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Maximally-warped metrics with harmonic curvature

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Abstract. We describe the local structure of Riemannian manifolds with harmonic curvature which admit a maximum number, in a well-defined sense, of local warped-product decompositions, and at the same time their Ricci tensor has, at some point, only simple eigenvalues. We also prove that, in every given dimension greater than two, the local-isometry types of such manifolds form a finite-dimensional moduli space, and a nonempty open subset of this moduli space is realized by locally irreducible complete metrics which are neither Ricci-parallel, nor – for dimensions greater than three – conformally flat.

Introduction

A Riemannian manifold is said to have *harmonic curvature* [**1**, Sect. 16.33] if

$$\operatorname{div} R \,=\, 0, \quad \text{or, in local coordinates,} \quad R_{ijl}{}^{k}{}_{,k} = 0, \tag{0.1}$$

R being the curvature tensor. We consider harmonic-curvature Riemannian manifolds (M,g) of dimensions $n \geq 3$ in which, with r denoting the Ricci tensor,

$$\begin{aligned}&\mathrm{r}\ \text{has } n \text{ distinct eigenvalues at some point, and an open submani-}\\&\text{fold of } (M,g) \text{ admits a nontrivial warped-product decomposition.}\end{aligned} \tag{0.2}$$

Such warped-product decompositions have one-dimensional fibres (Corollary 1.3), and are in a one-to-one correspondence with certain one-dimensional Lie subalgebras of $\mathfrak{isom}(M',g')$, the Lie algebra of Killing fields on the Riemannian universal covering (M',g') of (M,g) (see Remark 3.2). The number γ of these subalgebras cannot exceed $n-1$, cf. Corollary 4.2, and we refer to g as *maximally warped* if

$$\gamma \,=\, n-1. \tag{0.3}$$

Our Theorem 5.6 describes the local structure of Riemannian manifolds (M,g) of dimensions $n \geq 3$, satisfying (0.1) – (0.3). Their local-isometry types turn out to form a $(2n-3)$-dimensional moduli space (Remark 5.7), and we prove (in Theorem 7.3) that some nonempty open subset of the moduli space consists of local-isometry types of such manifolds which in addition are

$$\begin{aligned}&\text{complete, locally irreducible, and neither con-}\\&\text{formally flat (unless } n=3\text{), nor Ricci-parallel.}\end{aligned} \tag{0.4}$$

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We do not know if any compact manifold can have the properties (0.1) – (0.3). However, we observe (see Theorem 1.4) that compact Riemannian manifolds with harmonic curvature that admit *global* nontrivial warped-product decompositions must have fibre dimensions greater than one and, consequently, cannot be *Ricci-generic* in the sense of satisfying the distinct-eigenvalues clause of (0.2).

Harmonicity of the curvature always follows if the metric is *Ricci-parallel*, or *locally reducible with harmonic-curvature factors*, or *conformally flat and of constant scalar curvature* while, in dimension three, harmonic curvature amounts to conformal flatness plus constancy of the scalar curvature [**1**, Sect. 16.35 and 16.4].

Compact Riemannian manifolds with (0.1) have been studied extensively. All their known examples (aside from the three classes italicized above), listed as 2,3,4 in [**1**, p. 432], admit – at least locally – nontrivial warped-product decompositions with a fibre of dimension greater than one. Consequently (see Corollary 1.3 below), they are not Ricci-generic. However, for examples 2 and 4 of [**1**, p. 432] the warped-product structure, rather than being an Ansatz, is a consequence of geometric conditions involving multiplicities of eigenvalues of the Ricci tensor [**2**] or self-dual Weyl tensor [**3**]. The following questions about compact Riemannian manifolds with harmonic curvature, lying outside of the three italicized classes, are thus open and natural: can they be Ricci-generic? must they, locally, have a warped-product decomposition and if not, how to describe those among them which have one?

Our Theorem 7.3 yields an affirmative answer to a weaker version of the first question in which completeness replaces compactness. On the other hand, the second question provides an obvious motivation for studying condition (0.3).

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1. Preliminaries

Manifolds (always assumed connected), mappings and tensor fields are by definition C^∞-differentiable. By a *Codazzi tensor* [**1**, p. 435] on a Riemannian manifold one means a twice-covariant symmetric tensor field S with a totally symmetric covariant derivative ∇S. One then has two well-known facts [**1**, Sect. 16.4(ii)]:

(1.1)
i) $\operatorname{div} R = 0$ if and only if r is a Codazzi tensor,
ii) the condition $\operatorname{div} R = 0$ implies constancy of s,

s being the scalar curvature. As shown by DeTurck and Goldschmidt [**4**],

(1.2) metrics with $\operatorname{div} R = 0$ are real-analytic in suitable local coordinates.

We call two (connected) real-analytic Riemannian manifolds *locally isometric* if they have open submanifolds that are both isometric to open submanifolds of a third such manifold. One easily sees that this is an equivalence relation. In view of the extension theorem for analytic isometries [**9**, Corollary 6.4 on p. 256], for two complete real-analytic Riemannian manifolds, being locally isometric to each other means the same as having isometric Riemannian universal coverings.

