij} - g_{hj}R_{ik} - g_{ik}R_{hj}).$$

Hence and in view of (11), (12) and (7), there are only $n-1$ essential curvature components in our frame:

$$R_{hijk}c^h d^i c^j d^k = F^{-1}, \quad R_{hijk}a^h c^i c^j d^k = -(n-2)^{-1},$$

$$R_{hijk}b^h d^i c^j d^k = (n-2)^{-1} \quad \text{and} \quad R_{hijk}c^h e_x^i d^j e_x^k = -(n-2)^{-1}\varepsilon_x.$$

On the other hand, curvature components can be computed by means of (13) (cf. (2)). The comparison of both results yields our assertion. Thus, we can obtain (14) by calculating $R_{acc} = R(a, c)c = -(n-2)^{-1}b$, R_{adc}, $R_{acx} = R(a, c)e_x$ and R_{adx}, (15) from R_{bcc}, $R_{bdc} = (n-2)^{-1}b$, R_{bcx} and R_{bdx}, (16) from R_{cxc}, R_{dxd} and $R_{cxy} = (n-2)^{-1}\varepsilon_x\delta_{xy}b$, $R_{dxy} = (n-2)^{-1}\varepsilon_x\delta_{xy}a$, (17) and (18) from $R_{cdc} = F^{-1}b+(n-2)^{-1}c$. Finally, to obtain (19), (20) and (21) it is sufficient to consider R_{cdx}. This completes the proof.

A Riemannian manifold M (the metric definite or not) is said to be *locally homogeneous* if for any $p, q \in M$ there exists an isometry of a neighbourhood of p onto a neighbourhood of q, sending p onto q.

PROPOSITION 1. *Let M be a singular hyperbolic e.c.s. manifold with constant fundamental function. Then M is not locally homogeneous.*

Proof. If M were so, then the function μ of (i) of Lemma 5 would be constant, as it is determined by (8) and (10). In view of Lemma 6, this would contradict (17), which completes the proof.

LEMMA 7. *Under the hypothesis* (5), *there exists an S-special frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p such that, in the notation of Lemma* 6,

$$C_{xy} = G_{xy} = 0, \tag{22}$$

$$A_x = F_x = 0, \quad B_x = E_x, \tag{23}$$

$$D_cE_x = D_dE_x = 0, \tag{24}$$

$$\begin{gathered} D_c\xi = -\mu\xi+\frac{1}{2}\psi, \quad D_d\xi = -F^{-1}-\frac{1}{2}\sum_x \varepsilon_xE_x^2, \\ D_c\psi = \mu\psi+\frac{1}{2}\sum_x \varepsilon_xE_x^2, \quad D_d\psi = 0. \end{gathered} \tag{25}$$

Proof. We proceed by three steps.

(i) As in Lemma 6, choose an S-special frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ near p. The system of differential equations

$$\begin{gathered} D_c\tau_{xy} = -\sum_z \tau_{xz}C_{zy}, \quad D_d\tau_{xy} = -\sum_z \tau_{xz}G_{zy}, \\ D_{e_z}\tau_{xy} = D_b\tau_{xy} = D_a\tau_{xy} = 0 \end{gathered} \tag{26}$$

with unknown functions τ_{xy} is completely integrable in view of (13)-(16) and (21) (thus, e.g., the consistency relation $D_cD_d\tau_{xy} - D_dD_c\tau_{xy} - D_{[c,d]}\tau_{xy} = 0$ is immediate from (21)). Let τ_{xy} be the solution of (26) with initial value $\tau_{xy}(p) = \delta_{xy}$. By (26), $\sum_z \varepsilon_z\tau_{xz}\tau_{yz}$ is constant, whence it must equal $\varepsilon_x\delta_{xy}$. Therefore, $c, d, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a$, where

$$\bar{e}_x = \sum_y \tau_{xy}e_y, \tag{27}$$

is an S-special frame field near p and it is easily seen that it satisfies (22).

