

WEAKLY EINSTEIN CURVATURE TENSORS

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ABSTRACT. We classify weakly Einstein algebraic curvature tensors in an oriented Euclidean four-space, defined by requiring that the three-index contraction of the curvature tensor against itself be a multiple of the inner product. This algebraic formulation parallels the geometric notion of weakly Einstein Riemannian four-manifolds, which include conformally flat scalar-flat, and Einstein manifolds. Our main result provides a complete classification of non-Einstein weakly Einstein curvature tensors in dimension four, naturally dividing them into three disjoint five-dimensional families of algebraic types. These types are explicitly constructed using bases that simultaneously diagonalize both the Einstein tensor and the (anti)self-dual Weyl tensors, which consequently proves that such simultaneous diagonalizability follows from the weakly Einstein property. We also point out that our classification has immediate applications, and describe how some known geometric examples that are neither Einstein, nor conformally flat scalar-flat (namely, the EPS space and certain Kähler surfaces) fit within our classification framework.

1. INTRODUCTION

One says that a Riemannian four-manifold is *weakly Einstein* [8] when the three-index contraction of its curvature tensor against itself equals a function times the metric. This topic has been studied by several authors. Our Refs. [1], [3], [4], and [6] – [15] list articles on weakly Einstein manifolds that do not exclude dimension four or positive-definite metrics.

The present paper deals with algebraic curvature tensors R in an oriented Euclidean four-space $\mathcal{T}$ which are *weakly Einstein* in the sense that, analogously, the three-index contraction of R is a multiple of the inner product g . As we point out in Sect. 2,

$$(1.1) \quad R \text{ is weakly Einstein if and only if } 6We = -se,$$

W, s and e denoting, here and below, the Weyl component of R , its scalar curvature, and its Einstein (traceless Ricci) tensor, with We standing for the usual action of algebraic curvature tensors on symmetric $(0, 2)$ tensors.

Due to (1.1), the weakly Einstein property follows in the two “trivial” cases

$$(1.2) \quad e = 0, \text{ or } W = 0 \text{ and } s = 0,$$

which reflects two known facts about Riemannian four-manifolds: they are weakly Einstein whenever they are Einstein [2, Sect. 1.133], and, under the assumption of conformal flatness, being weakly Einstein amounts to having zero scalar curvature [11, Theorem 2(i)]; cf. also [6, the three lines following formula (4.6)].

Our use of the word “trivial” only refers to the purely algebraic focus of our discussion: there is nothing trivial in the problem of classifying Einstein metrics in dimension four.

We provide, in Theorems 1.3 – 1.5, an algebraic classification of nontrivial weakly Einstein curvature tensors, based on the spectra of the Einstein and Weyl tensors. Our classification

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leads to immediate applications. Namely, the spectral conditions are often easier to impose, or verify, than the original form of the weakly Einstein property. As a result, our paper [7] uses Theorems 1.3 – 1.5, rephrased as [7, Theorem 5.1 and Remark 5.2], which includes [7, formula (5.3)], both to show that certain metrics are weakly Einstein, and to classify some special types of weakly Einstein metrics. For the former, see [7, Sect. 10–11], which use [7, formulae (5.3-b), (5.3-c) and Remark 5.2]; for the latter, [7, Sect. 19–20].

Let $\Lambda^\pm$ be the spaces of self-dual and anti-self-dual bivectors in our oriented Euclidean four-space $\mathcal{T}$. Any positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$ leads to the length $\sqrt{2}$ orthogonal bases of $\Lambda^\pm$ given by

$$(1.3) \quad u_1 \wedge u_2 \pm u_3 \wedge u_4, \quad u_1 \wedge u_3 \pm u_4 \wedge u_2, \quad u_1 \wedge u_4 \pm u_2 \wedge u_3.$$

The restrictions $W^\pm : \Lambda^\pm \rightarrow \Lambda^\pm$ of W acting on bivectors are always diagonalized by some bases of the form (1.3). See Lemma 7.1(a). However, one cannot, in general, expect $u_1, \dots, u_4$ to be in any way related to the Einstein tensor e .

Our first result shows that the weakly Einstein case is quite exceptional in this regard.

Theorem 1.1. *Let R be a weakly Einstein algebraic curvature tensor in an oriented Euclidean four-space $\mathcal{T}$. Then there exists a positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$, consisting of eigenvectors of e , and such that the bases (1.3) of $\Lambda^\pm$ diagonalize $W^\pm$.*

We easily derive Theorem 1.1, at the end of Sect. 9, from the next three results, which constitute a complete classification of (non-Einstein) weakly Einstein algebraic curvature tensors in dimension four. The resulting algebraic-equivalence types form three disjoint classes, depending, first, on whether e does, or does not, have two distinct double eigenvalues. The latter case is further split into two subcases, with $s = 0$ and $s \neq 0$. Each of the three disjoint classes is five-dimensional, as we construct it using five free parameters.

In all three theorems below, $u_1, \dots, u_4$ is a fixed positive orthonormal basis of an oriented Euclidean four-space $\mathcal{T}$ and, invoking the Singer-Thorpe theorem [2, Sect. 1.126], to define an algebraic curvature tensor R in $\mathcal{T}$, we prescribe its scalar curvature s , and

$$(1.4) \quad \text{require the bases } u_1, \dots, u_4 \text{ of } \mathcal{T} \text{ and (1.3) of } \Lambda^\pm \text{ to diagonalize the Einstein tensor } e \text{ and, respectively, } W^\pm : \Lambda^\pm \rightarrow \Lambda^\pm, \text{ with some specific eigenvalues.}$$

Remark 1.2. In each of the next three theorems, to define W one can obviously use item (b) instead of prescribing the spectra of $W^\pm$.

