FLAT MANIFOLDS AND REDUCIBILITY
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Abstract. Hiss and Szczepański proved in 1991 that the holonomy group of any compact flat Riemannian manifold, of dimension at least two, acts reducibly on the rational span of the Euclidean lattice associated with the manifold via the first Bieberbach theorem. Geometrically, their result states that such a manifold must admit a nonzero proper parallel distribution with compact leaves. We study algebraic and geometric properties of the sublattice-spanned holonomy-invariant subspaces that exist due to the above theorem, and of the resulting compact-leaf foliations of compact flat manifolds. The class consisting of the former subspaces, in addition to being closed under spans and intersections, also turns out to admit (usually nonorthogonal) complements. As for the latter foliations, we provide descriptions, first – and foremost – of the intrinsic geometry of their generic leaves in terms of that of the original flat manifold and, secondly – as an essentially obvious afterthought – of the leaf-space orbifold. The general conclusions are then illustrated by examples in the form of generalized Klein bottles.

1. Introduction

As shown by Hiss and Szczepański [7, the corollary in Sect. 1], on any compact flat Riemannian manifold $\mathcal{M}$ with $\dim \mathcal{M} = n \geq 2$ there exists a parallel distribution $D$ of dimension $k$, where $0 < k < n$, such that the leaves of $D$ are all compact. Their result, in its original algebraic phrasing (see the Appendix), stated that the holonomy group $H$ of $\mathcal{M}$ must act reducibly on $L \otimes \mathbb{Q}$, for the Euclidean lattice $L$ corresponding to $\mathcal{M}$ (which is a maximal Abelian subgroup of the fundamental group $\Pi$ of $\mathcal{M}$).

The present paper explores the algebraic context and geometric consequences of this fact. We view $L$ as an additive subgroup of a Euclidean vector space $\mathcal{V}$ (so that $L \otimes \mathbb{Q}$ becomes identified with the rational span of $L$ in $\mathcal{V}$), and use the term $L$-subspace when referring to a vector subspace of $\mathcal{V}$ spanned by some subset of $L$.

Hiss and Szczepański’s theorem amounts to the existence a nonzero proper $H$-invariant $L$-subspace $\mathcal{V}' \subseteq \mathcal{V}$. We begin by observing that the class of $H$-invariant $L$-subspaces of $\mathcal{V}$ is closed under the span and intersection operations applied to its arbitrary subclasses (Lemma 4.4), while every $H$-invariant $L$-subspace of $\mathcal{V}$ has an $H$-invariant $L$-subspace complementary to it (Theorem 4.8).

The Bieberbach group of a given compact flat Riemannian manifold $\mathcal{M}$ is its fundamental group $\Pi$ treated as the deck transformation group acting via affine isometries on
the Euclidean affine space $\mathcal{E}$ that constitutes the Riemannian universal covering space of $\mathcal{M}$. The space $\mathcal{V}$ mentioned above is associated with $\mathcal{E}$ by being its translation vector space, that is, the space of parallel vector fields on $\mathcal{E}$, and the roles of the lattice $L$ and holonomy group $H$ are summarized by the short exact sequence $L \to \Pi \to H$. See Section 6. We proceed to describe, in Sections 7 – 10, the constituents $L', \Pi', H'$ appearing in the analog $L' \to \Pi' \to H'$ of this short exact sequence for any (compact, flat) leaf $\mathcal{M}'$ of a parallel distribution $D$, guaranteed to exist on $\mathcal{M}$ by the aforementioned result of [7]. Specifically, by Theorem 7.1(ii), $\Pi'$ (or, $L'$) may be identified with a subgroup of $\Pi$ (or, $\mathcal{V}$), and $H'$ with a homomorphic image of a subgroup of $H$. This description becomes particularly simple for leaves $\mathcal{M}'$ which we call generic (Theorem 10.1): their union is an open dense subset of $\mathcal{M}$, they all have the same triple $L', \Pi', H'$, and are mutually isometric. When all leaves of $D$ happen to be generic, they form a locally trivial bundle with compact flat manifolds serving both as the base and the fibre (the fibration case).

Aside from the holonomy group $H'$ of each individual leaf $\mathcal{M}'$ of $D$, forming a part of its intrinsic (submanifold) geometry, $\mathcal{M}'$ also gives rise to two “extrinsic” holonomy groups, one arising since $\mathcal{M}'$ is a leaf of the foliation $F_{\mathcal{M}}$ of $\mathcal{M}$ tangent to $D$, the other coming from the normal connection of $\mathcal{M}'$. Due to flatness of the normal connection, the two extrinsic holonomy groups coincide, and are trivial for all generic leaves. In Section 11 we briefly discuss the leaf space $\mathcal{M}/F_{\mathcal{M}}$, pointing out that (not surprisingly!)

$$\mathcal{M}/F_{\mathcal{M}}$$ forms a flat compact orbifold, canonically identified with the quotient of the torus $[\mathcal{E}/\mathcal{V}]/[L \cap \mathcal{V}]$ under the isometric action of the finite group $H$.

In the fibration case (see above), $\mathcal{M}/F_{\mathcal{M}}$ is the base manifold of the bundle.

We illustrate the above conclusions by examples (generalized Klein bottles, Section 14), where both the fibration and non-fibration cases occur, depending on the choice of $D$.

Section 12 provides a formula for the intersection number of generic leaves of the foliations of the compact flat manifold $\mathcal{M}$ arising from two mutually complementary $H$-invariant $L$-subspaces of $\mathcal{V}$ (cf. Theorem 4.8, mentioned earlier).

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2. Preliminaries

Manifolds, mappings and tensor fields, such as bundle and covering projections, submanifold inclusions, and Riemannian metrics, are by definition of class $C^\infty$. Submanifolds need not carry the subset topology, and a manifold may be disconnected (although, being required to satisfy the second countability axiom, it must have at most countably many connected components). Connectedness/compactness of a submanifold always refer to its own topology, and imply the same for its underlying set within the ambient manifold.
Thus, a compact submanifold is always endowed with the subset topology. By a distribution on a manifold $\mathcal{N}$ we mean, as usual, a (smooth) vector subbundle $D$ of the tangent bundle $T\mathcal{N}$. An integral manifold of $D$ is any submanifold $\mathcal{L}$ of $\mathcal{N}$ with $T_x\mathcal{L} = D_x$ for all $x \in \mathcal{L}$. The maximal connected integral manifolds of $D$ will also be referred to as the leaves of $D$. In the case where $D$ is integrable, its leaves form the foliation associated with $D$. We call $D$ projectable under a mapping $\psi : \mathcal{N} \to \hat{\mathcal{N}}$ onto a distribution $\hat{D}$ on the target manifold $\hat{\mathcal{N}}$ if $d\psi_x(D_x) = \hat{D}_{\psi(x)}$ whenever $x \in \mathcal{N}$.

**Remark 2.1.** The following well-known facts will be used below.

(a) Free diffeomorphic actions of finite groups on manifolds are properly discontinuous and thus give rise to covering projections onto the resulting quotient manifolds.

(b) Any locally-diffeomorphic mapping from a compact manifold into a connected manifold is a (surjective) finite covering projection.

(c) More generally, the phrases ‘locally-diffeomorphic mapping’ and ‘finite covering projection’ in (b) may be replaced with submersion and fibration.

**Lemma 2.2.** Let a distribution $\hat{D}$ on $\hat{\mathcal{M}}$ be projectable, under a locally diffeomorphic surjective mapping $\psi : \hat{\mathcal{M}} \to \mathcal{M}$ between manifolds, onto a distribution $D$ on $\mathcal{M}$.

(i) The $\psi$-image of any leaf of $\hat{D}$ is a connected integral manifold of $D$.

(ii) Integrability of $\hat{D}$ implies that of $D$.

(iii) For any compact leaf $\mathcal{L}$ of $\hat{D}$, the image $\mathcal{L}' = \psi(\mathcal{L})$ is a compact leaf of $D$, and the restriction $\psi : \mathcal{L} \to \mathcal{L}'$ constitutes a covering projection.

(iv) If the leaves of $\hat{D}$ are all compact, so are those of $D$.

**Proof.** Assertion (i) is immediate from the definitions of a leaf and projectability, while (i) implies (ii) since integrability amounts to the existence of an integral manifold through every point. Remark 2.1(b) combined with (i) yields (iii), and (iv) follows.

**Lemma 2.3.** Suppose that $F$ is a mapping from a manifold $\mathcal{W}$ into any set. If for every $x \in \mathcal{W}$ there exists a diffeomorphic identification of a neighborhood $B_x$ of $x$ in $\mathcal{W}$ with a unit open Euclidean ball centered at $0$ under which $x$ corresponds to $0$ and $F$ becomes constant on each open straight-line interval of length $1$ in the open ball having $0$ as an endpoint, then $F$ is locally constant on some open dense subset of $\mathcal{W}$.

**Proof.** We use induction on $n = \dim \mathcal{W}$. The case $n = 1$ being trivial, let us assume the assertion to be valid in dimension $n - 1$ and consider a function $F$ on an $n$-dimensional manifold $\mathcal{W}$, satisfying our hypothesis, along with an embedded open Euclidean ball $B_x \subseteq \mathcal{M}$ “centered” at a given point $x$, as in the statement of the lemma. Due to constancy of $F$ along the fibres of the normalization projection $\mu : B_x \setminus \{x\} \to \mathcal{S}$ onto the unit $(n - 1)$-sphere $\mathcal{S}$, we may view $F$ as a mapping $G$ having the domain $\mathcal{S}$. We may now fix $y \in B_x \setminus \{x\}$ with an embedded open Euclidean ball $B_y$ “centered” at


y, such that $F$ is constant on each radial open interval in $B_y$. The obvious submersion property of $\mu$ allows us to pass from $B_y$ to a smaller concentric ball and then choose a codimension-one open Euclidean ball $B'_y$ arising as a union of radial intervals within this smaller version of $B_y$, for which $\mu : B'_y \to S$ is an embedding. The assumption of the lemma thus holds when $W$ and $F$ are replaced by $S$ and $G$, leading to local constancy of $G$ (and $F$) on a dense open set in $S$ (and, respectively, in $B_x \setminus \{x\}$). Since the union of the latter sets over all $x$ is obviously dense in $W$, our claim follows. □

Remark 2.4. As a well-known consequence of the inverse mapping theorem combined with the Gauss lemma for submanifolds, given a compact submanifold $M'$ of a Riemannian manifold $M$ there exists $\delta \in (0, \infty)$ with the following properties.

(a) The normal exponential mapping restricted to the radius $\delta$ open-disk subbundle $N_\delta$ of the normal bundle of $M'$ constitutes a diffeomorphism $\text{Exp}^\perp : N_\delta \to M_\delta$ onto the open submanifold $M_\delta$ of $M$ equal to the preimage of $[0, \delta)$ under the function $\text{dist}(M', \cdot)$ of metric distance from $M'$.

(b) Every $x \in M_\delta$ has a unique point $y \in M'$ nearest to $x$, which is simultaneously the unique point $y$ of $M'$ joined to $x$ by a geodesic in $M_\delta$ normal to $M'$ at $y$, and the resulting assignment $M_\delta \ni x \mapsto y \in M'$ coincides with the composite mapping of the inverse diffeomorphism of $\text{Exp}^\perp : N_\delta \to M_\delta$ followed by the normal-bundle projection $N_\delta \to M'$.