(ii) Now let $c, d, e_3, \ldots, e_{n-2}, b, a$ be an S-special frame field in a neighbourhood of p, satisfying (22). Consider the system of differential equations

$$D_c\zeta_x = -\frac{1}{2}\iota_x - \mu\zeta_x - A_x, \qquad D_d\zeta_x = -E_x + \varkappa_x,$$

$$D_{e_y}\zeta_x = D_b\zeta_x = D_a\zeta_x = 0,$$

$$D_c\iota_x = \mu\iota_x - B_x + \varkappa_x, \qquad D_d\iota_x = -F_x, \qquad D_{e_y}\iota_x = D_b\iota_x = D_a\iota_x = 0,$$

$$D_c\varkappa_x = -(n-2)^{-1}\zeta_x, \qquad D_d\varkappa_x = -(n-2)^{-1}\iota_x,$$

$$D_{e_y}\varkappa_x = -(n-2)^{-1}\varepsilon_y\delta_{xy},$$

$$D_b\varkappa_x = D_a\varkappa_x = 0$$

with indeterminates $\zeta_x, \iota_x, \varkappa_x$. Its integrability conditions follow immediately from (13)-(17), (19), (20) and (22). Choosing a solution $\zeta_x, \iota_x, \varkappa_x$ and setting

$$\bar{c} = c - \sum_x \varepsilon_x\zeta_x e_x - \frac{1}{2}\sum_x \varepsilon_x\zeta_x^2 a, \tag{28}$$

$$\bar{d} = d - \sum_x \varepsilon_x\iota_x e_x - \frac{1}{2}\sum_x \varepsilon_x\iota_x^2 b - \sum_x \varepsilon_x\zeta_x\iota_x a, \qquad \bar{e}_x = e_x + \zeta_x a + \iota_x b,$$

it is easy to verify that $\bar{c}, \bar{d}, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a$ is an S-special frame field satisfying (22) and (23). By (19) and (20), it must satisfy (24).

(iii) Given an S-special frame field satisfying (22)-(24), the system of differential equations

$$D_c h = \alpha, \qquad D_d h = \beta, \qquad D_{e_x} h = D_b h = D_a h = 0,$$

$$D_c\alpha = D_c\xi + \mu(\xi - \alpha) + \frac{1}{2}(\beta - \psi),$$

$$D_d\alpha = D_d\xi - (n-2)^{-1}h + F^{-1} + \frac{1}{2}\sum_x \varepsilon_x E_x^2,$$

$$D_{e_x}\alpha = D_b\alpha = D_a\alpha = 0,$$

$$D_c\beta = D_c\psi - (n-2)^{-1}h + \mu(\beta - \psi) - \frac{1}{2}\sum_x \varepsilon_x E_x^2, \qquad D_d\beta = D_d\psi,$$

$$D_{e_x}\beta = D_b\beta = D_a\beta = 0$$

is completely integrable, which follows immediately from Lemma 6.

Thus, e.g., (14)-(17) yield

$$D_cD_da - D_dD_ca - D_{[c,d]}a = D_cD_d\xi - D_dD_c\xi - (n-2)^{-1}\xi - \mu D_d\xi,$$

which vanishes since $(n-2)^{-1}\xi + \mu D_d\xi = D_{[c,d]}\xi$ as $[c, d] = D_cd - D_dc$. Similarly,

$$D_{e_x}D_ca - D_cD_{e_x}a - D_{[e_x,c]}a = D_{e_x}D_c\xi = D_cD_{e_x}\xi + D_{[e_x,c]}\xi,$$

which vanishes by (23), etc. Given a solution h, α, β it is now easy to verify that the formulae $\bar{c} = c - hb$, $\bar{d} = d + ha$ define an S-special frame field with the desired properties. This completes the proof.

We can now describe the local structure of singular hyperbolic e.c.s. manifolds as follows:

THEOREM 2. (i) *Let (M, g) be an n-dimensional $(n \geqslant 4)$ singular hyperbolic e.c.s. manifold with fundamental function $F = \text{const} \neq 0$. Given a point $p \in M$ with $R_{ij,k}(p) \neq 0$, there exists a chart $u^1, \ldots, u^n$ in a neighbourhood of p such that the metric components are given by*

$$g_{11} = u^{n-1} + 2F^{-1}(n-2)^2B(u^1) - F^{-1}(u^2)^2 + 2(n-2)^{-1}u^2u^n +$$
$$+2u^2u^{n-1}B(u^1) - (n-2)^{-2}(u^2)^3u^{n-1} + [2B(u^1) - (n-2)^{-2}(u^2)^2]\sum_x \varepsilon_x(u^x)^2,$$

$$(29)\quad \begin{aligned} g_{12} &= -(n-2)F^{-1} - (n-2)^{-1}u^2u^{n-1} - (n-2)^{-1}\sum_x \varepsilon_x(u^x)^2,\\ g_{1,n-1} &= \frac{1}{2}(n-2)^{-1}(u^2)^2 - (n-2)B(u^1), \qquad g_{1n} = g_{2,n-1} = 1,\\ g_{xy} &= \varepsilon_x\delta_{xy}, \qquad g_{1x} = g_{22} = g_{2x} = g_{2n} = g_{x,n-1} = g_{xn} = 0,\\ &\qquad g_{n-1,n-1} = g_{n-1,n} = g_{nn} = 0,\end{aligned}$$

where $|\varepsilon_x| = 1$ and B is a function of the first variable u^1.