Theorem 1.3. *Given $\mu_1, \mu_2, \mu_3, \mu_4, c_2, c_3, c_4 \in \mathbb{R}$ with $\mu_1 + \mu_2 + \mu_3 + \mu_4 = c_2 + c_3 + c_4 = 0$, let $\mu_1, \mu_2, \mu_3, \mu_4$ and $\pm c_2, \pm c_3, \pm c_4$ be the eigenvalues in (1.4). If we set $s = 0$, then*

- (a) *the resulting algebraic curvature tensor R is weakly Einstein,*
- (b) *the only possibly-nonzero components of W in the frame $u_1, \dots, u_4$ are those algebraically related to $W_{1234}, W_{1342}, W_{1423}$, where $(W_{1234}, W_{1342}, W_{1423}) = (c_2, c_3, c_4)$,*
- (c) *every non-Einstein, weakly Einstein algebraic curvature tensor in dimension four, having zero scalar curvature, arises as above, unless e has two double eigenvalues.*

Theorem 1.4. *If $s, \lambda, \mu, c_2, c_3, c_4 \in \mathbb{R}$ have $c_2 + c_3 + c_4 = 0$ and $\lambda > \mu \geq 0$, and we choose the scalar curvature to be s , then, for the eigenvalues in (1.4) provided by $(\mu_1, \mu_2, \mu_3, \mu_4) = (-\lambda, -\mu, \mu, \lambda)$ and $\pm c_2 - s/12, \pm c_3 - s/12, \pm c_4 + s/6$,*

- (a) *the resulting algebraic curvature tensor R is weakly Einstein,*

- (b) *the only possibly-nonzero components of W in the frame $u_1, \dots, u_4$ are those algebraically related to the ones occurring in the ordered triples $(W_{1212}, W_{1313}, W_{1414}) = (W_{3434}, W_{2424}, W_{2323}) = (-s/12, -s/12, s/6)$ and $(W_{1234}, W_{1342}, W_{1423}) = (c_2, c_3, c_4)$,*
- (c) *every non-Einstein, weakly Einstein algebraic curvature tensor in dimension four, with $s \neq 0$, arises as above, except when e has two double eigenvalues.*

Theorem 1.5. *For $s, \lambda, \xi, c_2, c_3, c_4 \in \mathbb{R}$ with $c_2 + c_3 + c_4 = 0$ and $\lambda > 0$, let the scalar curvature be s . With the eigenvalues in (1.4) given by $(\mu_1, \mu_2, \mu_3, \mu_4) = (-\lambda, -\lambda, \lambda, \lambda)$ and $\pm c_2 - s/12, \pm c_3 + \xi - s/12, \pm c_4 - \xi + s/6$,*

- (a) *the resulting algebraic curvature tensor R is weakly Einstein,*
- (b) *the only possibly-nonzero components of W in the frame $u_1, \dots, u_4$ are algebraically related to those in the ordered triples $(W_{1212}, W_{1313}, W_{1414}) = (W_{3434}, W_{2424}, W_{2323}) = (-s/12, \xi - s/12, -\xi + s/6)$ and $(W_{1234}, W_{1342}, W_{1423}) = (c_2, c_3, c_4)$,*
- (c) *every weakly Einstein algebraic curvature tensor in dimension four such that e has two distinct double eigenvalues arises as described above.*

The multiplicities of the eigenvalues of e , listed in weakly descending order, form one of the five strings, of which the last four (the non-Einstein cases) are of interest to us:

$$(1.5) \quad \begin{array}{cccccc} \mathbf{4}, & \mathbf{31}, & \mathbf{22}, & \mathbf{211}, & \mathbf{1111}, & \text{so that} \\ d = 1, & d = 2, & d = 2, & d = 3, & d = 4, \end{array}$$

d being the number of distinct eigenvalues. As the statements of the last three results clearly indicate, Theorem 1.3 involves the multiplicities **31**, **211** and **1111**, Theorem 1.4 – only **211** and **1111**, while Theorem 1.5 deals solely with the multiplicities **22**.

Our proofs of Theorems 1.3 – 1.5 are split between Sect. 3, 6, 8 for the first two theorems, and Sect. 3, 9 for the last one, as we establish each item (c) by cases, based on (1.5).

An algebraic curvature tensor R in an oriented Euclidean four-space $\mathcal{T}$ is said to be of *Kähler type* if e and $W^+ : \Lambda^+ \rightarrow \Lambda^+$ are simultaneously diagonalized by some positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$ and the basis (1.3) for the $+$ sign with the respective eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4$ and $(s/6, -s/12, -s/12)$ such that $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$. This is well known [5, pp. 485–486] to be the case for the curvature tensor of every Kähler surface.

The following fact is an obvious consequence Theorems 1.1, 1.5 and Remark 1.2.

Corollary 1.6. *For $s, \lambda, \xi \in \mathbb{R}$, letting the scalar curvature and the spectra in (1.4) be*

$$(1.6) \quad s, \quad (-\lambda, -\lambda, \lambda, \lambda), \quad (s/6, -s/12, -s/12), \quad (-s/3, 2\xi - s/12, -2\xi + 5s/12),$$

we define a Kähler-type weakly Einstein algebraic curvature tensor.

Conversely, every non-Einstein Kähler-type weakly Einstein algebraic curvature tensor in dimension four is obtained as described here from some s, ξ and $\lambda > 0$.

Rather than prescribing the spectra of $W^\pm$ in (1.6) one can equivalently require that the condition (b) in Theorem 1.5 be satisfied with $(c_2, c_3, c_4) = (s/4, -\xi, \xi - s/4)$.