(c) The $\text{Exp}^\perp$ images of length $\delta$ radial line segments emanating from the zero vectors in the fibres of $N_\delta$ coincide with the length $\delta$ minimizing geodesic segments in $M_\delta$ emanating from $M'$. They are all normal to the levels of $\text{dist}(M', \cdot)$, and realize the minimum distance between any two such levels within $M_\delta$.

Lemma 2.5. In a complete metric space, any countable union of closed sets with empty interiors has an empty interior.

Proof. This is Baire’s theorem [6, p. 187] stating, equivalently, that the intersection of countably many dense open subsets is dense. □

3. Free Abelian groups

The following well-known facts, cf. [1], are gathered here for easy reference.

For a finitely generated Abelian group $G$, being torsion-free amounts to being free, in the sense of having a $\mathbb{Z}$-basis, by which one means an ordered $n$-tuple $e_1, \ldots, e_n$ of elements of $G$ such that every $x \in G$ can be uniquely expressed as an integer combination of $e_1, \ldots, e_n$. The integer $n \geq 0$, also denoted by $\dim_\mathbb{Z}G$, is an algebraic invariant of $G$, called its Betti number or $\mathbb{Z}$-dimension.

Any finitely generated Abelian group $G$ is isomorphic to the direct sum of its (necessarily finite) torsion subgroup $S$ and the free group $G/S$, and we set $\dim_\mathbb{Z}G = \dim_\mathbb{Z}[G/S]$. 
A subgroup $G'$ (or, a homomorphic image $G'$) of such $G$, in addition to being again finitely generated and Abelian, also satisfies the inequality $\dim_{\mathbb{Z}} G' \leq \dim_{\mathbb{Z}} G$, strict unless $G/G'$ is finite (or, respectively, the homomorphism involved has a finite kernel).

**Lemma 3.1.** A subgroup $G'$ of finitely generated a free Abelian group $G$ constitutes a direct summand of $G$ if and only if the quotient group $G/G'$ is torsion-free.

In fact, more generally, given a surjective homomorphism $\chi : P \to P'$ between Abelian groups $P, P'$ and elements $x_j, y_a$ (with $j, a$ ranging over finite sets), such that $x_j$ and $\chi(y_a)$ form $\mathbb{Z}$-bases of $\ker \chi$ and, respectively, of $P'$, the system consisting of all $x_j$ and $y_a$ is a $\mathbb{Z}$-basis of $P$. To see this, note that every element of $P$ then can be uniquely expressed as an integer combination of $x_j$ and $y_a$.

**Lemma 3.2.** For any finitely generated subgroup $G$ of the additive group of a finite-dimensional real vector space $V$, the intersection $G \cap V'$ with any vector subspace $V' \subseteq V$ forms a direct-summand subgroup of $G$. On the other hand, the class of direct-summand subgroups of $G$ is closed under intersections, finite or not.

Both claims are obvious from Lemma 3.1. The next lemma is a straightforward exercise:

**Lemma 3.3.** If normal subgroups $G', G''$ of a group $G$ intersect trivially and every $\gamma' \in G'$ commutes with every $\gamma'' \in G''$, then $G'G'' = \{ \gamma'\gamma'': (\gamma', \gamma'') \in G' \times G'' \}$ is a normal subgroup of $G$, and the assignment $(\gamma', \gamma'') \mapsto \gamma'\gamma''$ defines an isomorphism $G' \times G'' \to G'G''$.

4. Lattices and vector subspaces

Throughout this section $V$ denotes a fixed finite-dimensional real vector space, and

\[(4.1)\quad \text{we call subspaces } V', V'' \text{ of } V \text{ complementary to each other if } V = V' \oplus V''.\]

As usual, we define a (full) lattice in $V$ to be any subgroup $L$ of the additive group of $V$ generated by a basis of $V$ (which must consequently also be a $\mathbb{Z}$-basis of $L$). The quotient Lie group $V/L$ then is a torus, and we use the term subtori when referring to its compact connected Lie subgroups. Projectability of distributions under overing projections is generally equivalent to their deck-transformation invariance; this obvious fact, applied to the projection $V \to V/L$, shows that

\[(4.2)\quad \text{every parallel distribution on } V \text{ is projectable onto the torus } V/L.\]

**Definition 4.1.** Given a lattice $L$ in $V$, by an $L$-subspace of $V$ we will mean any vector subspace $V'$ of $V$ spanned by $L \cap V'$. One may equivalently require $V'$ to be the span of just a subset of $L$, rather than specifically of $L \cap V'$.

**Lemma 4.2.** The parallel distribution on $V$ tangent to any prescribed vector subspace $V'$ projects onto a parallel distribution $D$ on the torus group $V/L$. The leaves of $D$ must be either
all compact, or all noncompact, and they are compact if and only if $\mathcal{V}'$ is an $L$-subspace, in which case the leaf of $D$ through zero is a subtorus of $\mathcal{V}/L$.

**Proof.** Projectability is obvious from (4.2). The first claim about the leaves of $D$ follows as the leaves are one another’s translation images. For the second, let $\mathcal{N}$ be the leaf of $D$ through zero. Requiring $\mathcal{V}'$ to be (or, not to be) an $L$-subspace makes $L \cap \mathcal{V}'$, by Lemma 3.2, a direct-summand subgroup of $L$ spanning $\mathcal{V}'$ or, respectively, yields the existence of a nonzero linear functional $f$ on $\mathcal{V}'$, the kernel of which contains $L \cap \mathcal{V}'$. In the former case, $\mathcal{N}$ is a factor of a product-of-tori decomposition of $\mathcal{V}/L$, while in the latter $f$ descends to an unbounded function on $\mathcal{N}$. □

**Example 4.3.** For a finite group $H$, let $\mathcal{V} = \mathbb{R}^H$ and $L = \mathbb{Z}^H$, with the convention that, whenever $X, Y$ are sets, $Y^X$ is the set of all mappings $X \to Y$. Clearly, $\mathcal{V} \cong \mathbb{R}^n$ and $L \cong \mathbb{Z}^n$ if $H$ has $n$ elements, the isomorphic identifications $\cong$ coming from a fixed bijection $\{1, \ldots, n\} \to H$, and so $L$ constitutes a lattice in the real vector space $\mathcal{V}$. Denoting by $\tau_a : H \to H$ the left translation by $a \in H$, we define a right action $\mathcal{V} \times H \ni (f, a) \mapsto f \circ \tau_a \in \mathcal{V}$ of the group $H$ on $\mathcal{V}$. This action – obviously effective – turns $H$ into a finite group of linear automorphisms of $\mathcal{V}$, preserving both $L$ and the $\ell^2$ inner product. Any $d$-element subgroup $\tilde{H}$ of $H$ gives rise to two $H$-invariant $L$-subspaces $\mathcal{V}', \mathcal{V}'' \subseteq \mathcal{V}$, of dimensions $n/d$ and $n - n/d$, with $\mathcal{V}'$ consisting of all $f \in \mathcal{V}$ constant on each left coset $a\tilde{H}$, $a \in H$, and $\mathcal{V}''$, the $\ell^2$-orthogonal complement of $\mathcal{V}'$, formed by those $f \in \mathcal{V}$ having the sum of values over every coset $a\tilde{H}$ equal to zero. Thus, for the action of $\tilde{H}$ on $H$ via right translations, $\mathcal{V}'$ (or $\mathcal{V}''$) is the space of vectors in $\mathcal{V}$ that are $\tilde{H}$-invariant (or, have zero $\tilde{H}$-average). Both $\mathcal{V}', \mathcal{V}''$ are $L$-subspaces: a subset of $L$ spanning $\mathcal{V}'$ (or, $\mathcal{V}''$ when $d > 1$ and $\mathcal{V}'' \neq \{0\}$) consists of functions equal to 1 on one coset and to 0 on the others (or, respectively, of functions assuming the values 1 and $-1$ at two fixed points within the same coset, and vanishing everywhere else).

**Lemma 4.4.** Given a lattice $L$ in $\mathcal{V}$, the span and intersection of any family of $L$-subspaces are $L$-subspaces. The same is true if one replaces the phrase ‘$L$-subspaces’ with ‘$H$-invariant $L$-subspaces’ for any fixed group $H$ of linear automorphisms of $\mathcal{V}$ sending $L$ into itself.

**Proof.** The assertion about spans follows from the case of two $L$-subspaces, obvious in turn due to the second sentence of Definition 4.1. Next, the intersection of the family of subtori in $\mathcal{V}/L$, arising via Lemma 4.2 from the given family of $L$-subspaces, constitutes a compact Lie subgroup of $\mathcal{V}/L$, so that it is the union of finitely many cosets of a subtorus $\mathcal{N}$. Since subtori are totally geodesic relative to the flat affine connection on $\mathcal{V}/L$, while the projection $\mathcal{V} \to \mathcal{V}/L$ locally diffeomorphic, the tangent space of $\mathcal{N}$ at zero equals the intersection of the tangent spaces of the subtori forming the family, and each tangent space corresponds to an $L$-subspace from our family. The conclusion is now immediate from Lemma 4.2. □
Remark 4.5. For a lattice $L$ in $V$ generated by a basis $e_1, \ldots, e_n$ of $V$, the translational action of $L$ on $V$ has an obvious compact fundamental domain (a compact subset of $V$ intersecting all orbits of $L$): the parallelepiped $\{t_1 e_1 + \ldots + t_n e_n : t_1, \ldots, t_n \in [0,1]\}$.

Remark 4.6. We need the well-known fact [3] that
(a) lattices in $V$ are the same as discrete subgroups of $V$, spanning $V$.
(b) $L'$ is a lattice in the vector subspace spanned by it, and
(c) $L'$ constitutes a direct-summand subgroup of $L$.

as one sees using (a) and the first part of Lemma 3.2.

Lemma 4.7. Let $W$ be the rational vector subspace of a finite-dimensional real vector space $V$, spanned by a fixed lattice $L$ in $V$. The four sets formed, respectively, by

(i) $L$-subspaces $V'$ of $V$,
(ii) direct-summand subgroups $L'$ of $L$,
(iii) rational vector subspaces $W'$ of $W$,
(iv) subtori $N'$ of the torus group $V/L$, that is, its compact connected Lie subgroups,

then stand in mutually consistent, natural bijective correspondences with one another, obtained by declaring $V'$ to be the real span of both $L'$ and $W'$ as well as the identity component of the pre-image of $N'$ under the projection homomorphism $V \to V/L$. Furthermore, $W'$ equals $W \cap V'$ and, simultaneously, is the rational span of $L'$, while $N' = V'/L'$ and $L' = L \cap V' = L \cap W'$.

Finally, $\dim_{\mathbb{R}} V' = \dim_{\mathbb{Z}} L' = \dim_{\mathbb{Q}} W' = \dim N'$.

‘Mutual consistency’ means here that the above finite set of bijections is closed under the operations of composition and inverse.