On a manifold with a torsion-free connection ∇, the Ricci tensor r satisfies the Bochner identity $\mathrm{r}(\,\cdot\,, v) + d[\operatorname{div} v] = \operatorname{div} \nabla v$, where v is any vector field. Its coordinate form $R_{jk}v^k = v^k{}_{,jk} - v^k{}_{,kj}$ arises via contraction from the Ricci identity $v^l{}_{,jk} - v^l{}_{,kj} = R_{jkq}{}^l v^q$. (We use the sign convention for R such that $R_{jk} = R_{jqk}{}^q$.)

Applied to the gradient v of a function ϕ on a Riemannian manifold, this yields

$$\mathrm{r}(\nabla\phi,\,\cdot\,) \,+\, d\Delta\phi \,=\, \mathrm{div}[\nabla d\phi]. \tag{1.3}$$

The *warped product* of Riemannian manifolds $(\overline{M},\bar{g})$ and (Σ,η) with the *warping function* $\phi:\overline{M}\to(0,\infty)$ is the Riemannian manifold

$$(M,g) \,=\, (\overline{M}\times\Sigma,\, \bar{g}+\phi^2\eta). \tag{1.4}$$

(The same symbols $\bar{g},\eta,\phi$ stand here for also the pullbacks of $\bar{g},\eta,\phi$ to the product $M=\overline{M}\times\Sigma$.) One calls $(\overline{M},\bar{g})$ and (Σ,η) the *base* and *fibre* of (1.4), and refers to (1.4) as *nontrivial* if ϕ is nonconstant. From now on we assume that $\dim\Sigma\ge 1$.

REMARK 1.1. As $\bar{g}+\phi^2\eta \,=\, \phi^2[\phi^{-2}\bar{g}+\eta]$, a warped product is nothing else than a Riemannian manifold conformal to a Riemannian product via multiplication by a positive function which is constant along one of the factor manifolds.

A proof of the following well-known lemma [**7**] is given in the Appendix.

LEMMA 1.2. *A warped product* (1.4) *with a nonconstant warping function* ϕ *has harmonic curvature if and only if the Levi-Civita connection* $\overline{\nabla}$ *of* $(\overline{M},\bar{g})$*, its Ricci tensor* $\bar{\mathrm{r}}$ *and the* $\bar{g}$*-gradient* $\overline{\nabla}\phi$ *of* ϕ *satisfy three conditions*:

(a) (Σ,η) *is an Einstein manifold, with some Einstein constant* κ.

(b) $\bar{\mathrm{r}}-p\phi^{-1}\overline{\nabla}d\phi$ *is a Codazzi tensor on* $(\overline{M},\bar{g})$*, where* $p=\dim\Sigma\ge 1$.

(c) $\phi^3\,\overline{\mathrm{div}}[\phi^{-1}\overline{\nabla}d\phi]=[(p-1)\Lambda-\kappa]\,d\phi+(1-p)\phi\,d\Lambda/2$*, with* $\Lambda=\bar{g}(\overline{\nabla}\phi,\overline{\nabla}\phi)$

and the $\bar{g}$*-divergence* $\overline{\mathrm{div}}$*. Then, at each point of* M*, for the Ricci tensor* r *of* g*,*

(d) *the space tangent to the fibre factor is contained in an eigenspace of* r.

One may rewrite (c) *as a requirement involving the* $\bar{g}$*-Laplacian* $\overline{\Delta}\phi$*, namely,*

(e) $\phi^2[\bar{\mathrm{r}}(\overline{\nabla}\phi,\,\cdot\,)+d\overline{\Delta}\phi]=[(p-1)\Lambda-\kappa]\,d\phi+(1-p/2)\phi\,d\Lambda$.

Finally, when $p=1$*, and so* $\kappa=0$*,* (c) *reads* $\overline{\mathrm{div}}[\phi^{-1}\overline{\nabla}d\phi]=0$.

From (d) and, respectively, (c), we obtain two easy consequences:

COROLLARY 1.3. *In a warped-product Riemannian manifold with a nonconstant warping function, harmonic curvature, and a fibre of dimension greater than one, the Ricci tensor has, at every point, at least one multiple eigenvalue. The assumptions* (0.1) – (0.2) *thus imply one-dimensionality of the fibre for any warped-product decomposition in* (0.2).

THEOREM 1.4. *If a warped-product Riemannian manifold* (M,g) *has a compact base* $(\overline{M},\bar{g})$*, a nonconstant warping function, and harmonic curvature, then the Einstein constant* κ *of its fibre must be positive. Thus, the dimension of the fibre is greater than one and, in view of Corollary* 1.3*,* (M,g) *cannot satisfy the distinct-eigenvalues clause of* (0.2).