It is convenient to use the term *proper* when referring to weakly Einstein manifolds or metrics that are neither Einstein, nor conformally flat and scalar-flat. There are three narrow classes of known examples [6, 7, 9] of proper weakly Einstein Riemannian four-manifolds. They have abundant local self-isometries: some are locally homogeneous [9], the others have local cohomogeneity one [6, Remark 12.1] or two [7, Sect. 21]. It turns out that

$$(1.7) \quad \begin{array}{l} \text{the weakly Einstein Kähler surfaces constructed in [6, Sect. 12] real-} \\ \text{ize all parameter triples } (s, \lambda, \xi) \text{ of Corollary 1.6 for which } s = 8\xi, \end{array}$$

as we show in Sect. 10, where we explain how the curvature tensors of the examples in [6, 9] fit within our algebraic classification. The analogous question about the examples in [7] is answered in [7, Sect. 21].

Sect. 11 discusses the cases of self-duality and anti-self-duality (which, as usual, means vanishing of W^+ or W^-) within each of the three disjoint classes of non-Einstein, weakly Einstein algebraic curvature tensors, characterized by Theorems 1.3 – 1.5.

In [16, p. 32] Shapiro establishes a classification theorem for weakly Einstein algebraic curvature tensors R (in all dimensions), where R is also assumed [16, Definition 1.3] to be the Kulkarni-Nomizu square of some symmetric 2-tensor b . Her result states that R is weakly Einstein if and only if b is either a multiple of the inner product g , or has exactly two, mutually opposite eigenvalues relative to g . In dimension four, all such R are “trivial” in our sense: $d \in \{1, 2\}$ in (1.5), for b rather than e , and the possible multiplicities are **4**, **22** and **31**. They all yield (1.2) – the first two have $e = 0$, the third one $W = 0$ and $s = 0$. The last sentence is justified in Remark 10.1.

2. PRELIMINARIES

In a Euclidean n -space $\mathcal{T}$ with the inner product g , any algebraic curvature tensor R acts on arbitrary $(0, 2)$ tensors b by $[Rb]_{ij} = R_{ipq}b^{pq}$. This action preserves (skew)symmetry of b and, in view of the Bianchi identity, becomes $2[Rb]_{ij} = R_{ijpq}b^{pq}$ when b is skew-symmetric [2, Sect. 1.114, 1.131]. The *triple contraction* of R is the symmetric $(0, 2)$ tensor $\text{trc}R$ with $[\text{trc}R]_{ij} = R_{ikpq}R_j^{kpq}$. Our sign convention about the curvature tensor is such that the Ricci tensor has the components $R_{ij} = R_{ikj}^k$. Here (but nowhere else in the paper) we use g -index raising and summation over repeated indices.

As pointed out in [6, formula (4.6)], when $n = 4$,

$$(2.1) \quad \text{trc}R - 2Wr - sr/3 \text{ is a multiple of } g,$$

W, s, r being the Weyl component, scalar curvature, and Ricci contraction of R . This trivially implies (1.1). In [6, Theorem 5.1(c)], (1.1) is established for Kähler surfaces.

For the remainder of this section, R is an algebraic curvature tensor in an oriented Euclidean four-space $\mathcal{T}$, with W, s as above and the Einstein tensor $e = r - sg/4$.

Let a positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$ diagonalize e with some eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4$. Since W is well known [5, p. 647] to commute with the Hodge star, for the components of W in the frame $u_1, \dots, u_4$ one has

$$(2.2) \quad W_{ijkl} = W_{pqrs} \text{ if } (i, j, p, q) \text{ and } (k, l, r, s) \text{ are even permutations of } (1, 2, 3, 4).$$

Due to (1.1), the weakly Einstein property of R is equivalent to

$$(2.3) \quad \text{a) } W_{ikjk}(\mu_k - \mu_l) = 0, \quad \text{b) } W_{ijij}\mu_j + W_{ikik}\mu_k + W_{ilil}\mu_l = -s\mu_i/6,$$

whenever $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Here (2.3-a) arises from $W_{ikjk}\mu_k + W_{iljl}\mu_l = 0$ and (2.2): $W_{iljl} = -W_{ikjk}$ since the permutations $(i, l, j, k), (j, l, i, k)$ have opposite parities. Let us set

$$(2.4) \quad (a_2, a_3, a_4) = (W_{1212}, W_{1313}, W_{1414}) = (W_{3434}, W_{2424}, W_{2323}),$$

the second equality being obvious from (2.2). With (2.4), equation (2.3-b) reads

$$(2.5) \quad \begin{bmatrix} \mu_2 & \mu_3 & \mu_4 \\ \mu_1 & \mu_4 & \mu_3 \\ \mu_4 & \mu_1 & \mu_2 \\ \mu_3 & \mu_2 & \mu_1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ a_4 \end{bmatrix} = -\frac{s}{6} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}.$$

To simplify our discussion, here and in Sect. 4, we let

$$(2.6) \quad \rho \in \{0, 1, 2, 3\} \text{ denote the rank of the } 4 \times 3 \text{ matrix in (2.5).}$$

We will encounter some cases where, requiring that $\mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4$, one has

$$(2.7) \quad (\mu_1, \mu_2, \mu_3, \mu_4) = (-\lambda, -\mu, \mu, \lambda) \text{ with } \lambda \geq \mu \geq 0 \text{ and } \lambda > 0.$$

Lemma 2.1. *Under the assumption (2.7), the only solutions (a_2, a_3, a_4) of (2.5) such that $a_2 + a_3 + a_4 = 0$ are*

- (a) *just $(a_2, a_3, a_4) = (-s/12, -s/12, s/6)$, when $\lambda > \mu$,*
- (b) *$(a_2, a_3, a_4) = (-s/12, \xi - s/12, -\xi + s/6)$ for all $\xi \in \mathbb{R}$, if $\lambda = \mu$.*

Proof. Both (a)–(b) define (a_2, a_3, a_4) with (2.5) and $a_2 + a_3 + a_4 = 0$. If $\lambda > \mu$ (or, $\lambda = \mu$), then $\rho = 2$ (or, $\rho = 1$) in (2.6), and the kernel of the matrix operator in (2.5) is spanned by $(1, 1, 0)$ or, respectively, by $(1, 1, 0)$ and $(0, 1, -1)$, so that the only vectors (x_2, x_3, x_4) in the kernel having $x_2 + x_3 + x_4 = 0$ are $(0, 0, 0)$ or, respectively, multiples of $(0, 1, -1)$. $\square$

Remark 2.2. In an oriented Euclidean plane, any traceless symmetric $(0, 2)$ tensor a has, in a suitable positive orthonormal basis, a matrix with zeros on the diagonal: we start with a positive orthonormal basis v, w diagonalizing a , and replace it with $(v-w)/\sqrt{2}, (v+w)/\sqrt{2}$.