(ii) *Conversely, given a C^∞-function B of u^1 and real numbers F and ε_x with $F \neq 0$ and $|\varepsilon_x| = 1$, formulae (29) define a singular hyperbolic e.c.s. indefinite metric with fundamental function F.*

Proof. (i) Since (5) is satisfied, we may choose a frame field $c, d, e_3, \ldots$ $\ldots, e_{n-2}, b, a$ as in Lemma 7. In the notation of Lemma 6, it is easy to verify that

$$(30)\qquad D_dD_c\mu = -(n-2)^{-1}\mu, \quad D_{e_x}D_c\mu = D_bD_c\mu = D_aD_c\mu = 0.$$

Defining now u^1 to be any solution of the completely integrable system $D_cu^1 = 1$, $D_du^1 = D_{e_x}u^1 = D_bu^1 = D_au^1 = 0$ and setting

$$(31)\quad \begin{aligned} u^2 &= (n-2)\mu, \quad u^x = -(n-2)\varepsilon_xE_x, \; x = 3, \ldots, n-2,\\ u^{n-1} &= (n-2)\psi, \quad u^n = -(n-2)\xi,\end{aligned}$$

one verifies immediately that $u^1, \ldots, u^n$ is a coordinate system at p, whose basic vector fields $\partial_i = \partial/\partial u^i$ are given by

$$\partial_1 = c-(n-2)D_c\mu\cdot d-(n-2)\left(\mu\psi+\frac{1}{2}\sum_x \varepsilon_x E_x^2\right)b+$$

$$+\left(\frac{1}{2}(n-2)\psi+F^{-1}(n-2)^2D_c\mu+\frac{1}{2}(n-2)^2D_c\mu\sum_x \varepsilon_x E_x^2-(n-2)\mu\xi\right)a,$$

$$\partial_2 = d-(n-2)\left(F^{-1}+\frac{1}{2}\sum_x \varepsilon_x E_x^2\right)a,$$

$$\partial_x = e_x,\ x = 3,\ldots,n-2, \quad \partial_{n-1} = b, \quad \partial_n = a.$$

Relations (30) state now that the function $D_c\mu$ satisfies

$$\partial_2 D_c\mu = -(n-2)^{-2}u^2, \quad \partial_x D_c\mu = \partial_{n-1}D_c\mu = \partial_n D_c\mu = 0,$$

i.e. that it is of the form

$$D_c\mu = -\frac{1}{2}(n-2)^{-2}(u^2)^2+B(u^1)$$

for some function B. Computing now $g_{ij} = g(\partial_i, \partial_j)$ with the help of (11), (12) and (31), we obtain (30), as desired.

(ii) The non-trivial contravariant metric components are $g^{1n} = g^{2,n-1} = 1$, $g^{2n} = -g_{1,n-1}$, $g^{xx} = \varepsilon_x$, $g^{n-1,n} = -g_{12}$ and g^{nn}. We can now compute fhe Christoffel symbols

$$\Gamma_{11}^1 = -(n-2)^{-1}u^2 = -\Gamma_{12}^2,$$

$$\Gamma_{11}^2 = -(n-2)B'-\frac{1}{2}-2u^2B+(n-2)^{-2}(u^2)^3,$$

$$\Gamma_{11}^x = u^x[(n-2)^{-2}(u^2)^2-2B], \quad \Gamma_{12}^x = (n-2)^{-1}u^x,$$

$$\Gamma_{11}^{n-1} = -(n-2)^{-1}u^n-u^{n-1}B+\frac{1}{2}(n-2)^{-2}(u^2)^2u^{n-1},$$

$$\Gamma_{1x}^{n-1} = -(n-2)^{-1}\varepsilon_x u^x, \quad \Gamma_{1,n-1}^{n-1} = -(n-2)^{-1}u^2,$$