In fact, given a positive function ϕ on a Riemannian manifold $(\overline{M},\bar{g})$ and constants $\kappa,p\in\mathrm{I\!R}$, let us set $v=\overline{\nabla}\phi$ and $w=\phi^2u-[(p-1)\Lambda-\kappa]\,v+(p-2)\phi\overline{\nabla}_{\!v}v$, where $u=\overline{\mathrm{div}}\,\overline{\nabla}v$ and $\Lambda=\bar{g}(v,v)$. Then the function $\phi^{p-4}\bar{g}(v,w)$ differs by a $\bar{g}$-divergence from $-[(p-1)\Lambda-\kappa]\phi^{p-4}\Lambda-\phi^{p-2}\bar{g}(\overline{\nabla}v,\overline{\nabla}v)$, while (c) reads $w=0$, as $2\overline{\nabla}_{\!v}v=\overline{\nabla}\Lambda$. (An easy exercise.)

REMARK 1.5. The base and fibre factor distributions of any warped product are Ricci-orthogonal to each other. (See the equality $R_{ia}=0$ in formula (A.3) of the Appendix.) Thus, if the base, or fibre, is one-dimensional, nonzero vectors tangent to it constitute eigenvectors of the Ricci tensor.

2. Vector fields

LEMMA 2.1. *Let a maximal integral curve* $(a_-, a_+) \ni t \mapsto x(t)$ *of a vector field* v *on a manifold* M, *with* $-\infty \le a_- < a_+ \le \infty$, *some* $t' \in (a_-, a_+)$, *and some compact set* $C \subseteq M$, *have the property that* $x(t) \in C$ *for all* $t \in [t', a_+)$ *or, respectively, for all* $t \in (a_-, t']$. *Then* $a_+ = \infty$ *or, respectively,* $a_- = -\infty$.

Consequently, a maximal integral curve of a vector field on a manifold, lying within a compact set, must be complete, that is, defined on $\mathrm{I\!R}$.

PROOF. For a compactly supported function χ equal to 1 on an open set U containing C, the curve restricted to $[t', a_+)$, or to $(a_-, t']$, clearly remains half-maximal (not extendible beyond a_+, or a_-) when treated as an integral curve of χv. On the other hand, χv is complete due to compactness of its support. □

By a *section* of a locally trivial fibre bundle we mean, as usual, a submanifold Σ of the total space Q mapped diffeomorphically onto the base M by the bundle projection p. We also identify the section with the inverse $\psi : M \to \Sigma$ of the latter diffeomorphism, which makes it a mapping $\psi : M \to Q$ having $\mathrm{p} \circ \psi = \mathrm{Id}_M$. In the case of a vector bundle Q, a section ψ, and a zero $z \in M$ of ψ, the corresponding submanifold Σ of Q intersects the zero section M at z (that is, at the zero vector of the fibre Q_z), giving rise to the *differential* $\partial\psi_z$, defined to be the linear operator $T_zM \to Q_z$ obtained as the composite of the ordinary differential of $\psi : M \to Q$ at z (the inverse of $d\mathrm{p}_z : T_z\Sigma \to T_zM$), followed by the direct-sum projection $T_zQ = T_zM \oplus Q_z \to Q_z$. Relative to any local coordinates at z and a local trivialization of Q, the components of $A = \partial\psi_z$ form the matrix $[A_j^\lambda] = [\partial_j\psi^\lambda]$, with the partial derivatives of the components of ψ evaluated at z.

Two important examples are provided by zeros z of $\psi = v$, a vector field on M (with $Q = TM$) and of $\psi = df$, for a function $f : M \to \mathrm{I\!R}$ (here $Q = T^*M$). In the former case, $A = \partial v_z$ (in coordinates: $A_j^k = \partial_j v^k$), is the infinitesimal generator of the one-parameter group of linear transformations of T_zM arising as the differentials, at the fixed point z, of the local diffeomorphisms forming the local flow of v. In the latter, $\partial df_z = \mathrm{Hess}_z f$, the Hessian of f at the critical point z.

Let v be a vector field on a manifold M, having a zero at $z \in M$, where one assumes M either to be an open submanifold of a vector space Y, or to have a submanifold N with $z \in N$ such that v is tangent to N at each point of N. In this way v, or the restriction of v to N, becomes a mapping $v : M \to Y$, or a vector field w on N. The equality $A_j^k = \partial_j v^k$ of the last paragraph, evaluated in coordinates for M which are linear functionals on Y or, respectively, in which N is defined by equating some coordinate functions to 0, clearly implies that

(2.1) i) $\partial v_z = dv_z : Y \to Y$, ii) ∂w_z equals the restriction of ∂v_z to T_zN.

LEMMA 2.2. *Given a zero* $z \in M$ *of a vector field* v *on a manifold* M, *with the differential* $A = \partial v_z$, *let a function* $f : U \to \mathrm{I\!R}$ *on a neighborhood* U *of* z *have* $df_z = 0$. *Then* $d\sigma_z = 0$ *and* $(u,u)_\sigma = 2(Au,u)_f$ *for the directional derivative* $\sigma = d_v f : U \to \mathrm{I\!R}$, *all* $u \in T_zM$, *the Hessian* $(\,,\,)_f = \mathrm{Hess}_z f$, *and* $(\,,\,)_\sigma = \mathrm{Hess}_z\sigma$.