Proof. The mappings (ii) $\to$ (i) and (iii) $\to$ (i), as well as (iv) $\to$ (i), defined in the three lines following (iv), are all bijections, with the inverses given by $(L', W', N') = (L \cap V', W \cap V', V'/L')$. Namely, each of the three mappings and their purported inverses takes values in the correct set, and each of the six mapping-inverse compositions is the respective identity. To be specific, the claim about the values follows from Lemma 4.2 for (iv) $\to$ (i) and (i) $\to$ (iv), from Definition 4.1 and Lemma 3.2 for (ii) $\to$ (i) and (i) $\to$ (ii), while it is obvious for (i) $\to$ (iii) and, for (iii) $\to$ (i), immediate from Definition 4.1, since we are free to assume that

$$ (L, W, V) = (\mathbb{Z}^n, \mathbb{Q}^n, \mathbb{R}^n), \quad \text{where } n = \dim V, $$

and every rational vector subspace of $\mathbb{Q}^n$ has a basis contained in $\mathbb{Z}^n$. Next, the compositions (ii) $\to$ (i) $\to$ (ii) and (i) $\to$ (ii) $\to$ (i) are the identity mappings – the former due to the fact that $L \cap \text{span}_{\mathbb{R}} L' \subseteq L'$ (which one sees extending a $\mathbb{Z}$-basis of $L'$ to a $\mathbb{Z}$-basis of $L$) – the opposite inclusion being obvious, the latter, as Definition 4.1 gives $V' = \text{span}_{\mathbb{R}} (L \cap V')$. Similarly for (iii) $\to$ (i) $\to$ (iii) and (i) $\to$ (iii) $\to$ (i), as long as
one replaces the letters $L$ and $\mathbb{Z}$ with $W$ and $Q$, using (4.3) and the line following it. Finally, (iv) $\rightarrow$ (i) $\rightarrow$ (iv) and (i) $\rightarrow$ (iv) $\rightarrow$ (i) are the identity mappings as a consequence of Lemma 4.2, and the dimension equalities become obvious if one, again, chooses a $\mathbb{Z}$-basis of $L$ containing a $\mathbb{Z}$-basis of $L'$.

In the next theorem, as $H$ is finite, the $L$-preserving property of $H$ means that, whenever $A \in H$, one has $\det A = \pm 1$, and so $AL = L$ (rather than just $AL \subseteq L$).

**Theorem 4.8.** For a lattice $L$ in a finite-dimensional real vector space $V$, a finite group $H$ of $L$-preserving linear automorphisms of $V$, and an $H$-invariant $L$-subspace $V'$ of $V$, there exists an $H$-invariant $L$-subspace $V''$ of $V$, complementary to $V'$ in the sense of (4.1).

**Proof.** Let $W' = W \cap V'$, for the rational span $W$ of $L$ (see Lemma 4.7). Restricted to $W$, elements of $H$ act by conjugation on the rational affine space $P$ of all $Q$-linear projections $W \rightarrow W'$ (by which we mean linear operators $W \rightarrow W'$ equal to the identity on $W'$). The average of any orbit of the action of $H$ on $P$ is an $H$-invariant projection $W \rightarrow W'$ with a kernel $W''$ corresponding via Lemma 4.7 to our required $V''$.

**Corollary 4.9.** If $L, V, H$ satisfy the hypotheses of Theorem 4.8, every nonzero $H$-invariant $L$-subspace $V'_0$ of $V$ can be decomposed into a direct sum of one or more nonzero $H$-invariant $L$-subspaces, each of which is minimal in the sense of not containing any further nonzero proper $H$-invariant $L$-subspace.

**Proof.** Induction on the possible values of $\dim V'_0$. Assuming the claim true for subspaces of dimensions less than $\dim V'_0$, along with non-minimality of $V'_0$, we fix a nonzero proper $H$-invariant $L$-subspace $V'$ of $V$, contained in $V'_0$, and choose a complement $V''$ of $V'$, guaranteed to exist by Theorem 4.8. Since $V''$ intersects every coset of $V'$ in $V$, including cosets within $V'_0$, the subspace $V'_0 \cap V''$ is an $H$-invariant complement of $V'$ in $V'_0$, as well as an $L$-subspace (due to Lemma 4.4). We may now apply the induction assumption to both $V'$ and $V'_0 \cap V''$.

**Remark 4.10.** Given a lattice $L$ in a finite-dimensional real vector space $V$ and an $L$-subspace $V'$ of $V$, the restriction to $L$ of the quotient-space projection $V \rightarrow V/V'$ has the kernel $L' = L \cap V'$, and so it descends to an injective group homomorphism $L/L' \rightarrow V/V'$, the image of which is a (full) lattice in an $V/V'$ (which follows if one uses a $\mathbb{Z}$-basis of $L$ containing a $\mathbb{Z}$-basis of $L'$). From now on we will treat $L/L'$ as a subset of $V/V'$.

Discreteness of the lattice $L/L' \subseteq V/V'$ clearly implies the existence of an open subset $U'$ of $V$, containing $V'$ and forming a union of cosets of $V'$, such that $L \cap U' = L'$.

5. **Affine spaces**

We denote by $\text{End} \ V$ the space of linear endomorphisms of a given real vector space $V$. Scalars stand for the corresponding multiples of identity, so that the identity itself
becomes \( 1 \in \text{End} \mathcal{V} \). For a finite-dimensional real affine space \( \mathcal{E} \) with the translation vector space \( \mathcal{V} \), let \( \text{Aff} \mathcal{E} \) be the group of all affine transformations of \( \mathcal{E} \). The inclusion \( \mathcal{V} \subseteq \text{Aff} \mathcal{E} \) expresses the fact that \( \text{Aff} \mathcal{E} \) contains the normal subgroup consisting of all translations. Any vector subspace \( \mathcal{V}' \) of \( \mathcal{V} \) gives rise to a foliation of \( \mathcal{E} \), with the leaves formed by affine subspaces \( \mathcal{E}' \) parallel to \( \mathcal{V}' \), that is, orbits of the translational action of \( \mathcal{V}' \) on \( \mathcal{E} \) (which we may also refer to as cosets of \( \mathcal{V}' \) in \( \mathcal{E} \)). The resulting leaf (quotient) space \( \mathcal{E}/\mathcal{V}' \) constitutes an affine space having the translation vector space \( \mathcal{V}/\mathcal{V}' \). Clearly,

\[ (5.1) \quad \text{for cosets } \mathcal{E}', \mathcal{E}'' \text{ of subspaces } \mathcal{V}', \mathcal{V}'' \subseteq \mathcal{V} \text{ with } (4.1), \mathcal{E}' \cap \mathcal{E}'' \text{ is a one-point set.} \]

A fixed inner product in \( \mathcal{V} \) turns \( \mathcal{E} \) into a Euclidean affine space, with the isometry group \( \text{Iso} \mathcal{E} \subseteq \text{Aff} \mathcal{E} \). If \( \delta \in (0, \infty) \), we define the \( \delta \)-neighborhood of an affine subspace \( \mathcal{E}' \) of \( \mathcal{E} \) to be the set of points in \( \mathcal{E} \) lying at distances less than \( \delta \) from \( \mathcal{E}' \). Clearly, the \( \delta \)-neighborhood of \( \mathcal{E}' \) is a union of cosets of a vector subspace \( \mathcal{V}' \) of \( \mathcal{V} \) (one of them being \( \mathcal{E}' \) itself), as well as the preimage, under the projection \( \mathcal{E} \rightarrow \mathcal{E}/\mathcal{V}' \), of the radius \( \delta \) open ball centered at the point \( \mathcal{E}' \) in the quotient Euclidean affine space \( \mathcal{E}/\mathcal{V}' \) (for the obvious inner product on \( \mathcal{V}/\mathcal{V}' \)).

**Remark 5.1.** Given a Euclidean affine space \( \mathcal{E} \) and an affine subspace \( \mathcal{E}' \) parallel to a vector subspace \( \mathcal{V}' \) of the translation vector space \( \mathcal{V} \) of \( \mathcal{E} \), (affine) self-isometries \( \zeta \) of \( \mathcal{E} \) such that \( \zeta(x) = x \) for all \( x \in \mathcal{E}' \) are in an obvious one-to-one correspondence with linear self-isometries \( A \) of the orthogonal complement of \( \mathcal{V}' \). In this case we will refer to \( \zeta \) as an affine extension of \( A \), depending on \( \mathcal{E}' \).

**Remark 5.2.** Any choice of an origin \( o \in \mathcal{E} \) in an affine space \( \mathcal{E} \) leads to the obvious identification of \( \mathcal{E} \) with its translation vector space \( \mathcal{V} \), under which a vector \( v \in \mathcal{V} \) corresponds to the point \( x = o + v \in \mathcal{E} \). Affine mappings \( \gamma \in \text{Aff} \mathcal{E} \) are then represented by pairs \( (A, b) \) consisting of \( A \in \text{End} \mathcal{V} \) and \( b \in \mathcal{E} \), so that \( \gamma(o + v) = o + Av + b \). The pair associated in this way with \( \gamma \) and a new origin \( o + w \) is, obviously, \( (A, c) \), for the same \( A \) (the linear part of \( \gamma \)) and \( c = b + (A - 1)w \). Thus, the coset \( b + \mathcal{V} \subseteq \mathcal{V} \), where \( \hat{\mathcal{V}} \) denotes the image of \( A - 1 \), forms an invariant of \( \gamma \) (while \( b \) itself does not, except in the case of translations \( \gamma \), having \( A = 1 \)). For any fixed vector subspace \( \mathcal{V}' \) of \( \mathcal{V} \) and any \( \gamma \in \text{Aff} \mathcal{E} \) with a linear part \( A \) leaving \( \mathcal{V}' \) invariant, it now makes sense to require that \( A \) descend to the identity transformation of \( \mathcal{V}/\mathcal{V}' \) (i.e., \( (A - 1)(\mathcal{V}) \subseteq \mathcal{V}' \)) and, simultaneously, that the “translational part” \( b \) of \( \gamma \) lie in \( \mathcal{V}' \). More precisely, such a property of \( \gamma \) does not depend on the origin used to represent \( \gamma \) as a pair \( (A, b) \).

**Remark 5.3.** Given \( \mathcal{E}, \mathcal{V} \) and \( \mathcal{V}' \) as in Remark 5.2, the affine transformations \( \gamma \) of \( \mathcal{E} \) with linear parts leaving \( \mathcal{V}' \) invariant and descending to the identity transformation of \( \mathcal{V}/\mathcal{V}' \) obviously form a subgroup of \( \text{Aff} \mathcal{E} \) containing, as a normal subgroup, the set of such \( \gamma \) which have “translational parts” in \( \mathcal{V}' \). This follows since the latter set is the kernel of the obvious homomorphism from the original subgroup into \( \mathcal{V}/\mathcal{V}' \subseteq \text{Aff} [\mathcal{E}/\mathcal{V}'] \).
More precisely, $\gamma$ represented by the pair $(A, b)$ (see Remark 5.2) preserves each element of $E/V'$ if and only if $Av + b$ differs from $v$, for every $v \in V$, by an element $V'$ or, equivalently (as one sees setting $v = 0$), $V'$ contains both $b$ and the image of $A - 1$.

**Lemma 5.4.** Remark 2.4 has the following additional conclusions when $\mathcal{M}'$ is a compact leaf of a parallel distribution $D$ on a complete flat Riemannian manifold $\mathcal{M}$.

(a) Every level of $\text{dist}(\mathcal{M}', \cdot)$ in $\mathcal{M}_\delta$, and $\mathcal{M}_\delta$ itself, is a union of leaves of $D$.

(b) Restrictions of $\mathcal{M}_\delta \ni x \mapsto y \in \mathcal{M}'$ to leaves of $D$ in $\mathcal{M}_\delta$ are locally isometric.

(c) The local inverses of all the above locally-isometric restrictions correspond via the diffeomorphism $\text{Exp}^{-1}$ to all local sections of the normal bundle of $\mathcal{M}'$ obtained by restricting to $\mathcal{M}'$ local parallel vector fields of lengths $r \in [0, \delta)$ that are tangent to $\mathcal{M}$ and normal to $\mathcal{M}'$, with $r$ equal to the value of $\text{dist}(\mathcal{M}', \cdot)$ on the leaf.