3. PROOFS OF PARTS (a) AND (b) IN THEOREMS 1.3, 1.4 AND 1.5

In Theorem 1.3, W sends $u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4, u_3 \wedge u_4, u_4 \wedge u_2, u_2 \wedge u_3$ to, respectively, $c_2 u_3 \wedge u_4, c_3 u_4 \wedge u_2, c_4 u_2 \wedge u_3, c_2 u_1 \wedge u_2, c_3 u_1 \wedge u_3, c_4 u_1 \wedge u_4$, and (b) trivially follows. As $s = 0$, this yields (2.3), and hence (a).

Theorem 1.4: W sends $u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4, u_3 \wedge u_4, u_4 \wedge u_2, u_2 \wedge u_3$ to, respectively,

$$\begin{aligned} & c_2 u_3 \wedge u_4 - s u_1 \wedge u_2 / 12, c_3 u_4 \wedge u_2 - s u_1 \wedge u_3 / 12, c_4 u_2 \wedge u_3 + s u_1 \wedge u_4 / 6, \\ & c_2 u_1 \wedge u_2 - s u_3 \wedge u_4 / 12, c_3 u_1 \wedge u_3 - s u_4 \wedge u_2 / 12, c_4 u_1 \wedge u_4 + s u_2 \wedge u_3 / 6, \end{aligned}$$

proving (b); (a) follows from (2.4), as (2.3-a) is obvious and Lemma 2.1(a) yields (2.3-b).

Theorem 1.5: W sends $u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4, u_3 \wedge u_4, u_4 \wedge u_2, u_2 \wedge u_3$ to, respectively,

$$\begin{aligned} & c_2 u_3 \wedge u_4 - s u_1 \wedge u_2 / 12, c_3 u_4 \wedge u_2 + (\xi - s/12) u_1 \wedge u_3, c_4 u_2 \wedge u_3 - (\xi - s/6) u_1 \wedge u_4, \\ & c_2 u_1 \wedge u_2 - s u_3 \wedge u_4 / 12, c_3 u_1 \wedge u_3 + (\xi - s/12) u_4 \wedge u_2, c_4 u_1 \wedge u_4 - (\xi - s/6) u_2 \wedge u_3, \end{aligned}$$

which proves (b); with (2.3-a) obvious, Lemma 2.1(b) yields (2.3-b) and, by (2.4), (a).

4. SOME ALGEBRAIC FACTS

As before, R is an algebraic curvature tensor in an oriented Euclidean four-space $\mathcal{T}$. We use the notation of Sect. 2. Consider the following three conditions:

- (i) $s \neq 0$,
- (ii) $\rho < 3$ in (2.6),
- (iii) the set $\{|\mu_1|, |\mu_2|, |\mu_3|, |\mu_4|\}$ has at most two elements.

Lemma 4.1. *The condition (4.1-i) implies (4.1-ii), and (4.1-ii) implies (4.1-iii). Also,*

- (a) *if the set $\{|\mu_1|, |\mu_2|, |\mu_3|, |\mu_4|\}$ has more than two elements, then $\rho = 3$,*
- (b) *if the multiplicities in (1.5) are **31**, then $\rho = 3$,*
- (c) *if $\rho = 3$, then $s = a_2 = a_3 = a_4 = 0$ in (2.5).*

Proof. The formula $(a_2, a_3, a_4) = (s/6, s/6, s/6)$ defines a solution of (2.5). Since, by (2.4), our (a_2, a_3, a_4) has $a_2 + a_3 + a_4 = 0$, the assumption that $s \neq 0$ leads to nonuniqueness of solutions, and hence (4.1-ii). Next, (4.1-ii) amounts to the determinant conditions

$$\begin{aligned} 2(\mu_1^3 + \mu_2\mu_3\mu_4) &= \mu_1(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2), \\ 2(\mu_2^3 + \mu_1\mu_3\mu_4) &= \mu_2(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2), \\ 2(\mu_3^3 + \mu_1\mu_2\mu_4) &= \mu_3(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2), \\ 2(\mu_4^3 + \mu_1\mu_2\mu_3) &= \mu_4(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2), \end{aligned}$$

where it suffices to verify just one equality, as the other three then follow from invariance of (2.3-b) under permutations of $\{1, 2, 3, 4\}$. Multiplying the four lines above by, respectively, $\mu_1, \mu_2, \mu_3, \mu_4$, we see that each $\sigma = \mu_i^2$ is a root of the quadratic equation $2\sigma^2 - p\sigma + q = 0$, for $p = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2$ and $q = 2\mu_1\mu_2\mu_3\mu_4$, so that (4.1-iii) follows.