$$\Gamma_{ij}^1 = \Gamma_{ij}^2 = \Gamma_{ij}^x = \Gamma_{ij}^{n-1} = 0$$

tor pairs i, j not involved above. It is easy to verify that the only non-zero components of the curvature tensor, Weyl tensor, Ricci tensor and its covariant derivative are related to

$$R_{1212} = -F^{-1}-2(n-2)^{-2}u^2u^{n-1}-2(n-2)^{-2}\sum_x \varepsilon_x(u^x)^2,$$

$$R_{121,n-1} = B-\frac{1}{2}(n-2)^{-2}(u^2)^2, \quad R_{121n} = (n-2)^{-1} = -R_{122,n-1},$$

$$R_{1x1x} = \varepsilon_x\left(2B-(n-2)^{-2}(u^2)^2\right), \quad R_{1x2x} = -(n-2)^{-1}\varepsilon_x$$

and

$$C_{1212} = F^{-1},$$

$$R_{11} = -2(n-2)B + (n-2)^{-1}(u^2)^2, \quad R_{12} = 1 \quad \text{and} \quad R_{11,1} = 1,$$

respectively.

Now it is immediate that $C_{hijk,l} = 0$, which completes the proof.

4. The local structure in the regular case.

LEMMA 8. *Under the hypothesis* (6), *there exists an R-special frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p. For such a field, the covariant derivatives of the frame vectors are given by*

$$D_c c = \xi b - \sigma c - \varepsilon S d - \sum_x \varepsilon_x A_x e_x, \quad D_c d = -\xi a + \sigma d - \sum_x \varepsilon_x B_x e_x,$$

$$D_c e_x = A_x a + B_x b + \sum_v C_{xv} e_v, \quad C_{vx} = -\varepsilon_x \varepsilon_y C_{xy}, \quad D_c b = \varepsilon S a - \sigma b,$$

$$D_c a = \sigma a, \quad D_d c = \psi b - \lambda c - \sum_x \varepsilon_x E_x e_x,$$

$$D_d d = -\psi a - S c + \lambda d - \sum_x \varepsilon_x F_x e_x,$$

$$D_d e_x = E_x a + F_x b + \sum_v G_{xv} e_v, \quad G_{vx} = -\varepsilon_x \varepsilon_y G_{xv}, \quad D_d b = -\lambda b,$$

$$D_d a = \lambda a + S b, \quad D_{e_x} \ldots = D_b \ldots = D_a \ldots = 0,$$

where ... stands for any frame vector and $\xi, \psi, A_x, B_x, E_x, F_x, C_{xv}, G_{xv}$ are certain C^∞-functions, while λ and σ are determined by a, b as in (ii) *of Lemma* 5, *S is the fundamental invariant of M, and ε is its sign* (cf. Remark 2). *Moreover, these functions satisfy the following equations:*

$$(32)\quad \begin{cases} D_a \xi = -(n-2)^{-1}, \\ D_a \sigma = D_a S = D_a \psi = D_a \lambda = D_a A_x = D_a B_x = D_a E_x = 0, \\ D_a F_x = D_a C_{xy} = D_a G_{xy} = 0, \end{cases}$$

$$(33)\quad \begin{cases} D_b \psi = (n-2)^{-1}, \\ D_b \xi = D_b \sigma = D_b S = D_b \lambda = D_b A_x = D_b B_x = D_b E_x = 0, \\ D_b F_x = D_b C_{xy} = D_b G_{xy} = 0, \end{cases}$$

$$(34)\quad \begin{cases} D_{e_x} \xi = D_{e_x} \sigma = D_{e_x} S = D_{e_x} \psi = D_{e_x} \lambda = D_{e_x} A_y = D_{e_x} F_y = 0, \\ D_{e_x} B_y = D_{e_x} E_y = -(n-2)^{-1} \varepsilon_x \delta_{xy}, \quad D_{e_x} C_{yz} = D_{e_x} G_{yz} = 0, \end{cases}$$

$$(35)\quad D_c \lambda - D_d \sigma + \varepsilon S^2 - 2\sigma\lambda + (n-2)^{-1} = 0,$$

$$(36)\quad D_c S - 3S\sigma = 0, \quad D_d S + 3S\lambda = 0,$$

$$(37)\quad D_c \psi - D_d \xi - \lambda \xi - \sigma \psi + \sum_x \varepsilon_x (A_x F_x - E_x B_x) - F^{-1} = 0,$$

$$D_d A_x - D_c E_x + 2\lambda A_x - \varepsilon S F_x + \sum_y (C_{xy} E_y - G_{xy} A_y) = 0, \tag{38}$$

$$D_c F_x - D_d B_x - 2\sigma F_x - S A_x + \sum_y (G_{xy} B_y - C_{xy} F_y) = 0, \tag{39}$$

$$D_c G_{xy} - D_d C_{xy} - \lambda C_{xy} - \sigma G_{xy} + \sum_z (G_{xz} C_{zy} - C_{xz} G_{zy}) = 0. \tag{40}$$