This trivially follows from the fact the pullback of $D$ to the Euclidean affine space $\mathcal{E}$ constituting the Riemannian universal covering space of $\mathcal{M}$ is a distribution with the leaves provided by affine subspaces parallel to $V'$, for some vector subspace $V'$ of the translation vector space $V$ of $\mathcal{E}$.

**6. Bieberbach groups and flat manifolds**

Let $\mathcal{E}$ be a Euclidean affine $n$-space (Section 5), with the translation vector space $V$. By a Bieberbach group in $\mathcal{E}$ one means any torsion-free discrete subgroup $\Pi$ of $\text{Iso} \mathcal{E}$ for which there exists a compact fundamental domain (Remark 4.5). The lattice subgroup $L$ of $\Pi$, and its holonomy group $H \subseteq \text{Iso} V \cong O(n)$ then are defined by

$$L = \Pi \cap V, \quad H = \alpha(\Pi),$$

$\alpha : \text{Aff} \mathcal{E} \to \text{Aut} V \cong \text{GL}(n, \mathbb{R})$ being the linear-part homomorphism. Thus, $L$ is the set of all translations lying in $\Pi$ (which also makes it the kernel of the restriction $\alpha : \Pi \to H$), and $H$ consists of the linear parts of elements of $\Pi$. Note that $L \subseteq V$ is a (full) lattice in the usual sense [3], cf. Section 4. The relations involving $\Pi, L$ and $H$ are conveniently summarized by the short exact sequence

$$L \to \Pi \to H,$$

where the arrows are the inclusion homomorphism and $\alpha$.

**Remark 6.1.** The action of a Bieberbach group $\Pi$ on the Euclidean affine space $\mathcal{E}$ being always free and properly discontinuous, the quotient $\mathcal{M} = \mathcal{E}/\Pi$, with the projected metric, forms a compact flat Riemannian manifold, while $H$ must be finite [3].

**Remark 6.2.** As the normal subgroup $L$ of $\Pi$ is Abelian, the action of $\Pi$ on $L$ by conjugation descends to an action on $L$ of the quotient group $\Pi/L$, identified via (6.2) with $H$. This last action is clearly nothing else than the ordinary linear action of $H$ on $V$, restricted to the lattice $L \subseteq V$ (and so, in particular, $L$ must be $H$-invariant).
**Remark 6.3.** The assignment of \( M = \mathcal{E}/\Pi \) to \( \Pi \) establishes a well-known bijective correspondence [3] between equivalence classes of Bieberbach groups and isometry types of compact flat Riemannian manifolds. Bieberbach groups \( \Pi \) and \( \tilde{\Pi} \) in Euclidean affine spaces \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) are called *equivalent* here if some affine isometry \( \mathcal{E} \to \tilde{\mathcal{E}} \) conjugates \( \Pi \) onto \( \tilde{\Pi} \). Furthermore, \( \Pi \) and \( H \) in (6.2) serve as the fundamental and holonomy groups of \( M \), while \( \Pi \) also acts via deck transformations on the Riemannian universal covering space of \( M \), isometrically identified with \( \mathcal{E} \).

### 7. Lattice-reducibility

A Bieberbach group \( \Pi \) in a Euclidean affine space \( \mathcal{E} \) (or, the compact flat Riemannian manifold \( M = \mathcal{E}/\Pi \) corresponding to \( \Pi \), cf. Remark 6.3) will be called *lattice-reducible* if, for \( V, H \) and \( L \) associated with \( \mathcal{E} \) and \( \Pi \) as in Section 6, there exists \( V' \) such that

\[(7.1) \quad V' \text{ is a nonzero proper } H\text{-invariant } L\text{-subspace of } V.\]

(See Definition 4.1.) To emphasize the role of \( V' \) in (7.1), we also say that

\[(7.2) \quad \text{the lattice-reducibility condition (7.1) holds for the quadruple } (V, H, L, V').\]

As shown by Hiss and Szczepański [7], *every compact flat Riemannian manifold of dimension greater than one is lattice-reducible*. For more details, see the Appendix.

For a Bieberbach group \( \Pi \) in a Euclidean affine space \( \mathcal{E} \) and a fixed affine subspace \( \mathcal{E}' \) of \( \mathcal{E} \) parallel to a vector subspace \( V' \) of \( V \) satisfying (7.2), we denote by \( \Sigma' \) the *stabilizer subgroup of \( \mathcal{E}' \) relative to the action of \( \Pi \)*, meaning that

\[(7.3) \quad \Sigma' \text{ consists of all the elements of } \Pi \text{ mapping } \mathcal{E}' \text{ onto itself.}\]

Let \( \gamma \in \Pi \). From \( \Pi \)-invariance, cf. (8.1), of the foliation of \( \mathcal{E} \) formed by the cosets of \( V' \),

\[(7.4) \quad \gamma \in \Sigma' \text{ if and only if } \gamma(\mathcal{E}') \text{ intersects } \mathcal{E}'.\]

**Theorem 7.1.** Given a lattice-reducible Bieberbach group \( \Pi \) in a Euclidean affine space \( \mathcal{E} \) and a vector subspace \( V' \) of \( V \) with (7.1), the following three conclusions hold.

(i) **The affine subspaces of dimension** \( \dim V' \) **in** \( \mathcal{E} \), **parallel to** \( V' \), **are the leaves of a foliation** \( F_{\mathcal{E}} \) **on** \( \mathcal{E} \), **projectable under the covering projections** \( \text{pr} : \mathcal{E} \to M = \mathcal{E}/\Pi \) **and** \( \mathcal{E} \to T = \mathcal{E}/L \) **onto foliations** \( F_{M} \) **of** \( M \) **and** \( F_{T} \) **of the torus** \( T = \mathcal{E}/L \), **both of which have compact totally geodesic leaves, tangent to a parallel distribution.**

(ii) **The leaves** \( \mathcal{M}' \) **of** \( F_{M} \) **coincide with the** \( \text{pr} \)-**images of leaves** \( \mathcal{E}' \) **of** \( F_{\mathcal{E}} \), **and the restrictions** \( \text{pr} : \mathcal{E}' \to \mathcal{M}' \) **are covering projections.** **The same remains true if one replaces** \( M \) **and** \( \Pi \) **with** \( T \) **and the projection** \( \mathcal{E} \to T \). **Any such** \( \mathcal{M}' \), **being a compact flat Riemannian manifold, corresponds via Remark 6.3 to a Bieberbach group** \( \Pi' \) **in the Euclidean affine space** \( \mathcal{E}' \). **For** \( L', H' \) **appearing in the analog** \( L' \to \Pi' \to H' \) **of (6.2), this** \( \Pi' \), **and** \( \Sigma' \) **defined by (7.3),**

(a) **\( \Pi' \) consists of the restrictions to** \( \mathcal{E}' \) **of all the elements of** \( \Sigma' \).
(b) \( H' \) is formed by the restrictions to \( V' \) of the linear parts of elements of \( \Sigma' \).
(c) \( L' = \Pi' \cap V' \), as in (6.1), and \( L \cap V' \subseteq L' \).

We prove Theorem 7.1 in the next section.

**Remark 7.2.** The restriction homomorphism \( \Sigma' \to \Pi' \), cf. (ii-a) above, is an isomorphism: nontrivial elements of \( \Sigma' \), being fixed-point free (Remark 6.1), have nontrivial restrictions to \( E' \). The last inclusion of (ii-c) may be proper; see the end of Section 14.

8. **Proof of Theorem 7.1**

Projectablity of the foliation \( F_E \) under both covering projections \( \text{pr} : E \to M \) and \( E \to T \) follows as a trivial consequence of the fact that, due to \( H \)-invariance of \( V' \),

\[ F_E \text{ is } \Pi \text{-invariant and, obviously, } L \text{-invariant}, \]

while Lemma 2.2(ii) implies integrability of the image distribution. Next,

\[ \text{pr is the composite } E \to T \to M \text{ of two mappings: the} \]

universal-covering projection of the flat torus \( T = E/L \),
and the quotient projection for the action of \( \Pi \) on \( T \),

the latter action clearly becoming free if one replaces \( \Pi \) with \( \Pi/L \cong H \). Both factor mappings, \( E \to T \) and \( T \to M \), are covering projections – the first since \( L \) is a lattice in \( V \), the second due to Remark 2.1(a). Parts (iii)–(iv) of Lemma 2.2, along with Lemma 4.2, may now be applied to the foliations \( F_T \) and \( F_M \) of the torus \( T \) and of \( M \) obtained as projections of \( F_E \), proving the last (compact-leaves) claim of (i).

We now fix a leaf \( E' \) of \( F_E \), and choose a leaf \( M' \) of \( F_M \) containing \( \text{pr}(E') \), cf. Lemma 2.2(i). It follows that

\[ \text{pr : } E' \to M' \text{ is a (surjective) covering projection,} \]

since (8.2) decomposes \( \text{pr : } E' \to M' \) into the composition \( E' \to T' \to M' \), in which the first mapping is the universal-covering projection of the torus \( T' = E'/L' \), and the second one must be a covering due to Remark 2.1(b).

Two points of \( E' \) have the same \( \text{pr} \)-image if and only if one is transformed into the other by an element of the group \( \Pi' \) described in assertion (ii). (Namely, the ‘only if’ part follows since, given \( x, y \in E' \) with \( \text{pr}(x) = \text{pr}(y) \) in \( M = E/\Pi \), the element of \( \Pi \) sending \( x \) to \( y \) must lie in \( \Pi' \), or else, by (8.1), it would map \( E' \) onto a different leaf of the foliation \( F_E \).) Furthermore, the action of \( \Pi' \) on \( E' \) is free due to Remark 6.1. Thus, \( \Pi' \) coincides with the deck transformation group for the universal covering projection (8.3). Now (ii) is a consequence of Lemma 2.2, Remark 6.3 and the definitions of the data (6.2) for any Bieberbach group \( \Pi \), applied to our \( \Pi' \).
9. Geometries of Individual Leaves

Throughout this section we adopt the assumptions and notations of Theorem 7.1. The 
\( \Pi \)-invariance of the foliation \( F_e \), cf. (8.1), trivially gives rise to

\[
(9.1) \quad \text{the obvious isometric action of } \Pi \text{ on the quotient Euclidean affine space } \mathcal{E}/\mathcal{V}'
\]

(that is, on the leaf space of \( F_e \), the points of which coincide with the affine subspaces \( \mathcal{E}' \) of \( \mathcal{E} \) parallel to \( \mathcal{V}' \)). Whenever \( \mathcal{E}' \in \mathcal{E}/\mathcal{V}' \) is fixed, \( \Sigma' \) in (7.3) obviously coincides with the isotropy group of \( \mathcal{E}' \) for (9.1). The action (9.1) is not effective, as the kernel of the corresponding homomorphism \( \Pi \to \text{Iso}[\mathcal{E}/\mathcal{V}'] \) clearly contains the group \( L' = L \cap \mathcal{V}' \) forming a lattice in \( \mathcal{V}' \), cf. Remark 10.2 below. Let \( \text{pr} \) again stand for the covering projection \( E \to M \). The former is also bijective, its inverse

\[
\text{pr}^{-1} \quad \text{exists when one switches } r \text{ and } s. \quad \text{As the composite } M'_{su} \to M'_{ru} \to M' \text{ clearly equals}
\]
the analogous covering projection \( M'_u \to M' \) (with \( s \) rather than \( r \)), the coverings \( M'_{ru} \to M' \) and \( M'_{su} \to M' \) have the same multiplicity, which completes the proof. \( \square \)