Now (a) is obvious, for (b) we may require that $(\mu_1, \mu_2, \mu_3, \mu_4) = (\mu, \mu, \mu, -3\mu)$ with some $\mu \neq 0$, which gives $\rho = 3$ in (2.6), while for (c) we have $(\rho, s) = (3, 0)$, and the resulting injectivity of the matrix operator in (2.5) yields $(a_2, a_3, a_4) = (0, 0, 0)$. $\square$

5. STEP ONE TOWARD THE PROOFS OF THEOREMS 1.3 – 1.5

The components of W in a positive orthonormal basis $u_1, \dots, u_4$, diagonalizing e with some eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4$, are of three kinds:

$$(5.1) \quad \text{a) } W_{ijk}, \quad \text{b) } W_{ijij}, \quad \text{c) } W_{ijkl},$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. According to (2.3), the three kinds are subject to three separate conditions: (2.3-a) for (5.1-a) and (2.3-b) for (5.1-b), with no conditions at all for (5.1-c), so that *the components (5.1-c) are completely arbitrary*:

$$(5.2) \quad (W_{1234}, W_{1342}, W_{1423}) = (c_2, c_3, c_4) \text{ with } c_2 + c_3 + c_4 = 0.$$

As for (5.1-a), equation (2.3-a) states that some of them vanish, and the others are arbitrary.

Lemma 5.1. *For the multiplicities **1111**, **211** and **31** in (1.5), if $\mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4$,*

$$(5.3) \quad (W_{1212}, W_{1313}, W_{1414}) = (W_{3434}, W_{2424}, W_{2323}) = (-s/12, -s/12, s/6),$$

*while in the cases **1111** and **211**, when $s \neq 0$, one must have (2.7) with $\lambda > \mu \geq 0$.*

Proof. If the multiplicities are **31**, we get (5.3) from Lemma 4.1(b)–(c) and (2.4). When $\{|\mu_1|, |\mu_2|, |\mu_3|, |\mu_4|\}$ has more than two elements, $(s, a_2, a_3, a_4) = (0, 0, 0, 0)$ by Lemma 4.1(a), (c), and (5.3) follows from (2.4). Let $\{|\mu_1|, |\mu_2|, |\mu_3|, |\mu_4|\}$ now have at most two elements. For the multiplicities **1111** or **211**, this clearly implies (2.7) with $\lambda > \mu > 0$ or, respectively, $\lambda > \mu = 0$, and, due to (2.4), Lemma 2.1(a) gives (5.3).

Finally, if one has $s \neq 0$, and **1111** or **211**, the first line of Lemma 4.1 gives (4.1-iii), and the preceding two sentences yield the last line of the lemma. $\square$

6. STEP TWO: FOUR OR THREE DIFFERENT EIGENVALUES

We are assuming the multiplicities in (1.5) to be **1111** or **211**.

For **1111**, (2.3-a) gives $W_{ikjk} = 0$ whenever $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Thus, if $s = 0$ (or, $s \neq 0$), (5.2) – (5.3) imply (b) in Theorem 1.3 (or, respectively, in Theorem 1.4).

Due to Remark 1.2 and Sect. 3, along with the final clause of Lemma 5.1, this proves Theorems 1.3 – 1.4 when the multiplicities are **1111**.

In the case **211**, for a positive orthonormal basis $u_1, \dots, u_4$ diagonalizing the Einstein tensor e with some eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4$, let us order $\mu_1, \mu_2, \mu_3, \mu_4$ in such a way that

$$(6.1) \quad \begin{aligned} &\mu_3 \neq \mu_1 \neq \mu_2 \neq \mu_3 = \mu_4, \text{ and so, by (2.2) – (2.3-a),} \\ &W_{1323} = W_{1442} = b, \text{ for some } b \in \mathbb{R}, \\ &W_{1213} = W_{1214} = W_{1223} = W_{1224} = W_{1314} \\ &\quad = W_{1334} = W_{1434} = W_{2324} = W_{2334} = W_{2434} = 0. \end{aligned}$$

Thus, W has in the basis $u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4, u_3 \wedge u_4, u_4 \wedge u_2, u_2 \wedge u_3$ the matrix

$$\begin{bmatrix} -s/12 & 0 & 0 & c_2 & 0 & 0 \\ 0 & -s/12 & 0 & 0 & c_3 & b \\ 0 & 0 & s/6 & 0 & b & c_4 \\ c_2 & 0 & 0 & -s/12 & 0 & 0 \\ 0 & c_3 & b & 0 & -s/12 & 0 \\ 0 & b & c_4 & 0 & 0 & s/6 \end{bmatrix},$$

cf. (5.2), the matrices of $W^\pm$ in the bases (1.3) of $\Lambda^\pm$ are

$$(6.2) \quad \begin{bmatrix} \pm c_2 - s/12 & 0 & 0 \\ 0 & \pm c_3 - s/12 & \pm b \\ 0 & \pm b & \pm c_4 + s/6 \end{bmatrix},$$

and the matrix of $a = W(u^1 \otimes u^2 + u^2 \otimes u^1)$ in the basis $u_1, \dots, u_4$ is

$$(6.3) \quad \begin{bmatrix} 0 & s/12 & 0 & 0 \\ s/12 & 0 & 0 & 0 \\ 0 & 0 & 2b & c_4 - c_3 \\ 0 & 0 & c_4 - c_3 & -2b \end{bmatrix},$$

$u^1, \dots, u^4$ being the basis of $\mathcal{T}^*$ dual to $u_1, \dots, u_4$. By suitably rotating u_3, u_4 , while leaving u_1, u_2 and u^1, u^2 unchanged, we may require (Remark 2.2) that, in addition, $b = 0$ in (6.3). Now (6.2) and the last line of Lemma 5.1 establish Theorems 1.3 – 1.4 for **211**.