Proof. We proceed just as in the proof of Lemma 6, using (ii) of Lemma 5. The essential curvature components are given again by the same formulae which we may compare with those derived by means of (2). Thus, we obtain (32) by computing $R_{adc} = R(a, d)c$, $R_{acc} = -(n-2)^{-1}b$, $R_{acx} = R(a, c)e_x$, R_{adx}, (33) by $R_{bdc} = (n-2)^{-1}b$, R_{bcc}, R_{bcx}, R_{bdx}, (34) by R_{cxc}, $R_{dxc} = -(n-2)^{-1}e_x$, $R_{cxy} = (n-2)^{-1}\varepsilon_x \delta_{xy} b$, $R_{dxy} = (n-2)^{-1}\varepsilon_x \delta_{xy} a$, (35)-(37) by R_{cda} and R_{cdc}, (38)-(40) by R_{cdx}. This completes the proof.

LEMMA 9. *Under the hypothesis* (6), *there exists an R-special frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p such that, in the notation of Lemma* 8, *we have* (22)-(24) *and*

$$\begin{aligned} D_c \xi &= -\sigma \xi + \varepsilon S \psi, \quad D_d \xi = -\lambda \xi - \frac{1}{2} \sum_x \varepsilon_x E_x^2 - \frac{1}{2} F^{-1}, \\ D_c \psi &= \sigma \psi + \frac{1}{2} \sum_x \varepsilon_x E_x^2 + \frac{1}{2} F^{-1}, \quad D_d \psi = S \xi + \lambda \psi, \end{aligned} \tag{41}$$

$$\xi(p) = \psi(p) = E_x(p) = 0. \tag{42}$$

Proof. Our argument is a replica of the proof of Lemma 7. Given an R-special frame field $c, d, e_3, \ldots, e_{n-2}, b, a$, system (26) (notation of Lemma 8) is again completely integrable, so that choosing its solution τ_{xy} with initial value $\tau_{xy}(p) = \delta_{xy}$, and setting (27), we obtain an R-special frame field $c, d, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a$ satisfying (22). Now, given any R-special frame field satisfying (22), consider the system (notation adapted to the frame)

$$\begin{gathered} D_c \zeta_x = -\sigma \zeta_x - \varepsilon S \iota_x - A_x, \quad D_d \zeta_x = -\lambda \zeta_x - E_x + \varkappa_x, \\ D_{e_y} \zeta_x = D_b \zeta_x = D_a \zeta_x = 0, \quad D_c \iota_x = \sigma \iota_x - B_x + \varkappa_x, \\ D_d \iota_x = \lambda \iota_x - S \zeta_x - F_x, \quad D_{e_y} \iota_x = D_b \iota_x = D_a \iota_x = 0, \\ D_c \varkappa_x = -(n-2)^{-1} \zeta_x, \\ D_d \varkappa_x = -(n-2)^{-1} \iota_x, \quad D_{e_y} \varkappa_x = -(n-2)^{-1} \varepsilon_x \delta_{xy}, \quad D_b \varkappa_x = D_a \varkappa_x = 0, \end{gathered}$$

which is completely integrable in view of Lemma 8 and (22). Choosing its solution with initial value $\varkappa_x(p) = 0$, we can define by (28) a new R-special frame field whose coefficient functions satisfy $\bar{A}_x = \bar{F}_x = 0$,