**Remark 9.3.** Replacing \( \delta \) of (9.2) with \( 1/4 \) times its original value, we can also require it to have the following property: *if \( \gamma \in \Pi \) and \( x \in E \) are such that both \( x \) and \( \gamma(x) \) lie in the \( \delta \)-neighborhood of \( E' \), cf. Section 5, then \( \gamma \in \Sigma' \) for the stabilizer group \( \Sigma' \) of \( E' \) defined by (7.3).* In fact, letting \( E'' \) be the leaf of \( F_E \) through \( x \), we see from (8.1) that its \( \gamma \)-image \( \gamma(E'') \) is also a leaf of \( F_E \), while both leaves are within the distance \( \delta \) from \( E' \), which gives \( \text{dist}(E'', \gamma(E'')) < 2\delta \) and so, due to the triangle inequality, \( \text{dist}(E', \gamma(E')) \leq \text{dist}(E', E'') + \text{dist}(E'', \gamma(E'')) + \text{dist}(\gamma(E''), \gamma(E')) < \delta + 2\delta + \delta = 4\delta \). Thus, \( x + ru \in \gamma(E') \) for some \( x \in E' \), some unit vector \( u \in V \) orthogonal to \( V' \), and \( r = \text{dist}(E', \gamma(E')) \in [0, 4\delta) \). Assuming now (9.2) with \( \delta \) replaced by \( 4\delta \), one gets \( r = 0 \), that is, \( \gamma(E') = E' \) and \( \gamma \in \Sigma' \). Namely, the \( \text{pr} \)-image of the curve \([0, 4\delta] \ni t \to x + tu \) is a geodesic in the image of the diffeomorphism \( \text{Exp}^\perp \) of Remark 2.4(a), which intersects \( M' \) only at \( t = 0 \), while \( M' = \text{pr}(E') = \text{pr}(\gamma(E')) \), since \( M = E/\Pi \).

**Lemma 9.4.** *Let there be given \( V', E' \) as in Lemma 9.2, \( \delta \) having the additional property of Remark 9.3, any \( r \in (0, \delta) \), and any unit vector \( u \in V \) orthogonal to \( V' \).*

(a) The isotropy group \( \Sigma'_u \), cf. (9.2), does not depend on \( r \in (0, \delta) \).

(b) The linear part of each element of \( \Sigma'_u \) keeps \( u \) fixed.

(c) \( \Sigma'_u \) is a subgroup of \( \Sigma_0 \) with the finite index \( k = k(u) \geq 1 \) of Lemma 9.2.

(d) \( \text{pr} : E \to M \) maps the \( \delta \)-neighborhood \( E_\delta \) of \( E' \) in \( E \) onto \( M_\delta \) of Remark 2.4(a).

(e) \( E_\delta \) and \( M_\delta \) are unions of leaves of, respectively, \( F_E \) and \( F_M \).

(f) The preimage under \( \text{pr} : E_\delta \to M_\delta \) of the leaf \( M'_{ru} = \text{pr}(E' + ru) \) of \( F_M \) equals the union of the images \( \gamma(E' + ru) \) over all \( \gamma \in \Sigma'_u \).

**Proof.** By (8.3) and (9.2), \( M'_v = (E' + v)/\Pi'_v \), if one lets \( \Pi'_v \) denote the image of \( \Sigma'_v \) under the injective homomorphism of restriction to \( E' \), cf. Remark 7.2. Fixing \( s \in (0, \delta) \) and \( r \in (0, \delta) \) we therefore conclude from Lemma 9.2 and (7.4) that, whenever \( x \in E' + ru \) and \( \gamma \in \Sigma'_u \), there exists \( \hat{\gamma} \in \Sigma'_s \) satisfying the condition

\[
(9.3) \quad \gamma(x) + v = \hat{\gamma}(x + v), \text{ where } v = (s-r)u, \text{ and } \hat{\gamma} = \gamma \text{ when } s = r,
\]

the last clause being obvious since \( \gamma, \hat{\gamma} \in \Pi \) and the action of \( \Pi \) is free. With \( u \) and \( \gamma \) fixed as well, for each given \( \hat{\gamma} \in \Sigma'_s \) the set of all \( x \in E' + ru \) having the property (9.3) is closed in \( E' + ru \) while, as we just saw, the union of these sets over all \( \hat{\gamma} \in \Sigma'_s \) equals \( E' + ru \). Thus, by Baire’s theorem (Lemma 2.5), some \( \hat{\gamma} \in \Sigma'_s \) satisfies (9.3) with all \( x \) from some nonempty open subset of \( E' + ru \), and hence – by real-analyticity – for all \( x \in E' + ru \). In terms of the translation \( \tau_v \) by the vector \( v \), we consequently have \( \hat{\gamma} = \tau_v \circ \gamma \circ \tau_v^{-1} \) on \( E' + su \), so that \( \gamma \) uniquely determines \( \hat{\gamma} \) (due to the injectivity claim of Remark 7.2), the assignment \( \gamma \mapsto \hat{\gamma} \) is a homomorphism \( \Sigma'_u \to \Sigma'_s \subseteq \Pi \), and \( \zeta = \hat{\gamma} \circ \tau_v \circ \gamma^{-1} \circ \tau_v^{-1} \) equals the identity on \( E' + su \). If we now allow \( s \) to vary from \( r \)
to 0, the resulting curve $s \mapsto \zeta$ consists, due to Remark 5.1, of affine extensions of linear self-isometries of the orthogonal complement of $\mathcal{V}'$, and $\hat{\gamma} = \zeta \circ \tau_v \circ \gamma \circ \tau_v^{-1}$ on $\mathcal{E}$. As $\Pi$ is discrete, the curve $s \mapsto \hat{\gamma} \in \Pi$, with $v = (s-r)u$, must be constant, and can be evaluated by setting $s = r$ (or, $v = 0$). Thus, $\hat{\gamma} = \gamma$ on $\mathcal{E}$ from the last clause of (9.3), and so $\Sigma'_v \subseteq \Sigma'_u$. For $s > 0$, switching $r$ with $s$ we get the opposite inclusion, and (a) follows. Also, taking the linear part of the resulting relation $\gamma = \zeta \circ \tau_v \circ \gamma \circ \tau_v^{-1}$, we see that $\zeta$ equals the identity, for all $s$. Hence $\gamma = \tau_v \circ \gamma \circ \tau_v^{-1}$ commutes with $\tau_v$, which amounts to (b). Setting $s = 0$, we obtain the first part of (c): $\Sigma'_v \subseteq \Sigma'_0$. Assertion (d) is clear as $\text{pr}$, being locally isometric, maps line segments onto geodesic segments. Lemma 5.4(a) for $D = F_M$ yields (e). Since $\text{pr} : \mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$, the additional property of $\delta$ (see Remark 9.3) implies (f). Finally, for $k = k(u)$, the geodesic $[0,r] \ni t \mapsto \text{pr}(x + tu)$, normal to $\mathcal{M}'$ at $y = \text{pr}(x)$, is one of $k$ such geodesics $[0,r] \ni t \mapsto \text{pr}(x + tw)$, joining $y$ to points of its preimage under the projection $\mathcal{M}'_v \to \mathcal{M}'$ of Lemma 9.2, where $w$ ranges over a $k$-element set $\mathcal{R}$ of unit vectors in $\mathcal{V}$, orthogonal to $\mathcal{V}'$. The union of the corresponding subset $C = \{\mathcal{E}' + rw : w \in \mathcal{R}\}$ of the leaf space of $F_\mathcal{E}$ equals the preimage in (f) – and hence an orbit for the action of $\Sigma'_0$ – as every leaf in the preimage contains a point nearest $x$. Due to the already-established inclusion $\Sigma'_v \subseteq \Sigma'_0$ and (9.2), $\Sigma'_v$ is the isotropy group of $\mathcal{E}' + ru$ relative to the transitive action of $\Sigma'_0$ on $C$, and so $k$, the cardinality of $C$, equals the index of $\Sigma'_v$ in $\Sigma'_0$, which proves the second part of (c). $\square$

10. THE GENERIC ISOTROPY GROUP

Given a Bieberbach group $\Pi$ in a Euclidean affine space $\mathcal{E}$ with the translation vector space $\mathcal{V}$, let us fix a vector subspace $\mathcal{V}'$ of $\mathcal{V}$ satisfying (7.1). As long as $\dim \mathcal{E} \geq 2$, such $\mathcal{V}'$ always exists (Section 7). An element $\mathcal{E}'$ of $\mathcal{E}/\mathcal{V}'$, that is, a coset of $\mathcal{V}'$ in $\mathcal{E}$, will be called generic if its stabilizer (isotropy) subgroup $\Sigma' \subseteq \Pi$, defined by (7.3), equals

\[(10.1) \quad \text{the kernel of the homomorphism } \Pi \to \text{Iso}[\mathcal{E}/\mathcal{V}'] \text{ corresponding to (9.1)}.\]

The pr-images of generic cosets of $\mathcal{V}'$ will be called generic leaves of $F_M$.

Still using the symbols $\text{pr}, L$ and $H$ for the universal-covering projection $\mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$ and the groups appearing in (6.1) – (6.2), let us also
denote by $K' \subseteq H$ the normal subgroup consisting of all elements

\[(10.2) \quad \text{of } H \text{ that act on the orthogonal complement of } \mathcal{V}' \text{ as the identity, and by } U' \text{ the subset of } \mathcal{E}/\mathcal{V}' \text{ formed by all generic cosets of } \mathcal{V}' \text{ in } \mathcal{E}.\]

**Theorem 10.1.** For $\Sigma'$ equal to (10.1), our assumptions yield the following conclusions.

(i) $U'$ constitutes an open dense subset of $\mathcal{E}/\mathcal{V}'$.

(ii) The normal subgroup $\Sigma'$ of $\Pi$ is contained as a finite-index subgroup in the isotropy group of every $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ for the action (9.1), and equal to this isotropy group if $\mathcal{E}' \in U'$.

(iii) The pr-images $\mathcal{M}', \mathcal{M}''$ of any $\mathcal{E}', \mathcal{E}'' \in U'$ are isometric to each other.
(iv) If one identifies \( E \) with its translation vector space \( V \) via a choice of an origin, \( \Sigma' \) becomes the set of all elements of \( \Pi \) having, for \( K' \) given by (10.2), the form

\[
V \ni x \mapsto Ax + b \in V, \quad \text{in which } b \in V' \text{ and the linear part } A \text{ lies in } K'.
\]

(v) Whenever \( E' \in U' \), the homomorphism which restricts elements of the generic isotropy group \( \Sigma' \) to \( E' \) is injective, and the resulting isomorphic image \( \Pi' \) of \( \Sigma' \) constitutes a Bieberbach group in the Euclidean affine space \( E' \). The lattice subgroup of \( \Pi' \) and its holonomy group \( H' \) are the intersection \( L' = L \cap V' \) and the image \( H' \) of the group \( K' \) defined in (10.2) under the injective homomorphism of restriction to \( V' \).