PROOF. With commas denoting, this time, partial derivatives relative to fixed local coordinates on a neighborhood of z, we have $\sigma = v^j f_{,j}$ as well as $\sigma_{,k} = v^j f_{,jk} + v^j{}_{,k} f_{,j}$ and $\sigma_{,kl} = v^j f_{,jkl} + v^j{}_{,l} f_{,jk} + v^j{}_{,k} f_{,jl} + v^j{}_{,kl} f_{,j}$. At z, both v^j and $f_{,j}$ vanish, while $v^j{}_{,k} = A_k^j$. This proves our claim. □

LEMMA 2.3. *Let $z \in N$ be a zero of a vector field w on a manifold N such that, for some $\varepsilon = \pm 1$, some Euclidean inner product $\langle\,,\rangle$ in T_zN, and $A = \varepsilon\partial w_z$, the bilinear form $\langle A\cdot\,,\,\cdot\rangle$ on T_zN is negative definite. In this case there exist arbitrarily small neighborhoods U of z with the following property: if a maximal integral curve $(a_-,a_+) \ni t \mapsto x(t)$ of w and $t' \in (a_-,a_+)$ satisfy the condition $x(t') \in U$, where $-\infty \le a_- < a_+ \le \infty$, then, denoting by $\pm$ the sign of ε, one has $a_\pm = \pm\infty$, and $x(t) \in U$ whenever $\varepsilon(t-t') \ge 0$.*

PROOF. We fix a Riemannian metric g on a neighborhood of z in N having $\langle\,,\rangle = g_z$. The required neighborhoods U of z are g-metric balls centered at z, small enough so as to have compact closures and be diffeomorphic images, under the g-exponential mapping at z, of the corresponding Euclidean balls around 0 in T_zN. This gives smoothness of the function $f : U \to \mathrm{I\!R}$ such that $2f$ equals $\mathrm{dist}^2(z,\,\cdot\,)$, the squared g-distance from z, and using normal coordinates one obtains $\mathrm{Hess}_z f = \langle\,,\rangle$. If the g-metric ball U is sufficiently small, Lemma 2.2 for $v = \varepsilon w$ implies negativity of $\sigma = \varepsilon d_w f$ on $U \smallsetminus \{z\}$, as σ assumes at z the critical value 0 with a negative-definite Hessian. Our claim now easily follows from Lemma 2.1. □

REMARK 2.4. *The same* neighborhoods U of z will still satisfy the assertion of Lemma 2.3 if one replaces w by w/c for a constant $c > 0$ and $(a_-,a_+) \ni t \mapsto x(t)$ by $(ca_-,ca_+) \ni t \mapsto x(t/c)$.

REMARK 2.5. For any Killing field v on a Riemannian manifold (M,g), the pair $(v,\nabla v)$ constitutes a parallel section of the vector bundle $TM \oplus \mathfrak{so}(TM)$ endowed with a suitable linear connection [**5**, Remark 17.25 on p. 547]. Therefore,

(i) a Killing field on M is uniquely determined by its restriction to any nonempty open subset of M, while

(ii) assuming (M,g) to be simply connected and real-analytic, we conclude that any Killing field v on a nonempty connected open subset of M has a unique extension to a Killing field on M.

Given a nontrivial Killing vector field v on a Riemannian manifold and a function θ, the obvious equality $\pounds_{\theta v} g = \theta\pounds_v g + 2d\theta \odot g(v,\,\cdot\,)$ clearly implies that

(2.2) if θv is also a Killing field, θ must be constant, cf. Remark 2.5(i).

3. Integrable-complement Killing fields

This section presents a well-known correspondence – see, for instance, the Appendix in [**10**] – between warped-product decompositions with a one-dimensional fibre and certain special Killing fields.

Let v a nontrivial Killing field on a Riemannian manifold (M,g) such that, on the dense (by Remark 2.5(i)) complement of its zero set, the distribution $v^\perp$ is integrable. In other words, locally, at points with $v \ne 0$, multiplying v by a suitable positive function one obtains a gradient vector field. Equivalently,

(3.1) the 1-form $g(v,\,\cdot\,)/g(v,v)$, defined wherever $v \ne 0$, is closed.

Namely, (3.1) is necessary: for $\xi = g(v,\,\cdot\,)$, due to skew-symmetry of $\nabla\xi$, the integrability condition $\xi \wedge d\xi = 0$ has the local-coordinate expression $\xi_{i,j}\xi_k + \xi_{j,k}\xi_i + \xi_{k,i}\xi_j = 0$, which transvected with v^k yields $v^k\xi_k\xi_{i,j} = v^k\xi_{k,j}\xi_i - v^k\xi_{k,i}\xi_j$, or

$$2\beta\xi_{i,j} = \beta_{,j}\xi_i - \beta_{,i}\xi_j, \quad \text{where } \beta = v^k\xi_k = g(v,v). \tag{3.2}$$

Closedness of ξ/β amounts to symmetry of $\nabla(\xi/\beta)$, and so it now follows since (3.2) with $\xi_{i,j} = -\xi_{j,i}$ implies symmetry of $\beta^2(\xi_i/\beta)_{,j} = \beta\xi_{i,j} - \beta_{,j}\xi_i$ in i, j.