7. SELF-DUALITY IN TERMS OF QUATERNIONS

Since a connected Lie group has no open proper subgroups, a homomorphism between two connected Lie groups of the same dimension, having a finite kernel, must be a covering projection. Consequently, for the group $S^3 = \text{SU}(2) \subseteq \mathbb{H}$ of unit quaternions we have the two-fold covering homomorphisms

$$(7.1) \quad S^3 \longrightarrow \text{SO}(1^\perp), \quad S^3 \times S^3 \longrightarrow \text{SO}(\mathbb{H}), \quad \text{SO}(\mathbb{H}) \longrightarrow \text{SO}(1^\perp) \times \text{SO}(1^\perp)$$

given by $p \mapsto [1^\perp \ni x \mapsto px\bar{p} \in 1^\perp]$ and, respectively, $(p, q) \mapsto [\mathbb{H} \ni x \mapsto px\bar{q} \in \mathbb{H}]$, $\pm(p, q) \mapsto (\pm p, \pm q)$. Here $1^\perp \approx \mathbb{R}^3$ is the three-space of pure quaternions. (In fact, as the multiplicative center of $\mathbb{H}$ is $\mathbb{R}$, all three homomorphisms have the kernel $\mathbb{Z}_2$.)

The two-to-one homomorphisms (7.1) justify the notation, with $p, q \in S^3$,

$$(7.2) \quad \pm(p, q) \in \mathrm{SO}(\mathbb{H}), \quad \pm p \in \mathrm{SO}(1^\perp).$$

For an oriented Euclidean four-space $\mathcal{T}$ with the inner product g , we use the identification

$$(7.3) \quad \mathcal{T}^{\wedge 2} = \mathfrak{so}(\mathcal{T})$$

provided by g . Thus, $u \wedge v$, for $u, v \in \mathcal{T}$, becomes the endomorphism of $\mathcal{T}$ given by

$$(7.4) \quad w \mapsto (u \wedge v)w = g(u, w)v - g(v, w)u.$$

Both summands $\Lambda^\pm$ of $\mathcal{T}^{\wedge 2}$ are canonically oriented by the bases (1.3): as $\mathrm{SO}(4)$ is connected, these bases form connected sets.

Lemma 7.1. *Let $\mathcal{T}$ be an oriented Euclidean four-space.*

- (a) *Every pair of length $\sqrt{2}$ positive orthogonal bases of $\Lambda^\pm$ has the form (1.3) for a positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$, unique up to an overall sign change.*
- (b) *The action of $\mathrm{SO}(u^\perp)$ on bases $u_1, \dots, u_4$ in (a) having a fixed first vector $u_1 = u$ results in the standard $\mathrm{SO}(1^\perp)$ actions on the two bases (1.3), for the two signs $\pm$.*
- (c) *The $\mathrm{SO}(2) \times \mathrm{SO}(2)$ action on a positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$, which independently rotates the pairs u_1, u_2 and u_3, u_4 , leads to the $\mathrm{SO}(2) \times \mathrm{SO}(2)$ action on (1.3), independently rotating the last two bivectors.*

Proof. We may set $\mathcal{T} = \mathbb{H} = \mathbb{R}^4$. Due to (7.1), any positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$ equals $p\bar{q}, pi\bar{q}, pj\bar{q}, pk\bar{q}$ for a unique $\pm(p, q)$ with $p, q \in S^3$, cf. (7.2). The identification (7.4) clearly turns each $u_a \wedge u_b \pm u_c \wedge u_d$ in (1.3) into

$$(7.5) \quad \text{a complex structure in } \mathcal{T}, \text{ sending } u_a \text{ to } u_b \text{ and } u_c \text{ to } \pm u_d.$$

As a consequence of (7.5), with the identification (7.3),

$$(7.6) \quad \begin{aligned} &\text{the plus-sign (or, minus-sign) triple in (1.3) consists of} \\ &\text{the left quaternion multiplications by } pi\bar{p}, pj\bar{p}, pk\bar{p} \text{ (or,} \\ &\text{respectively, the right multiplications by } qi\bar{q}, qj\bar{q}, qk\bar{q}). \end{aligned}$$

Thus, for the last two-fold covering homomorphism $\pm(p, q) \mapsto (\pm p, \pm q)$ in (7.1),

$$(7.7) \quad \begin{aligned} &\text{the mapping that sends } u_1, \dots, u_4 \text{ to (1.3) is} \\ &\text{equivariant relative to } \pm(p, q) \mapsto (\pm p, \pm q). \end{aligned}$$

This proves (a). For (b), we may require that, under our identification $\mathcal{T} = \mathbb{H}$, the vector u_1 correspond to 1. The isotropy group $\mathrm{SO}(u^\perp)$ in (b) thus consists of $\pm(p, q) \in \mathrm{SO}(\mathbb{H})$ with $p = q$, and (b) follows from (7.7).

Finally, (c) becomes obvious if we note that replacing (p, q) with $(pe^{i\phi}, qe^{-i\theta})$, where $e^{i\phi} = \cos \phi + i \sin \phi$, subjects the first (or, second) pair in the quadruple $(p\bar{q}, pi\bar{q}, pj\bar{q}, pk\bar{q})$ to a rotation by the angle $\phi + \theta$ (or, $\phi - \theta$) in the plane spanned by it while, at the same time, the last two vectors in the triples $pi\bar{p}, pj\bar{p}, pk\bar{p}$ and $qi\bar{q}, qj\bar{q}, qk\bar{q}$ undergo rotations by the angles 2ϕ and -2θ . $\square$

Under the identifications (7.3) for $\mathcal{T} = \mathbb{H}$, (7.6) clearly implies that

$$(7.8) \quad \begin{aligned} &\Lambda^+ \text{ (or, } \Lambda^-) \text{ equals the space of all left (or, right)} \\ &\text{quaternion multiplications by pure quaternions.} \end{aligned}$$

8. STEP THREE: ONE TRIPLE AND ONE SIMPLE EIGENVALUE

The multiplicities in (1.5) are now **31**, and we are free to assume that

$$(8.1) \quad \mu_1 \neq \mu_2 = \mu_3 = \mu_4.$$

Lemma 4.1(b)–(c) gives $s = a_2 = a_3 = a_4 = 0$ in (2.5).