$\bar{B}_x = \bar{E}_x = \varkappa_x$ and, therefore, $\bar{E}_x(p) = 0$. Finally, let $c, d, e_3, \ldots, e_{n-2}, b, a$ be any R-special frame field satisfying (22), (23) and $E_x(p) = 0$. Note that (38) and (39) imply then (24). The system of differential equations

$$D_c h = a, \quad D_d h = \beta, \quad D_{e_x} h = D_b h = D_a h = 0,$$

$$D_c a = D_c \xi + \sigma(\xi - a) + \varepsilon S(\beta - \psi),$$

$$D_d a = D_d \xi - (n-2)^{-1} h + \lambda(\xi - a) + \frac{1}{2}\sum_x \varepsilon_x E_x^2 + \frac{1}{2} F^{-1},$$

$$D_{e_x} a = D_b a = D_a a = 0,$$

$$D_c \beta = D_c \psi - (n-2)^{-1} h + \sigma(\beta - \psi) - \frac{1}{2}\sum_x \varepsilon_x E_x^2 - \frac{1}{2} F^{-1},$$

$$D_d \beta = D_d \psi + S(a - \xi) + \lambda(\beta - \psi), \quad D_{e_x} \beta = D_b \beta = D_a \beta = 0,$$

with unknown functions h, a, β, is completely integrable in view of Lemma 8 and (22)-(24). Taking its solution h, a, β with initial values $a(p) = \xi(p)$, $\beta(p) = \psi(p)$, and putting $\bar{c} = c - hb$, $\bar{d} = d + ha$, we obtain an R-special frame field $\bar{c}, \bar{d}, e_3, \ldots, e_{n-2}, b, a$ satisfying our assertion, which completes the proof.

We are now in a position to prove the local structure theorem for regular hyperbolic e.c.s. manfolds with constant fundamental function.

THEOREM 3. (i) *Let (M, g) be an n-dimensional ($n \geqslant 4$) regular hyperbolic e.c.s. manifold with fundamental function $F = \text{const} \neq 0$. Given a point $p \in M$ at which $R_{hi,j}R_{kl,m} \neq R_{hi,m}R_{kl,j}$, there exists a coordinate system $u^1, \ldots, u^n$ in a neighbourhood of p such that $u^1(p) = \ldots = u^n(p) = 0$ and the components of g are the following functions of coordinates:*

$$
\begin{aligned}
g_{11} &= -2u^n e^T \partial_1 T + 2\varepsilon u^{n-1} e^{-T},\\
g_{12} &= u^n e^T \partial_2 T + u^{n-1} e^T \partial_1 T - (n-2)^{-1} e^{2T} \sum_x \varepsilon_x (u^x)^2 - (n-2) F^{-1} e^{2T},\\
g_{22} &= 2u^n e^{-T} - 2u^{n-1} e^T \partial_2 T,\\
g_{1n} &= g_{2,n-1} = e^T, \quad g_{xy} = \varepsilon_x \delta_{xy}, \quad g_{1x} = g_{2x} = g_{1,n-1} = g_{2n} = 0,\\
g_{x,n-1} &= g_{xn} = g_{n-1,n-1} = g_{n-1,n} = g_{nn} = 0,
\end{aligned}
\tag{43}
$$

where $|\varepsilon_x| = 1$, ε is the sign of M, and T is a function of the first two variables u^1, u^2, related to the fundamental invariant S of M (cf. Remark 2) *by*

$$S = e^{-3T}, \tag{44}$$

and satisfying the quasi-linear hyperbolic partial differential equation

$$2\partial_1\partial_2 T + \varepsilon e^{-4T} + (n-2)^{-1}e^{2T} = 0. \tag{45}$$

(ii) *Conversely, given real numbers $F \neq 0$, $\varepsilon = \pm 1$, $\varepsilon_x = \pm 1$, and a function $T = T(u^1, u^2)$ satisfying* (45), *formulae* (43) *define an e.c.s. Riemannian metric with fundamental function F which is regular and hyperbolic. Moreover, its sign equals ε, while its fundamental invariant S is given by* (44).

Proof. (i) By Lemma 9, we may choose an R-special frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p, satisfying (22)-(24), (41) and (42). Defining T by (44), from (32)-(34) and (36) we obtain

$$D_c T = -\sigma, \quad D_d T = \lambda, \quad D_{e_x} T = D_b T = D_a T = 0. \tag{46}$$

The systems of partial differential equations

$$D_c u^1 = e^{-T}, \quad D_d u^1 = D_{e_x} u^1 = D_b u^1 = D_a u^1 = 0$$

and

$$D_d u^2 = e^{-T}, \quad D_c u^2 = D_{e_x} u^2 = D_b u^2 = D_a u^2 = 0,$$

with unknown functions u^1, u^2, are completely integrable in view of (46). Choosing their solutions u^1, u^2 with initial values $u^1(p) = u^2(p) = 0$ and setting

$$u^x = -(n-2)\varepsilon_x E_x, \quad u^{n-1} = (n-2)\psi, \quad u^n = -(n-2)\xi, \tag{47}$$

we obtain a chart $u^1, \ldots, u^n$ in a neighbourhood of p, whose basic fields ∂_i are given by

$$\partial_1 = (n-2)e^T D_c\xi \cdot a - (n-2)e^T D_c\psi \cdot b + e^T c,$$
$$\partial_2 = (n-2)e^T D_d\xi \cdot a - (n-2)e^T D_d\psi \cdot b + e^T d,$$
$$\partial_x = e_x, \quad \partial_{n-1} = b, \quad \partial_n = a$$