**Proof.** Lemma 9.4(a) states that the assumptions of Lemma 2.3 are satisfied by the Euclidean affine space \( \mathcal{W} = E/\mathcal{V}' \) and the mapping \( F \) that sends \( E' \in E/\mathcal{V}' \) to its isotropy group \( \Sigma' \) with (7.3). The assignment \( E' \mapsto \Sigma' \) is thus locally constant on some open dense set \( U' \subseteq E/\mathcal{V}' \). Letting \( \Sigma' \) be the constant value of this assignment assumed on a nonempty connected open subset \( W' \) of \( U' \), and fixing \( \gamma \in \Sigma' \), we obtain \( \gamma(E') = E' \) for all \( E' \in W' \), and hence, from real-analyticity, for all \( E' \in E/\mathcal{V}' \). Thus, our \( \Sigma' \) is contained in the isotropy group of every \( E' \in E/\mathcal{V}' \). Since the same applies also to another constant value \( \Sigma'' \) assumed on a nonempty connected open set, \( \Sigma'' = \Sigma' \) and the phrase ‘locally constant’ may be replaced with constant. By Lemma 9.4(c), such \( \Sigma' \) must be a finite-index subgroup of each isotropy group. As \( \Sigma' \) consists of the elements of \( \Pi \) preserving every \( E' \in U' \), real-analyticity implies that they preserve all \( E' \in E/\mathcal{V}' \), and so \( \Sigma' \) coincides with (10.1), which also shows that \( \Sigma' \) is a normal subgroup of \( \Pi \), and (i) – (ii) follow.

Assertions (iv) – (v) are in turn immediate from Remark 5.3 and, respectively, Theorem 7.1(ii) combined with Remark 7.2, while (v) implies (iii) via Remark 6.3. \( \square \)

**Remark 10.2.** An element of \( \Pi \) acting trivially on \( E/\mathcal{V}' \) need not lie in \( L' \). An example arises when the compact flat manifold \( \mathcal{M} = E/\Pi \) is a Riemannian product \( \mathcal{M} = \mathcal{M}' \times \mathcal{M}'' \) with \( E = E' \times E'' \) and \( \Pi = \Pi' \times \Pi'' \) for two Bieberbach groups \( \Pi', \Pi'' \) in Euclidean affine spaces \( E', E'' \) having the translation vector spaces \( \mathcal{V}', \mathcal{V}'' \), while \( \mathcal{M}' \) is not a torus. The \( H \)-invariant subspace \( \mathcal{V}' \times \{0\} \) then gives rise to the \( \mathcal{M}' \) factor foliation \( F_\mathcal{E} \) of the product manifold \( \mathcal{M} \), and the action of the group \( \Pi' \times \{1\} \) on its leaf space is obviously trivial, even though \( \Pi' \times \{1\} \) contains some elements that are not translations.

**Remark 10.3.** In Theorem 10.1, if \( E' \in U' \), we may treat \( \Pi' \) (or, \( H' \)) as a subgroup of \( \Pi \) (or, respectively, \( H \)), by identifying \( \Sigma' \) with \( \Pi' \) (or, \( H' \) with \( K' \)) via the isomorphism \( \Sigma' \to \Pi' \) and \( K' \to H' \) resulting from Theorem 10.1(v). Note that these subgroups \( \Pi' \subseteq \Pi \) and \( H' \subseteq H \) do not depend on the choice of \( E' \in U' \), and neither does the mapping degree \( d = |H'| \) of the \( d \)-fold covering projection \( \mathcal{T}' \to \mathcal{M}' = \mathcal{T}'/H' \), cf. (8.2) and the line following it.

**Remark 10.4.** Any lattice \( L \) in the translation vector space \( \mathcal{V} \) of a Euclidean affine space \( \mathcal{E} \) is, obviously, a Bieberbach group in \( \mathcal{E} \). In the case of a fixed vector subspace \( V' \)
of $\mathcal{V}$ with (7.1), all general facts established about any given Bieberbach group $\Pi$ in $\mathcal{E}$, the compact flat manifold $\mathcal{M} = \mathcal{E}/\Pi$, and the leaves $\mathcal{M}'$ of $F_M$ (see Theorem 7.1) thus remain valid for the torus $\mathcal{T} = \mathcal{E}/L$ and the leaves $\mathcal{T}'$ of $F_T$. Every coset of $\mathcal{V}'$ becomes generic if we declare the lattice $L$ of $\Pi$ to be the new Bieberbach group.

11. The leaf space

By a crystallographic group \([10]\) in a Euclidean affine space one means a discrete group of isometries having a compact fundamental domain, cf. Remark 4.5.

**Proposition 11.1.** Under the assumptions listed at the beginning of Section 10, with $\Sigma'$ denoting the normal subgroup (10.1) of $\Pi$, the quotient group $\Pi/\Sigma'$ acts effectively by isometries on the quotient Euclidean affine space $\mathcal{E}/\mathcal{V}'$ and, when identified a subgroup of $\text{Iso}[\mathcal{E}/\mathcal{V}']$, it constitutes a crystallographic group.

**Proof.** A compact fundamental domain exists by Remark 4.5 since, according to Remark 4.10, $\Pi/\Sigma'$ contains the lattice subgroup $L/L'$ of $\mathcal{E}/\mathcal{V}'$. To verify discreteness of $\Pi/\Sigma'$, suppose that, on the contrary, some sequence $\gamma_k \in \Pi$, $k = 1, 2, \ldots$, has terms representing mutually distinct elements of $\Pi/\Sigma'$ which converge in $\text{Iso}[\mathcal{E}/\mathcal{V}']$. As $L'$ is a lattice in $\mathcal{V}'$, fixing $x \in \mathcal{E}$ and suitably choosing $v_k \in L'$ we achieve boundedness of the sequence $\hat{\gamma}_k(x) = \gamma_k(x) + v_k$, while $\hat{\gamma}_k$ represent the same (distinct) elements of $\Pi/\Sigma'$ as $\gamma_k$. The ensuing convergence of a subsequence of $\hat{\gamma}_k$ contradicts discreteness of $\Pi$. \(\Box\)

The resulting quotient of $\mathcal{E}/\mathcal{V}'$ under the action of $\Pi/\Sigma'$ is thus a flat compact orbifold \([4]\), which may clearly be identified both with the leaf space $\mathcal{M}/F_M$, and with the quotient of the torus $[\mathcal{E}/\mathcal{V}']/[L \cap \mathcal{V}']$ under the action of $H$, mentioned in (1.1). The latter identification clearly implies the Hausdorff property the leaf space $\mathcal{M}/F_M$.

On the other hand, for an $H$-invariant subspace $\mathcal{V}''$ of $\mathcal{V}$ not assumed to be an $L$-subspace, there exists an $L$-closure of $\mathcal{V}''$, meaning the smallest $L$-subspace $\mathcal{V}'$ of $\mathcal{V}$ containing $\mathcal{V}''$, which is obviously obtained by intersecting all such $L$-subspaces (Lemma 4.4). The leaf space $\mathcal{M}/F_M$ corresponding to $\mathcal{V}'$ then forms a natural “Hausdorffization” of the leaf space of $\mathcal{V}''$, and may also be described in terms of Hausdorff-Gromov limits. See the forthcoming paper \([2]\).

12. Intersections of generic complementary leaves

For a homology interpretation of parts (a) and (c) in Theorem 12.1, see Remark 13.2.

Throughout this section $\Pi$ is a given Bieberbach group in a Euclidean affine space $\mathcal{E}$ of dimension $n \geq 2$, while $\mathcal{V}', \mathcal{V}''$ are two mutually complementary $H$-invariant $L$-subspaces of the translation vector space $\mathcal{V}$ of $\mathcal{E}$, in the sense of (4.1) and Definition 4.1, for $L$ and $H$ associated with $\Pi$ via (6.1). We also fix generic cosets $\mathcal{E}'$ of $\mathcal{V}'$ and $\mathcal{E}''$ of $\mathcal{V}''$ (see the beginning of Section 10), which leads to the analogs $L' \to \Pi' \to H'$ and $L'' \to \Pi'' \to H''$.
of (6.2), described by Theorem 7.1(ii) and, $\mathcal{E}', \mathcal{E}''$ being generic, Theorem 10.1(v) yields $L' = L \cap \mathcal{V}'$ and $L'' = L \cap \mathcal{V}''$. Furthermore, for these $\Pi, \Pi', \Pi'', L, L', L'', H, H', H''$, 

(12.1) the conclusions of Lemma 3.3 hold if we replace the letter $G$ with $\Pi, L$ or $H,$ since so do the assumptions of Lemma 3.3, provided that one uses Remark 10.3 to treat $\Pi'$ and $\Pi''$ (or, $H'$ and $H''$) as subgroups of $\Pi$ (or, respectively, $H$). In fact, (10.3) and the description of $K'$ in (10.2) show that all $A \in K'$ (and, among them, the linear parts of all elements of $\Sigma' = \Pi'$) leave invariant both $\mathcal{V}'$ and $\mathcal{V}''$, and act via the identity on the latter. (We have the obvious isomorphic identifications of $\mathcal{V}/\mathcal{V}'$ with $\mathcal{V}''$ on the one hand, and with the orthogonal complement of $\mathcal{V}'$ in $\mathcal{V}$ on the other, while such $A$ descend to the identity automorphism of $\mathcal{V}/\mathcal{V}'$.) The same is clearly the case if one switches the primed symbols with the double-primed ones, while elements of $\Sigma' = \Pi'$ now commute with those of $\Sigma''$ in view of (10.3). This yields (12.1), so that we may form 

(12.2) the quotient groups $\hat{\Pi} = \Pi/(\Pi'\Pi'')$, $\hat{L} = L/(L'L'')$, $\hat{H} = H/(H'H'')$.

Finally, let $\Pr : \mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$ and $\mathcal{M}', \mathcal{M}'', \mathcal{T}', \mathcal{T}''$ denote, respectively, the covering projection of Theorem 7.1(i), the pr-image $\mathcal{M}'$ of $\mathcal{E}'$ (or, $\mathcal{M}''$ of $\mathcal{E}''$), and the tori $\mathcal{E}'/L'$ and $\mathcal{E}''/L''$, contained in the torus $\mathcal{T} = \mathcal{E}/L$ of (8.2). Note that $\mathcal{M}'$ and $\mathcal{M}''$ are (compact) leaves of the parallel distributions arising, due to Theorem 7.1(i), on $\mathcal{M} = \mathcal{E}/\Pi$, which itself is a compact flat Riemannian manifold (Remark 6.1).

**Theorem 12.1.** Under the above hypotheses, the following conclusions hold.

(a) The intersection $\mathcal{M}' \cap \mathcal{M}''$, or $\mathcal{T}' \cap \mathcal{T}''$, is a finite subset of $\mathcal{M}$, or $\mathcal{T}$, and stands in a bijective correspondence with the quotient group $\hat{\Pi}$ or, respectively, $\hat{L}$, of (12.2),

(b) The projection $\mathcal{T} \to \mathcal{M}$ of (8.2) maps $\mathcal{T}' \cap \mathcal{T}''$ injectively into $\mathcal{M}' \cap \mathcal{M}''$.

(c) The cardinality $|\mathcal{M}' \cap \mathcal{M}''|$ of $\mathcal{M}' \cap \mathcal{M}''$ equals $|\mathcal{T}' \cap \mathcal{T}''|$ times $|\hat{H}|$.

(d) The claim about $\mathcal{T}' \cap \mathcal{T}''$ in (a) remains true whether or not $\mathcal{E}', \mathcal{E}''$ are generic.

(e) The two bijective correspondences in (a) may be chosen so that, under the resulting identifications, the injective mapping $\Pr : \mathcal{T}' \cap \mathcal{T}'' \to \mathcal{M}' \cap \mathcal{M}''$ of (b) coincides with the group homomorphism $\hat{L} \to \hat{\Pi}$ induced by the inclusion $L \to \Pi$.