If v is a Killing field, $g(v, \dot{x})$ is constant along any geodesic $t \mapsto x = x(t)$, as $d[g(v, \dot{x})]/dt = g(\nabla_{\dot{x}} v, \dot{x}) = 0$. Then, with the orthogonal complement $v^\perp$ only defined away from the zero set of v, one easily sees that

(3.3) v is orthogonal to any geodesic passing through a zero of v, while whenever (3.1) holds, the distribution $v^\perp$ has totally geodesic leaves.

Remark 3.1. Local Killing fields v satisfying (3.1), outside of their zero sets, if treated as defined only up to multiplication by nonzero constants, stand in a natural one-to-one correspondence with local warped-product decompositions of g that have a one-dimensional fibre. Here v is tangent to the fibre direction.

Namely, such a local decomposition is uniquely determined by the base and fibre factor distributions. Just one of them suffices, the other being its (necessarily integrable) orthogonal complement. That v locally spans the fibre factor distribution of a warped product follows from Remark 1.1 and the local version of de Rham's decomposition theorem: in view of (3.2), rewritten as $2\beta v^i{}_{,j} = v^i\beta_{,j} - \beta^{,i}\xi_j$, where $\beta = v^k\xi_k = g(v, v)$, and [**1**, Theorem 1.159], v is $\hat{g}$-parallel for the conformally related metric $\hat{g} = g/\beta$, with $d_v\beta = 2g(\nabla_v v, v) = 0$ due to skew-adjointness of ∇v. Conversely, for a warped product with a one-dimensional fibre, the required Killing field v comes from a local flow of local isometries of the fibre (cf. formula (A.2) in the Appendix), (2.2) implying uniqueness of v up to a constant factor.

Remark 3.2. From Remarks 2.5 and 3.1 it follows that, in the case of a real-analytic Riemannian manifold (M, g), denoting by $\mathfrak{isom}(M', g')$ the Lie algebra of Killing fields on the Riemannian universal covering (M', g') of (M, g), one has a natural bijective correspondence between the one-dimensional Lie subalgebras of $\mathfrak{isom}(M', g')$ spanned by Killing fields v satisfying (3.1), and the local warped-product decompositions, with one-dimensional fibres, of g restricted to the dense open set where $v \neq 0$. As before, v is tangent to the fibre direction.

Lemma 3.3. *Let an open ball B around 0 in a Euclidean n-space, $n \geq 2$, admit a connection ∇ such that all line segments through 0 in B are ∇-totally geodesic and tangent at all points $x \in B \smallsetminus \{0\}$ to some codimension-one foliation $\mathcal{F}$ on $B \smallsetminus \{0\}$ having ∇-totally geodesic leaves. Then $n = 2$.*

Proof. Fix a leaf L of $\mathcal{F}$ and $x \in L$ such that the ∇-exponential mapping $\exp_x$ sends a Euclidean open ball B' centered at 0 in T_xB, diffeomorphically, onto a neighborhood $\exp_x(B')$ of 0 in B. Thus, $\exp_x(B' \cap T_xL) \smallsetminus J \subseteq L$, for $J = B \cap \{qx : q \in (-\infty, 0]\}$, and so $J \subseteq L$. (The leaves of $\mathcal{F}$ are locally closed, being, locally, the level sets of a submersion.) Hence $L \cup \{0\}$ is a smooth ∇-totally geodesic submanifold of B, with some tangent space V at 0, meaning in turn that $L \cup \{0\} = B \cap V$. Consequently, $n = 2$, for otherwise any two such codimension-one subspaces V of our Euclidean n-space would have a nontrivial intersection. □

When $\mathcal{F}$ is real-analytic, we can also obtain the above assertion by applying, to a sphere Σ around 0 in B, Haefliger's theorem [**6**] which states that a transversally orientable real-analytic codimension-one foliation may exist on a compact manifold Σ only if the fundamental group of Σ has an element of infinite order.

REMARK 3.4. Kobayashi [**8**] showed that the zero set of any Killing vector field on a Riemannian manifold (M, g) is either empty, or its connected components are mutually isolated totally geodesic submanifolds of even codimensions.

For a nontrivial Killing field v with (3.1), *the above codimensions must all equal* 2. This is immediate if one fixes a zero z of v and applies Lemma 3.3 to a small ball B in the normal space at z of the connected component through z such that $\exp_z$ maps B diffeomorphically onto a submanifold N of M, with ∇ and $\mathcal{F}$ denoting the $\exp_z$-pullback of the Levi-Civita connection of the submanifold metric h on N and, respectively, of the foliation on $N \smallsetminus \{z\}$ the leaves of which are intersections of N and the leaves of $v^\perp$, the latter defined wherever $v \neq 0$. (The local flow of v preserves N and h, and so v is tangent to N.) The restriction of v to N now constitutes an h-Killing field w having just one zero, at z, and satisfying (3.1) (for h, w rather than g, v), so that (3.3) allows us to use Lemma 3.3.