By default, this – combined with Sect. 3 and 6, completes the proof of Theorem 1.4.

According to (2.4), it also clearly yields

$$(W_{1212}, W_{1313}, W_{1414}, W_{3434}, W_{2424}, W_{2323}) = (0, 0, 0, 0, 0, 0),$$

while, by (2.2) – (2.3-a) and (8.1), for some b_2, b_3, b_4 ,

$$\begin{aligned} (W_{1242}, W_{1223}, W_{1323}) &= (W_{1334}, W_{1434}, W_{1442}) = (b_4, b_3, b_2), \\ (W_{1213}, W_{1214}, W_{1314}, W_{2324}, W_{2334}, W_{2434}) &= (0, 0, 0, 0, 0, 0). \end{aligned}$$

Thus, the matrix of W in the basis $u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4, u_3 \wedge u_4, u_4 \wedge u_2, u_2 \wedge u_3$ is

$$\begin{bmatrix} 0 & 0 & 0 & c_2 & b_4 & b_3 \\ 0 & 0 & 0 & b_4 & c_3 & b_2 \\ 0 & 0 & 0 & b_3 & b_2 & c_4 \\ c_2 & b_4 & b_3 & 0 & 0 & 0 \\ b_4 & c_3 & b_2 & 0 & 0 & 0 \\ b_3 & b_2 & c_4 & 0 & 0 & 0 \end{bmatrix},$$

cf. (5.2), and the matrices of $W^\pm$ in the bases (1.3) of $\Lambda^\pm$ are

$$\pm \begin{bmatrix} c_2 & b_4 & b_3 \\ b_4 & c_3 & b_2 \\ b_3 & b_2 & c_4 \end{bmatrix}.$$

We have the $\text{SO}(3)$ freedom of rotating u_2, u_3, u_4 , leading – see Lemma 7.1(b) – to the same simultaneous rotation of the bases (1.3) of $\Lambda^\pm$, which allows us to simultaneously diagonalize W^+ and W^- , with opposite spectra and, in view of Sect. 3 and 6, proves Theorem 1.3.

9. STEP FOUR: TWO DOUBLE EIGENVALUES

We may assume (2.7) with $\lambda = \mu > 0$. By (2.4) and Lemma 2.1(b),

$$(W_{1212}, W_{1313}, W_{1414}) = (W_{3434}, W_{2424}, W_{2323}) = (-s/12, \xi - s/12, -\xi + s/6)$$

for some ξ . From (2.2) – (2.3-a) and (2.7) with $\lambda = \mu > 0$, for some b_2, b_3 ,

$$\begin{aligned} W_{1323} = W_{1442} &= b_3, & W_{1314} = W_{2342} &= b_2, \\ (W_{1213}, W_{1214}, W_{1223}, W_{1224}, W_{1334}, W_{1434}, W_{2334}, W_{2434}) &= (0, 0, 0, 0, 0, 0, 0, 0). \end{aligned}$$

Thus, W has in the basis $u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4, u_3 \wedge u_4, u_4 \wedge u_2, u_2 \wedge u_3$ the matrix

$$\begin{bmatrix} -s/12 & 0 & 0 & c_2 & 0 & 0 \\ 0 & \xi - s/12 & b_2 & 0 & c_3 & b_3 \\ 0 & b_2 & -\xi + s/6 & 0 & b_3 & c_4 \\ c_2 & 0 & 0 & -s/12 & 0 & 0 \\ 0 & c_3 & b_3 & 0 & \xi - s/12 & b_2 \\ 0 & b_3 & c_4 & 0 & b_2 & -\xi + s/6 \end{bmatrix},$$

cf. (5.2), and the matrices of $W^\pm$ in the bases (1.3) of $\Lambda^\pm$ are

$$(9.1) \quad \begin{bmatrix} \pm c_2 - s/12 & 0 & 0 \\ 0 & \pm c_3 + \xi - s/12 & b_2 \pm b_3 \\ 0 & b_2 \pm b_3 & \pm c_4 - \xi + s/6 \end{bmatrix}.$$

We have the $\mathrm{SO}(2) \times \mathrm{SO}(2)$ freedom of independently rotating the pairs u_1, u_2 and u_3, u_4 . Lemma 7.1(c) now allows us to require that $b_2 = b_3 = 0$, which proves Theorem 1.5.

Finally, Theorem 1.1 is a trivial consequence of Theorems 1.3 – 1.5 for the multiplicities **1111**, **211**, **31**, **22** in (1.5), and of Lemma 7.1(a) in the case **4**.

10. KNOWN GEOMETRIC EXAMPLES

Due to (1.1), obvious examples of weakly Einstein Riemannian four-manifolds are provided by those having $e = 0$ (the Einstein ones) or $W = 0$ and $s = 0$ (the case of conformal flatness with zero scalar curvature). There are only three other, quite narrow, classes of examples known: the EPS space [9], certain special Kähler surfaces [6], and some warped products [7, Sect. 14–15].

The goal of this section is to explain where the examples of [6, 9] stand with regard to our algebraic classification. For the examples of [7], this question is settled in [7, Sect. 21].