**Proof.** We first prove (a) for $\mathcal{M}' \cap \mathcal{M}''$. Finiteness of $\mathcal{M}' \cap \mathcal{M}''$ follows as $\mathcal{M}', \mathcal{M}'', \mathcal{M}''$, and hence also $\mathcal{M}' \cap \mathcal{M}'', \mathcal{M}' \cap \mathcal{M}''$, are compact totally geodesic submanifolds of $\mathcal{M}$, while $\mathcal{M}' \cap \mathcal{M}'', \mathcal{M}' \cap \mathcal{M}''$, nonempty by (5.1), has $\dim(\mathcal{M}' \cap \mathcal{M}'') = 0$ due to (4.1). The mapping $\Psi : \Pi \to \mathcal{M}' \cap \mathcal{M}''$ with $\Pr(\mathcal{E}' \cap \gamma(\mathcal{E}'')) = \{\Psi(\gamma)\}$ is well defined in view of (4.1) applied to $\gamma(\mathcal{E}'')$ rather than $\mathcal{E}'$, and clearly takes values in both $\mathcal{M}' = \Pr(\mathcal{E}')$ and $\mathcal{M}'' = \Pr(\mathcal{E}'') = \Pr(\gamma(\mathcal{E}''))$. Surjectivity of $\Psi$ follows: if $\Pr(x'') \in \mathcal{M}' \cap \mathcal{M}''$, where $x'' \in \mathcal{E}''$ then, obviously, $\Pr(x') = \Pr(x') \cap x' = \gamma(x'')$ for some $x' \in \mathcal{E}'$ and $\gamma \in \Pi$, so that $x' \in \mathcal{E}' \cap \gamma(\mathcal{E}'')$ and $\Pr(x'') = \Pr(x') = \Psi(\gamma)$, the unique element of $\Pr(\mathcal{E}' \cap \gamma(\mathcal{E}''))$. Furthermore, $\Psi$-pre-images of elements of $\mathcal{M}' \cap \mathcal{M}''$ are precisely the cosets of the normal subgroup $\Pi'\Pi''$ of
Now let $\gamma_1, \gamma_2$ lie in the same set of $\Pi' \cap \Pi''$. For $\gamma', \gamma''$ with (12.3), $\gamma'(\mathcal{E}') = \mathcal{E}'$ and $\gamma''(\mathcal{E}'') = \mathcal{E}''$ by the definition (7.3) of $\Sigma', \Sigma''$ and their identification with $\Pi', \Pi''$ (see above). Thus, $\{\Psi(\gamma_1) = pr(\mathcal{E}' \cap \gamma_1(\mathcal{E}'')) = pr(\gamma'(\mathcal{E}') \cap \gamma''(\gamma_1(\mathcal{E}''))) = pr(\mathcal{E}' \cap \gamma_1(\mathcal{E}''))) = pr(\mathcal{E}' \cap \gamma_2(\mathcal{E}'')) = pr(\mathcal{E}' \cap \gamma_2(\mathcal{E}'')) = pr(\mathcal{E}' \cap \gamma_2(\mathcal{E}'')) = \{\Psi(\gamma_2)\}$. Conversely, if $\gamma_1, \gamma_2 \in \Pi$ and $\Psi(\gamma_1) = \Psi(\gamma_2)$, the unique points $x_1$ of $\mathcal{E}' \cap \gamma_1(\mathcal{E}'')$ and $x_2$ of $\mathcal{E}' \cap \gamma_2(\mathcal{E}'')$ both lie in the same $\Pi$-orbit, and hence $x_2 = \gamma(x_1)$ with some $\gamma \in \Pi$. For $\gamma' = \gamma$ and $\gamma'' = \gamma_2^{-1} \circ \gamma \circ \gamma_1$, the image $\gamma'(\mathcal{E}')$ (or, $\gamma''(\mathcal{E}'')$) intersects $\mathcal{E}'$ (or, $\mathcal{E}''$), the common point being $x_2 = \gamma(x_1)$ or, respectively, $\gamma_2^{-1}(x_2) = \gamma_2^{-1}(\gamma(x_1))$. From (7.4) we thus obtain $\gamma' \in \Sigma' = \Pi'$ and $\gamma'' \in \Sigma'' = \Pi''$, which yields (12.3).

Assertion (a) for $T' \cap T''$, along with (d), now follows as a special case; see Remark 10.4.

Except for the word ‘injective’ the claim made in (e) is immediate if one uses the mapping $\Psi : \Pi \to \mathcal{M}' \cap \mathcal{M}''$ defined above and its analog $L \to T' \cap T''$ obtained by replacing $\Pi, \mathcal{M}', \mathcal{M}''$ and $pr$ with $\Pi, T', T''$ and the projection $\mathcal{E} \to T = \mathcal{E}/L$. This yields (b), injectivity of the homomorphism $\hat{L} \to \hat{H}$ being immediate: if an element of $L$ lies in $\Pi' \Pi''$ (and hence has the form $\gamma' \circ \gamma''$, where $(\gamma', \gamma'') \in \Pi' \times \Pi''$), (10.3) implies that $\gamma', \gamma''$ are translations with $\gamma' \in \Pi' \cap \Pi''$, and $\gamma'' \in \Pi'' = L \cap \Pi''$ (see the lines preceding (12.1)); in other words, $\gamma' \circ \gamma''$ represents zero in $\hat{L}$.

Finally, $\hat{L}$ identified as above with a subgroup of $\hat{H}$ equals the kernel of the clearly-surjective homomorphism $\hat{L} \to \hat{H}$, induced by $\Pi \to H$ in (6.2) (which, combined with (e), proves (c)). Namely, $\hat{L}$ contains the kernel (the other inclusion being obvious): if the linear part of $\gamma \in \Pi$ lies in $H' \Pi''$, and so equals the linear part of $\gamma' \circ \gamma''$ for some $(\gamma', \gamma'') \in \Pi' \times \Pi''$, then $\gamma = \lambda \circ \gamma' \circ \gamma''$, where $\lambda \in L$, as required.

13. Leaves and integral homology

This section once again employs the assumptions and notations of Theorem 7.1, with $\dim \mathcal{V} = n$ and $\dim \mathcal{V}' = k$, where $0 < k < n$. As the holonomy group $H \subseteq \text{Iso} \mathcal{V} \cong O(n)$ is finite (Remark 6.1), $\text{det}(H) \subseteq \{1, -1\}$. In other words, the elements of $H$ have the determinants $\pm 1$. Using the covering projection $T \to \mathcal{M} = T/H$, cf. (8.2) and the line following it, we see that

$$\text{det}(H) = \{1\}$$

By Theorem 10.1(iii), the generic leaves of $F_{\mathcal{M}}$, defined as in the line following (10.1), are either all orientable or all nonorientable.

**Lemma 13.1.** Let $\mathcal{M}$ be orientable. Then all the generic leaves $\mathcal{M}'$ of $F_{\mathcal{M}}$ may be oriented so as to represent the same nonzero $k$-dimensional real homology class $[\mathcal{M}'] \in H_k(\mathcal{M}, \mathbb{R})$. 

Proof. A fixed orientation of $\mathcal{V}'$, being preserved, due to (10.2) – (10.3) and (13.1), by the generic isotropy group $\Sigma'$, gives rise to orientations of all leaves $\mathcal{T}'$ of $F_T$ and all generic leaves $\mathcal{M}'$ of $F_M$, so as to make the covering projections $\mathcal{T}' \to \mathcal{M}'$ in the line following (8.3) orientation-preserving. Since the torus group $\mathcal{V}/L$ acts transitively on the oriented leaves $\mathcal{T}'$, they all represent a single real homology class $[T'] \in H_k(\mathcal{T}, \mathbb{R})$, equal to the image of the fundamental class of $\mathcal{T}'$ under the inclusion $\mathcal{T}' \to \mathcal{T}$. At the same time, for generic leaves $\mathcal{M}'$, the $d$-fold covering projection $\mathcal{T}' \to \mathcal{M}'$ (where $d = |H'|$) does not depend on the choice of $\mathcal{M}'$, cf. Remark 10.3) sends the fundamental class of $\mathcal{T}'$ to $d$ times the fundamental class of $\mathcal{M}'$. Thus, by functoriality, $d[\mathcal{M}'] \in H_k(\mathcal{M}, \mathbb{R})$ is the image of $[\mathcal{T}'] \in H_k(\mathcal{T}, \mathbb{R})$ under the covering projection $\mathcal{T} \to \mathcal{M}$, which makes it the same for all generic leaves $\mathcal{M}'$. Finally, $[\mathcal{M}'] \neq 0$, since a fixed constant positive differential $k$-form on the oriented space $\mathcal{V}'$ descends, in view of the first line of this proof, to a parallel positive volume form on each oriented generic leaf $\mathcal{M}'$, which yields a positive value when integrated over $[\mathcal{M}']$.

Remark 13.2. If $\mathcal{M}$ is orientable, the first two cardinalities in Theorem 12.1(c) equal the intersection numbers of the real homology classes $[\mathcal{M}'], [\mathcal{M}'', \mathcal{T}']$, or $[\mathcal{T}'], [\mathcal{T}'', \mathcal{M}]$, arising via Lemma 13.1, which is consistent with the fact that – according to Theorem 12.1(a) and Remark 10.3 – they depend just on the two mutually complementary $H$-invariant $L$-subspaces $\mathcal{V}', \mathcal{V}''$ of $\mathcal{V}$, and not on the individual generic leaves $\mathcal{M}', \mathcal{M}'', \mathcal{T}'$ or $\mathcal{T}''$.

14. Generalized Klein bottles

This section presents some known examples [3, p. 163] to illustrate our discussion.

Let $\Sigma$ and $\rho : \Sigma \to \Sigma$ denote the unit circle in $\mathbb{C}$ and the rotation by angle $\theta$ (multiplication by $e^{i\theta}$). For $(t, \psi) \in \mathbb{R} \times \mathbb{Z}^\Sigma$ and $f \in \mathbb{R}^\Sigma$, cf. Example 4.3, the assignment

\[(t, \psi, f) \mapsto f \circ r_{2\pi t} + t + \psi,\]

defines a left action on $\mathbb{R}^\Sigma$ of the group $\mathbb{R} \times \mathbb{Z}^\Sigma$, with the group operation $(t, \psi)(t', \psi') = (t + t', \psi \circ r_{2\pi t'} + \psi)$. The term $t$ in (14.1) is the constant function $t : \Sigma \to \mathbb{R}$, and one has the obvious short exact sequence $\mathbb{Z}^\Sigma \to \mathbb{R} \times \mathbb{Z}^\Sigma \to \mathbb{R}$, the arrows being $\psi \mapsto (0, \psi)$ and, respectively, $(t, \psi) \mapsto t$.

The functions $f : \Sigma \to \mathbb{R}$ are not assumed continuous and, whenever $H \subseteq \Sigma$, we treat $\mathbb{R}^H$ (and $\mathbb{Z}^H$) as subsets of $\mathbb{R}^\Sigma$ (and $\mathbb{Z}^\Sigma$) via the zero extension of functions $H \to \mathbb{R}$ to $\Sigma$. For $n \geq 2$ and the group $H = \mathbb{Z}_n \subseteq \Sigma$ of $n$th roots of unity, $\mathbb{Z}^H \cong \mathbb{Z}^n$ is a lattice in the Euclidean space $\mathcal{V} = \mathbb{R}^H \cong \mathbb{R}^n$, and the action (14.1) has a restriction to an affine isometric action of the subgroup $\Pi = [(1/n)\mathbb{Z}] \times \mathbb{Z}_0^H \subseteq \mathbb{R} \times \mathbb{Z}^\Sigma$ on $\mathcal{V}$, with the subgroup $\mathbb{Z}_0^H \cong \mathbb{Z}^{n-1}$ of $\mathbb{Z}^H$ given by $\{\psi \in \mathbb{Z}^H : \psi_{\text{avg}} = 0\}$, where $\cdot_{\text{avg}}$ denotes the averaging functional $\mathcal{V} \to \mathbb{R}$. Note that, in the right-hand side of (14.1) for $(t, \psi) \in \Pi$,

\begin{align*}
t_{\text{avg}} = t, & \quad \psi_{\text{avg}} = 0, \quad (f \circ r_{2\pi t})_{\text{avg}} = f_{\text{avg}}.
\end{align*}
**Lemma 14.1.** These $H, V$ and $\Pi$ have the following properties.