(which can easily be verified with the aid of (32)-(34) and (24)). Clearly, $u^i(p) = 0$. Now (46) yields

$$\partial_1 T = -e^T\sigma, \quad \partial_2 T = e^T\lambda, \quad \partial_x T = \partial_{n-1}T = \partial_n T = 0, \tag{48}$$

and (45) is an immediate consequence of (35), (44) and (48). Finally, since $g_{ij} = g(\partial_i, \partial_j)$, formulae (41), (44), (47), (48), (11) and (12) imply (43), as desired.

(ii) The non-zero contravariant metric components are given by

$$g^{1n} = g^{2,n-1} = e^{-T}, \quad g^{xx} = \varepsilon_x,$$
$$g^{n-1,n-1} = -e^{-2T}g_{22}, \quad g^{n-1,n} = -e^{-2T}g_{12}, \quad g^{nn} = -e^{-2T}g_{11}.$$

Now we can compute the following connection components:

$$\Gamma^1_{11} = 2\partial_1 T, \quad \Gamma^1_{22} = -e^{-2T}, \quad \Gamma^2_{11} = -\varepsilon e^{-2T}, \quad \Gamma^2_{22} = 2\partial_2 T,$$

$$\Gamma^x_{12} = (n-2)^{-1} u^x e^{2T}, \quad \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^x_{11} = \Gamma^x_{22} = 0,$$

$$\Gamma^1_{ij} = \Gamma^2_{ij} = \Gamma^x_{ij} = 0 \quad \text{whenever } i > 2 \text{ or } j > 2,$$

$$\Gamma^{n-1}_{11} = 2u^n(\partial_1\partial_2 T + \varepsilon e^{-4T}) + u^{n-1}[\partial_1\partial_1 T - (\partial_1 T)^2 - \varepsilon e^{-2T}\partial_2 T],$$

$$\Gamma^{n-1}_{12} = -u^n e^{-2T}\partial_1 T - u^{n-1}(\partial_1 T\cdot\partial_2 T + \partial_1\partial_2 T),$$

$$\Gamma^{n-1}_{1x} = -(n-2)^{-1}e^T\varepsilon_x u^x, \quad \Gamma^{n-1}_{1,n-1} = \partial_1 T, \quad \Gamma^{n-1}_{1n} = 0,$$

$$\Gamma^{n-1}_{22} = -4u^n e^{-2T}\partial_2 T + u^{n-1}[3(\partial_2 T)^2 - \partial_2\partial_2 T + e^{-2T}\partial_1 T] - \\ -(n-2)^{-1}e^{-T}\sum_x \varepsilon_x (u^x)^2 - (n-2)F^{-1}e^{-T},$$

$$\Gamma^{n-1}_{2x} = 0, \quad \Gamma^{n-1}_{2,n-1} = -\partial_2 T, \quad \Gamma^{n-1}_{2n} = e^{-2T},$$

$$\Gamma^n_{11} = u^n[3(\partial_1 T)^2 - \partial_1\partial_1 T + e^{-2T}\varepsilon\partial_2 T] - 4\varepsilon u^{n-1}e^{-2T}\partial_1 T - \\ -(n-2)^{-1}\varepsilon e^{-T}\sum_x \varepsilon_x (u^x)^2 - (n-2)F^{-1}\varepsilon e^{-T},$$

$$\Gamma^n_{12} = -u^n(\partial_1\partial_2 T + \partial_1 T\cdot\partial_2 T) - \varepsilon u^{n-1}e^{-2T}\partial_2 T,$$

$$\Gamma^n_{1x} = 0, \quad \Gamma^n_{1,n-1} = \varepsilon e^{-2T}, \quad \Gamma^n_{1n} = -\partial_1 T, \quad \Gamma^n_{2x} = -(n-2)^{-1}e^T\varepsilon_x u^x,$$

$$\Gamma^n_{2,n-1} = 0, \quad \Gamma^n_{2n} = \partial_2 T, \quad \Gamma^{n-1}_{ij} = \Gamma^n_{ij} = 0 \quad \text{if } i, j > 2.$$