4. Multiply-warped metrics with $\operatorname{div} R = 0$

LEMMA 4.1. *Suppose that the Ricci tensor of a real-analytic Riemannian n-manifold (M, g) has n distinct eigenvalues at some point and, with the notation of Remark* 3.2, $\mathfrak{a}_2, \dots, \mathfrak{a}_m$ *are distinct one-dimensional Lie subalgebras of* $\mathfrak{isom}(M', g')$ *spanned by Killing fields* $v_2, \dots, v_m$ *such that each* $v = v_j$ *satisfies* (3.1). *Then* $m \leq n$, *and* $g(v_j, v_k) = 0$ *as well as* $[v_j, v_k] = 0$ *if* $j \neq k$. *Finally,* $g(\nabla_u v_j, v_k) = 0$ *whenever* $j, k, l \in \{2, \dots, m\}$ *and* $u = v_l$.

PROOF. Remarks 3.2 and 1.5 imply that all v_j, wherever nonzero, are mutually nonproportional eigenvectors of the Ricci tensor, which makes them pointwise orthogonal to one another, as well as invariant, up to constant factors – by (2.2) – under each other's local flows. Thus, $m \leq n+1$ and, as $[v, w] = \pounds_v w$ for $v = v_j$ and $u = v_k$, one gets $[v_j, v_k] = c v_k$ with some constant c depending on j and k. Switching j and k, we see that $c = 0$. Now let $u = v_l$, $v = v_j$ and $u = v_k$, where $j, k, l \in \{2, \dots, m\}$. We have $g(\nabla_u v, w) = 0$ if $u = w$ (due to the Killing property of v) and, therefore, also when $v = w$ (since u, v commute). Also, $g(\nabla_u v, w) = 0$ in the remaining case, with u, v different from w (and hence orthogonal to w): as a consequence of (3.3), outside of the zero set of w the distribution $w^\perp$ has totally geodesic leaves. This proves the final claim of the lemma, implying in turn that, if one had $m = n+1$, all v_j would be parallel, leading to flatness of g, and contradicting the Ricci-eigenvalues assumption. □

Due to DeTurck and Goldschmidt's real-analyticity theorem (1.2), we may combine Lemma 4.1 with Remark 3.2 and Corollary 1.3, obtaining

COROLLARY 4.2. *Under the assumptions* (0.1) – (0.2)*, the integer* γ *defined in the Introduction does not exceed* $n-1$.

5. The local structure

Given an open interval $I \subseteq \mathrm{I\!R}$, we introduce a Riemannian metric g on the open set $I \times \mathrm{I\!R}^{n-1} \subseteq \mathrm{I\!R}^n$, $n \geq 2$, by declaring its component functions in the Cartesian coordinates $x^1, x^2, \dots, x^n$ to be

$$(5.1) \qquad g_{kl} = 0 \text{ if } k \neq l, \quad g_{11} = 1, \quad g_{jj} = g_{jj}(t) \text{ for } t = x^1 \text{ and } j \geq 2,$$

where $I \ni t \mapsto (g_{22}(t), \ldots, g_{nn}(t)) \in (0,\infty)^{n-1}$ is any prescribed smooth curve. We also define the functions $y_2, \ldots, y_n$ and $\mathbf{y} = \operatorname{diag}(y_2, \ldots, y_n)$ of the variable $t \in I$, valued in $\mathrm{I\!R}$ and, respectively, in the real vector space $\mathrm{I\!E} \cong \mathrm{I\!R}^{n-1}$ of all diagonal $(n-1) \times (n-1)$ matrices, by

$$2y_j g_{jj} = -\dot{g}_{jj} \quad \text{(no summation), with } (\;)^{\dot{}} = d/dt. \tag{5.2}$$

Remark 5.1. If $I = \mathrm{I\!R}$ while $\mp y_j(t) \geq \delta$ whenever $\pm t$ is sufficiently large and positive, for both signs $\pm$, some constant $\delta > 0$, and all $j \geq 2$, then the above metric g is complete. In fact, (5.2) gives $\log g_{jj}(t) \to \infty$ as $|t| \to \infty$, so that $g_{jj}(t) \geq a$ with some constant $a \in (0,1]$ and all $t \in \mathrm{I\!R}$, which in turn gives $g \geq ag'$ (positive semidefiniteness of $g - ag'$) for the standard Euclidean metric g'. Completeness of g' now implies that of g, as g-bounded sets have compact closures due to the resulting inequality $\operatorname{dist} \geq a \operatorname{dist}'$ between distance functions.