(i) The action of $\Pi$ on $V$ is effective.

(ii) $\Pi$ constitutes a Bieberbach group in the underlying Euclidean affine $n$-space of $V$.

(iii) The holonomy group and lattice subgroup of $\Pi$ are our $H \cong \mathbb{Z}_n$, acting on $V$ linearly by $H \times V \ni \left(e^{i\theta}, f\right) \mapsto f \circ r_{\theta} \in V$, and $L = \mathbb{Z} \times \mathbb{Z}_0^H$. Conversely, $\Pi \cap \mathbb{V} \subseteq L$. To verify this, suppose that $f \circ r_{k} + t + \psi = f + \psi'$ for all $f \in \mathbb{V} = \mathbb{R}^H$, some $(t, \psi) \in \Pi$, and some $\psi' \in V$. Taking the linear parts of both sides, we see that $t \in \mathbb{Z}$ and $(t, \psi) \in L$, as required.

Our $\Pi$ has a compact fundamental domain in $\mathbb{V}$, since so does the lattice $L \subseteq \Pi$. Also, $\Pi$ must be torsion-free: as $\Pi \ni (t, \psi) \mapsto t \in \mathbb{R}$ is a group homomorphism, any finite-order element $(t, \psi)$ of $\Pi$ has $t = 0$, and so, by (14.1), it acts via translation by $\psi$, which gives $\psi = 0$. Next, to establish discreteness of the subset $\Pi$ of $\text{Iso} \mathbb{V}$ (and, consequently, (ii)), suppose that a sequence $(t_k, \psi_k) \in \Pi$ with pairwise distinct terms yields, via (14.1), a sequence convergent in $\text{Iso} \mathbb{V}$. Evaluating (14.1) on $f = 0$, we get $(t_k, \psi_k) \rightarrow (t, \psi)$ in $\mathbb{R} \times \mathbb{R}^H$ as $k \rightarrow \infty$, for some $(t, \psi)$ and, since $(t_k, \psi_k) \in [(1/n)\mathbb{Z}] \times \mathbb{Z}_0^H$, the sequence $(t_k, \psi_k)$ becomes eventually constant, contrary to its terms’ being pairwise distinct.

The final clause of the lemma follows since, by (iv), a $\mathbb{Z}$-basis of $L \cap \mathbb{V}'$ (or, $L \cap \mathbb{V}''$) may be defined to consist just of the constant function $1$ (or, respectively, of the $n-1$ functions $\psi_q : H \rightarrow \mathbb{Z}$, labeled by $q \in H \setminus \{1\}$, where $\psi_q(1) = -\psi(1)$ and $\psi_q = 0$ on $H \setminus \{1, q\}$. Specifically, $\psi' = \sum_q \psi'_q(q) \psi_q$ whenever $\psi' \in \mathbb{Z}_0^H$ and $\psi'_0 = 0$. Furthermore, the claims about $\Sigma''$ and genericity of cosets follow from (14.1) – (14.2). The description of $\Pi$ is in turn immediate if one identifies $V/\mathbb{V}'$ with $V''$ and observes that the quotient action of $\Pi$ then becomes $\Pi \times \mathbb{V}'' \ni ((t, \psi), f) \mapsto f \circ r_{2nt} + \psi \in \mathbb{V}''$.

□

The compact flat Riemannian manifold $V/\Pi$ arising from our Bieberbach group $\Pi$ as in Section 6 is called the $n$-dimensional **generalized Klein bottle** [3, p. 163]. The linear functional $V \ni f \mapsto f_{\text{avg}} \in \mathbb{R}$ is equivariant, due to (14.2), with respect to the actions
of $\Pi$ and $\mathbb{Z}$ (the latter, on $\mathbb{R}$, via translations by multiples of $1/n$), relative to the homomorphism $\Pi \ni (t, \psi) \mapsto t \in (1/n)\mathbb{Z}$. Thus, it descends, in view of Remark 2.1(c), to a bundle projection $V/\Pi \to \mathbb{R}/[(1/n)\mathbb{Z}]$, making $V/\Pi$ a bundle of tori over the circle. The fibres of this bundle are, obviously, the images, under the projection $V \to V/\Pi$, of cosets of the $L$-subspace $V'' \subseteq V$ mentioned in the final clause of Lemma 14.1, all of them generic. On the other hand, $V'$ has some nongeneric cosets – an example is $V'$ itself, with the isotropy group easily seen to be $[(1/n)\mathbb{Z}] \times \{0\}$.

The $n$-dimensional generalized Klein bottle, for any $n \geq 2$, shows that the last inclusion of Theorem 7.1(ii-c) may be proper. In fact, the isotropy group $[(1/n)\mathbb{Z}] \times \{0\}$ of the preceding paragraph, although not contained in the lattice $L$, acts on $V'$ by translations.

15. Remarks on holonomy

The correspondence – Remark 6.3 – between Bieberbach groups and compact flat manifolds has an extension to almost-Bieberbach groups and infra-nilmanifolds [5] obtained by using (instead of the translation vector space of a Euclidean affine space) a connected, simply connected nilpotent Lie group $G$ acting simply transitively on a manifold $E$, and replacing the Bieberbach group with a torsion-free uniform discrete subgroup $\Pi$ of $\text{Diff} E$ contained in a semidirect product, canonically transplanted so as to act on $E$, of $G$ and a maximal compact subgroup of $\text{Aut} G$. Here ‘uniform’ means admitting a compact fundamental domain, cf. Remark 4.5. The analogs of (6.2) and (8.2) remain valid, reflecting the fact that any infra-nilmanifold is the quotient of a nilmanifold under the action of a finite group $H$.

A somewhat similar picture may arise in some cases where $G$ is not assumed nilpotent. As an example, one has $G \cong \text{Spin}(m, 1)$, the universal covering group of the identity component $G/\mathbb{Z}_2 \cong \text{SO}^+(m, 1)$ of the pseudo-orthogonal group in an $(m+1)$-dimensional Lorentzian vector space $\mathcal{L}$, $m \geq 3$. Here $E$ is the (two-fold) universal covering manifold of the orthonormal-frame bundle of the future unit pseudosphere $S \subseteq \mathcal{L}$, isometric to the hyperbolic $m$-space, and $G/\mathbb{Z}_2$ acts on $S$ via hyperbolic isometries, leading to an action of $G$ on $E$. The orthonormal-frame bundles of compact hyperbolic manifolds obtained as quotients of $S$ give rise to the required torsion-free uniform discrete subgroups $\Pi$.

The resulting compact quotient manifolds $\mathcal{M} = E/\Pi$ can be endowed with various interesting Riemannian metrics coming from $\Pi$-invariant metrics on $E$. For $\Pi$ and $E$ of the preceding paragraph, a particularly natural choice of an invariant indefinite metric is provided by the Killing form of $G$, turning $\mathcal{M}$ into a compact locally symmetric pseudo-Riemannian Einstein manifold.

Outside of the Bieberbach-group case, however, these metrics are not flat, and finite groups $H$ such as mentioned above cannot serve as their holonomy groups. The holonomy interpretation of $H$ still makes sense, though, if – instead of metrics – one uses $\Pi$-invariant flat connections, with (parallel) torsion, on $E$. Two such standard connections are
naturally induced by bi-invariant connections on $G$, characterized by the property of making all left-invariant (or, right-invariant) vector fields parallel. Both of these connections are, due to their naturality, invariant under all Lie-group automorphisms of $G$.

**Appendix: Hiss and Szczepański’s reducibility theorem**

Let us consider an abstract Bieberbach group, that is, any torsion-free group $\Pi$ containing a finitely generated free Abelian normal subgroup $L$ of a finite index, which is at the same time a maximal Abelian subgroup of $\Pi$. As shown by Zassenhaus [11], up to isomorphisms these groups coincide with the Bieberbach groups of Section 6, and one can again summarize their structure using the short exact sequence

\[(A.1)\quad L \to \Pi \to H, \quad \text{where } H = \Pi/L.\]

For the tensor product $G \otimes G'$ of Abelian groups $G, G'$ one has canonical isomorphisms

\[(A.2)\quad \mathbb{Z} \otimes G \cong G, \quad (G_1 \oplus G_2) \otimes G' \cong (G_1 \otimes G') \oplus (G_2 \otimes G'), \quad L \otimes \mathbb{Q} \cong \text{Hom}(L^*, \mathbb{Q}),\]

where $L^* = \text{Hom}(L, \mathbb{Z})$ and, for simplicity, $L$ is assumed to be finitely generated and free. In the last case, with a suitable integer $n \geq 0$, there are noncanonical isomorphisms

\[(A.3)\quad \begin{align*}
a) & \quad L \cong \mathbb{Z}^n, \\
b) & \quad L \otimes \mathbb{Q} \cong \mathbb{Q}^n,
\end{align*}\]

while, using the injective homomorphism $L \ni \lambda \mapsto \lambda \otimes 1 \in L \otimes \mathbb{Q}$ to treat $L$ as a subgroup of $L \otimes \mathbb{Q}$, we see that, under suitably chosen identifications (A.3),

\[(A.4)\quad \text{the inclusion } L \subseteq L \otimes \mathbb{Q} \text{ corresponds to the standard inclusion } \mathbb{Z}^n \subseteq \mathbb{Q}^n.\]

Finally, if $L$ as above is a (full) lattice in an finite-dimensional real vector space $V$ (cf. Remark 4.6), a further canonical isomorphic identification arises:

\[(A.5)\quad L \otimes \mathbb{Q} \cong \text{Span}_\mathbb{Q} L,\]

that is, we may view $L \otimes \mathbb{Q}$ as the rational vector subspace of $V$ spanned by $L$.

Let $\Pi$ now be an abstract Bieberbach group. Hiss and Szczepański [7, the corollary in Sect. 1] proved that, if $L$ in (A.1) satisfies (A.3.a) with $n \geq 2$, then the (obviously $\mathbb{Q}$-linear) action of $H$ on $L \otimes \mathbb{Q}$ must be reducible, in the sense of admitting a nonzero proper invariant rational vector subspace $W$.

Next, using (A.4), we may write $L = \mathbb{Z}^n \subseteq \mathbb{Q}^n = L \otimes \mathbb{Q}$, so that $W \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n$, and the closure $V'$ of $W$ in $\mathbb{R}^n$ has the real dimension $\dim_{\mathbb{Q}} W$ (any $\mathbb{Q}$-basis of $W$ being, obviously, an $\mathbb{R}$-basis of $V'$). By clearing denominators, one can replace such a $\mathbb{Q}$-basis with one consisting of vectors in $L = \mathbb{Z}^n$, and so, by Remark 4.6(b), the intersection $L' = L \cap W = L \cap V'$ is a lattice in $V'$. In other words, we obtain (7.2).

A stronger version of Hiss and Szczepański’s reducibility theorem was more recently established by Lutowski [8].
References


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