In the case of the EPS space constructed by Euh, Park, and Sekigawa [9, Example 3.7], which – according to [7, formula (4.6)] – forms a single local-homothety type, R arises as in Theorem 1.4 from any given $\lambda > 0$ and $(s, \mu, c_2, c_3, c_4) = (-2\lambda, 0, 0, 0, 0)$. Thus, $W^\pm$ both have the same spectrum $(\lambda/6, \lambda/6, -\lambda/3)$. In fact, using the description of the curvature tensor R of the EPS space in [6, Sect. 3], and declaring our (u_1, u_2, u_3, u_4) and λ equal to their (e_1, e_3, e_4, e_2) and $2a^2$, one gets $R_{1212} = R_{1313} = R_{1414} = R_{2323} = -R_{2424} = -R_{3434} = -\lambda/2$, so that $s = -2\lambda$ and $u_1, \dots, u_4$ diagonalize the Einstein tensor e with the eigenvalues $(-\lambda, 0, 0, \lambda)$, while $c_2 = c_3 = c_4 = 0$.

The construction of weakly Einstein Kähler surfaces in [6, Sect. 12] uses orthogonal vector fields e_i . We choose u_i related to their e_i by $(u_1, u_2, u_3, u_4) = (\hat{e}_1, \hat{e}_3, \hat{e}_2, \hat{e}_4)$, with $\hat{e}_i = e_i/|e_i|$. Thus, $u_1, \dots, u_4$ diagonalize e with some eigenvalues $(-\lambda, -\lambda, \lambda, \lambda)$, while combining the final clause of Corollary 1.6 with formula (12.6) in [6, Sect. 12] we obtain, in our notation,

$$\begin{aligned} (s/4, -\xi, \xi - s/4) &= (c_2, c_3, c_4) = (W_{1234}, W_{1342}, W_{1423}) \\ &= (R_{1234}, R_{1342}, R_{1423}) = (2p\eta'\theta, -p\eta'\theta, -p\eta'\theta), \end{aligned}$$

so that $s = 8\xi = 8p\eta'\theta$. (The construction uses positive functions ζ, η of a real variable and nonzero constants p, θ .) Formula (12.6) of [6, Sect. 12] also gives, in our notation, $r_{11} = 2[3p\eta' - \zeta\theta\eta'']\theta$. Hence $e_{11} = r_{11} - s/4 = 2[2p\eta' - \zeta\theta\eta'']\theta$. Next, ζ has the constant derivative $-p/\theta$, and the function $Q = 4\zeta\eta\theta^2$, besides being positive, is only subject to a second-order linear ordinary differential equation [6, formulae (12.4) (12.9)], while $|\lambda| = |e_{11}|$. Consequently, a suitable choice of initial data yields the claim made in (1.7).

Remark 10.1. We now justify the claim made at the end of the Introduction. If R is the Kulkarni-Nomizu square of b , and b corresponds via g to a $(1, 1)$ tensor B , then r is analogously related to $(\mathrm{tr} B)B - B^2$ and, clearly, $e = 0$ when b is a multiple of g or has two mutually opposite eigenvalues of multiplicities **22**. In the case **31**, if an orthonormal basis diagonalizes b with the eigenvalues $\delta, \delta, \delta, -\delta$, the corresponding eigenvalues of r are $\delta^2, \delta^2, \delta^2, -3\delta^2$, and so $s = 0$, while the only possibly-nonzero components of R are, with

$i, j \in \{1, 2, 3\}$ assumed distinct, those algebraically related to $R_{ijij} = \delta^2$ and $R_{i4i4} = -\delta^2$, resulting in $W_{ijij} = R_{ijij} - (\delta^2 + \delta^2)/2 = 0$ and $W_{i4i4} = R_{i4i4} - (\delta^2 - 3\delta^2)/2 = 0$.

11. THE SELF-DUAL AND ANTI-SELF-DUAL CASES

Theorems 1.3 – 1.5 divide the algebraic types of non-Einstein, weakly Einstein algebraic curvature tensors in dimension four into three disjoint classes. This section provides a closer look at the self-dual and anti-self-dual components $W^\pm$ of the Weyl tensor in each class, focusing on the cases of self-duality ($W^+ = 0$) and anti-self-duality ($W^- = 0$). We use

$$(11.1) \quad \text{arbitrary parameters } c_2, c_3, c_4 \in \mathbb{R} \text{ such that } c_2 + c_3 + c_4 = 0.$$

The first class is defined by requiring the Ricci tensor not to have two double eigenvalues, and the scalar curvature s to be zero. The spectra of $W^\pm$ acting on self-dual and anti-self-dual bivectors are $\pm c_2, \pm c_3, \pm c_4$. Consequently, both self-duality and anti-self-duality amount in this case to conformal flatness.

In the second class the Ricci tensor, again, does not have two double eigenvalues, but now $s \neq 0$, and the eigenvalues of $W^\pm$ are $\pm c_2 - s/12, \pm c_3 - s/12, \pm c_4 + s/6$. Self-duality is realized by choosing $(c_2, c_3, c_4) = (-s/12, -s/12, s/6)$, anti-self-duality by $(c_2, c_3, c_4) = (s/12, s/12, -s/6)$, and so conformal flatness is not possible, as $s \neq 0$.

The third class uses an additional real parameter ξ , and the Ricci tensor now does have two double eigenvalues. The spectra of $W^\pm$ are $\pm c_2 - s/12, \pm c_3 + \xi - s/12, \pm c_4 - \xi + s/6$. (Anti)self-duality amounts to (c_2, c_3, c_4) being $(-s/12, \xi - s/12, -\xi + s/6)$ or, respectively, $(s/12, -\xi + s/12, \xi - s/6)$, so that either can be achieved by suitable choices of (11.1) and ξ , while conformal flatness amounts to $s = \xi = c_2 = c_3 = c_4 = 0$.

An analogous description for Kähler-type non-Einstein, weakly Einstein algebraic curvature tensors follows readily from Corollary 1.6: self-duality is equivalent to having $s = \xi = 0$, anti-self-duality to $s = 0$. As mentioned just before Corollary 1.6, this last equivalence, valid for all Kähler-surface metrics, has really nothing to do with the weakly Einstein property.

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