Let us consider the second-order autonomous ordinary differential equation

$$\ddot{\mathbf{y}} - (\operatorname{tr}\mathbf{y} + \mathbf{y})\dot{\mathbf{y}} = (\operatorname{tr}\mathbf{y}^2)\mathbf{y} - (\operatorname{tr}\mathbf{y})\mathbf{y}^2 \tag{5.3}$$

imposed on a C^2 curve $I \ni t \mapsto \mathbf{y} \in \mathrm{I\!E}$, in which $\mathbf{y}\dot{\mathbf{y}}$ and $\mathbf{y}^2 = \mathbf{y}\mathbf{y}$ represent diagonal-matrix products, while $\operatorname{tr}\mathbf{y}$ also denotes $\operatorname{tr}\mathbf{y}$ times the identity.

Lemma 5.2. *For a metric g on $I \times \mathrm{I\!R}^{n-1}$ defined by* (5.1) *and the corresponding curve $I \ni t \mapsto \mathbf{y} = \operatorname{diag}(y_2, \ldots, y_n)$ with* (5.2), *at every point $(t, \mathbf{x}) \in I \times \mathrm{I\!R}^{n-1}$, each of the coordinate vectors ∂_k, $k = 1, \ldots, n$, is an eigenvector of the Ricci tensor of g with an eigenvalue μ_k depending on t.*

(a) *Specifically, $\mu_1 = \operatorname{tr}\dot{\mathbf{y}} - \operatorname{tr}\mathbf{y}^2$ and $\mu_j = \dot{y}_j - y_j \operatorname{tr}\mathbf{y}$ if $j \geq 2$.*
(b) *The scalar curvature s of g equals $2\operatorname{tr}\dot{\mathbf{y}} - \operatorname{tr}\mathbf{y}^2 - (\operatorname{tr}\mathbf{y})^2$.*
(c) *$\partial_2, \ldots, \partial_n$ are g-Killing fields with integrable orthogonal complements.*
(d) *Given any fixed $\mathbf{x} \in \mathrm{I\!R}^{n-1}$, the curve $I \ni t \mapsto (t, \mathbf{x})$ is a g-geodesic.*
(e) *g has harmonic curvature if and only if* (5.3) *holds.*

Proof. We assume j, k, l to range over $\{2, \ldots, n\}$ and be mutually distinct. Repeated indices are *not* summed over. First, (c) is obvious as $g_{11}, g_{1j}, g_{jj}, g_{jk}$ only depend on $t = x^1$. Also, $\Gamma_{11}^1 = \Gamma_{11}^j = 0$, proving (d), while $\Gamma_{1j}^1 = \Gamma_{1j}^k = \Gamma_{jj}^j = \Gamma_{jj}^k = \Gamma_{jk}^j = \Gamma_{jk}^l = 0$ and $g^{jj}\Gamma_{jj}^1 = -\Gamma_{1j}^j = y_j$. Hence $R_{11} = \mu_1$ and $g^{jj}R_{jj} = \mu_j$ for μ_1, μ_j as in (a). This yields (a), and hence (b). (Each ∂_k spans the fibre direction of a warped-product decomposition, and we may use Remark 1.5.) Next, $R_{11,j} = R_{1j,1} = R_{1j,k} = R_{jk,1} = R_{jk,j} = R_{jj,k} = R_{jk,l} = 0$. Finally, $g^{jj}R_{j1,j} = y_j(\mu_j - \mu_1)$ and $g^{jj}R_{jj,1} = \dot{\mu}_j$, so that (1.1.i) implies (e), □

We refer to a solution $I \ni t \mapsto \mathbf{y} \in \mathrm{I\!E}$ of (5.3) as *maximal* if it cannot be extended to a larger open interval, and call it *Ricci-generic* whenever the n values $\mu_k = \mu_k(t)$ of Lemma 5.2(a) are all distinct at some $t \in I$ (or, equivalently, no two among the functions $\mu_1, \ldots, \mu_n$ coincide everywhere in I).

Example 5.3. Two non-Ricci-generic maximal solutions of (5.3) are defined by $\mathbf{y} = -2\tanh nt$ and $\mathbf{y} = 2\tan nt$ (times the identity $\mathbf{1}$), with $I = \mathrm{I\!R}$ or $I = (-\pi/(2n), \pi/(2n))$. In fact, $2\dot{\mathbf{y}} = n(\mathbf{y}^2 \mp 4)$ and so $\ddot{\mathbf{y}} = n\mathbf{y}\dot{\mathbf{y}}$, while for multiples $\mathbf{y}$ of $\mathbf{1}$ the right-hand side of (5.3) vanishes and $\operatorname{tr}\mathbf{y} + \mathbf{y} = n\mathbf{y}$.

Example 5.4. Any solution $\mathbf{y} = \operatorname{diag}(y_2, \ldots, y_n)$ of (5.3), where $n \geq 2$, can be *trivially extended* to the solution $\operatorname{diag}(y_2, \ldots, y_n, 0, \ldots, 0)$ with a number

$m > 0$ of additional zero components. The new metric defined using (5.1) – (5.2) is isometric to the Riemannian product of the original g and a flat metric on $\mathrm{I\!R}^m$.