This implies

$$R_{121}{}^1 = -2\partial_1\partial_1 T - \varepsilon e^{-4T}, \quad R_{121}{}^2 = 0,$$

$$R_{121}{}^{n-1} = (u^{n-1}\partial_1 T + u^n\partial_2 T)(2\partial_1\partial_2 T + \varepsilon e^{-4T} + (n-2)^{-1}e^{2T}) - \\ -u^{n-1}\partial_1[2\partial_1\partial_2 T + \varepsilon e^{-4T} + (n-2)^{-1}e^{2T}] - u^n\partial_2[2\partial_1\partial_2 T + \varepsilon e^{-4T} + \\ +(n-2)^{-1}e^{2T}] - (n-2)^{-2}e^{3T}\sum_x \varepsilon_x (u^x)^2 + (n-2)^{-1}e^{2T}(u^{n-1}\partial_1 T + u^n\partial_2 T).$$

Using now (45), we infer that the only non-zero components of R_{hijk}, R_{ij}, $R_{ij,k}$ and C_{hijk} are related to

$$R_{1212} = 2u^n(n-2)^{-1}e^{3T}\partial_2 T + 2u^{n-1}(n-2)^{-1}e^{3T}\partial_1 T - \\ -2(n-2)^{-2}e^{4T}\sum_x \varepsilon_x (u^x)^2 - F^{-1}e^{4T},$$

$$R_{121n} = -R_{122,n-1} = (n-2)^{-1}e^{3T}, \quad R_{1x2x} = -(n-2)^{-1}\varepsilon_x e^{2T},$$

$$R_{12} = e^{2T}, \quad R_{11,1} = 2\varepsilon, \quad R_{22,2} = 2, \quad C_{1212} = F^{-1}e^{4T}.$$

It is easy to verify that $C_{hijk,l} = 0$. Our assertion is now immediate ((7) and (9) are satisfied by $a = \partial_n$, $b = \partial_{n-1}$). This completes the proof.

THEOREM 4. *Let M be a hyperbolic e.c.s. manifold with constant fundamental function. Then the following two conditions are equivalent:*

(i) *M is locally homogeneous.*

(ii) *M is regular and its fundamental invariant* (see Remark 2) *is constant.*

Proof. Assume (i). As the fundamental invariant S is determined by R_{ij} and $R_{ij,k}$ in an algebraic manner (see (7) and (9)), (ii) follows immediately from Proposition 1. Now let S be constant. In the notation of Theorem 3, (44) and (45) imply

$$S = e^{-3T} = (n-2)^{-1/2}, \quad \varepsilon = -1. \tag{49}$$

For $p, q \in M$ choose charts centered at p and q as in (i) of Theorem 3. The expressions for the metric so obtained are identical (the number of minuses among the ε_x is an algebraic invariant of the metric), which defines a local isometry sending p onto q. This completes the proof.

Remark 3. Theorems 3 and 4 imply the following local description of locally homogeneous hyperbolic e.c.s. manifolds. For any point p of such a manifold M, there exists a local coordinate system $u^1, \ldots, u^n$ in a neighbourhood of p such that $u^i(p) = 0, i = 1, \ldots, n$ $(n = \dim M \geqslant 4)$, and the only non-zero components of the metric are

$$g_{11} = -2u^{n-1}e^{-T}, \quad g_{12} = -(n-2)^{-1}e^{2T}\sum_x \varepsilon_x(u^x)^2 - (n-2)F^{-1}e^{2T},$$

$$g_{22} = 2u^n e^{-T}, \quad g_{1n} = g_{2,n-1} = e^T, \quad g_{xy} = \varepsilon_x\delta_{xy},$$

where $T = \frac{1}{6}\log(n-2)$, $|\varepsilon_x| = 1$ and $F \neq 0$ is the (constant) fundamental function of M.

Conversely, the metric so defined is locally homogeneous, e.c.s., hyperbolic, and its fundamental function equals F.

Remark 3 implies, in particular, that two locally homogeneous hyperbolic e.c.s. manifolds of equal dimensions, metric signatures and fundamental functions must be locally isometric to each other. Since a homothetic change of the metric changes the fundamental function in an obvious manner, the preceding statement allows us to deduce that two locally homogeneous hyperbolic e.c.s. manifolds of equal dimensions and signatures are locally homothetic.

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