The set of maximal solutions of (5.3) is obviously preserved by the group K acting on it via replacement of $\mathbf{y}$ with $t \mapsto \pm\mathbf{y}(b \pm t)$, where $b \in \mathrm{I\!R}$ and $\pm$ is either sign, combined with permutations of the components $y_2, \dots, y_n$. We will use the term K-*equivalence* when two maximal solutions lie in the same K-orbit.

REMARK 5.5. Nonzero real numbers a act on maximal solutions $t \mapsto \mathbf{y}(t)$ of (5.3) by sending them to $t \mapsto a\mathbf{y}(at)$. (The new metric arising via (5.1) – (5.2) is isometric to g/a^2.) The group K defined above, obviously isomorphic to the direct product of the isometry group of $\mathrm{I\!R}$ and the symmetric group S_{n-1}, along with the multiplicative group $\mathrm{I\!R} \smallsetminus \{0\}$ acting as described here, together generate an action of a semidirect product of K and $(0, \infty)$.

THEOREM 5.6. *For any* $n \geq 3$, *the construction summarized by* (5.1) – (5.2) *provides a bijective correspondence between two sets consisting, respectively, of*

(i) *all* K*-equivalence classes of maximal Ricci-generic solutions to* (5.3), *and*
(ii) *all local-isometry types of Riemannian* n*-manifolds with* (0.1) – (0.3).

For the meaning of local-isometry types, see (1.2) *and the paragraph following it.*

PROOF. We need to show that the mapping from (i) to (ii) is: (A) well-defined, (B) injective, and (C) surjective.

Part (A) easily follows from Lemma 5.2 combined with the comment on g/a^2 in Remark 5.5, the latter applied to $a = \pm 1$. To obtain (B), note that the local-isometry type of a metric g arising from (5.1) – (5.3) determines the K-equivalence class of the maximal Ricci-generic solution $t \mapsto \mathbf{y}$ of (5.3). Namely, the g-Killing fields $\partial_2, \dots, \partial_n$, valued in eigenvectors of the Ricci tensor of g (see Lemma 5.2), are – due to the Ricci-generic condition and (2.2) – unique up to permutations and multiplication by nonzero constants, which makes $y_2, \dots, y_n$, defined by (5.1) with $g_{jj} = g(\partial_j, \partial_j)$, also unique up to permutations. The variable t, being an arc-length parameter of g-geodesics orthogonal to $\partial_2, \dots, \partial_n$, cf. Lemma 5.2(d) and (5.1), is in turn unique up to substitutions by $b \pm t$, for constants b, as required.

Finally, to prove (C), we fix (M, g) of dimension $n \geq 3$ satisfying (0.1) – (0.3). Corollary 1.3 and (1.2), along with Remarks 3.2 and 2.5(ii), allow us to choose $\mathfrak{a}_2, \dots, \mathfrak{a}_n$ and $v_2, \dots, v_n$ as in Lemma 4.1 for $m = n$, and a point $x \in M$ at which all v_j are nonzero. (From now on j ranges over $\{2, \dots, n\}$.) By the Lie-bracket assertion of Lemma 4.1, the local flow of each v_j preserves all v_j and, consequently, also a unit vector field v_1 on a neighborhood of x, orthogonal to all v_j. Since v_1 and all v_j commute with one another, they constitute the coordinate vector fields of a local coordinate system $x^1 = t, x^2, \dots, x^n$ on a neighborhood of x, in which the metric g has the form (5.1) as a consequence of the last two lines of Lemma 4.1, with $m = n$. (In particular, the assertion $g(\nabla_u v_j, v_k) = 0$, for $u = v_l$ and $j, k, l \in \{2, \dots, n\}$, applied to $j = k$, shows that $g_{jj} = g(v_j, v_j)$ only depend on the variable $t = x^1$.) Now Lemma 5.2(e) yields (C). □

REMARK 5.7. The component version of (5.3) states that $\ddot{y}_j - (\operatorname{tr}\mathbf{y} + y_j)\dot{y}_j$ equals $y_j[\operatorname{tr}\mathbf{y}^2 - (\operatorname{tr}\mathbf{y})y_j]$. A solution $t \mapsto \mathbf{y}$ of (5.3) for $n \geq 3$, with *any* prescribed value at $t = 0$, may be chosen so as to make the values $\mu_1(0), \dots, \mu_n(0)$ mutually distinct. (By Lemma 5.2(a), this amounts to using $\dot{\mathbf{y}}(0)$ that realizes $(\mu_2(0), \dots, \mu_n(0))$

lying outside a finite union of specific hyperplanes in $\mathrm{I\!E}$.) Consequently, the local-isometry types in Theorem 5.6(ii) form a *moduli space* of dimension $2n-3$.

6. The scalar-curvature integral

Not surprisingly, in the light of (1.1.ii) and parts (b), (e) of Lemma 5.2,

$$\mathrm{s} \,=\, 2\,\mathrm{tr}\,\dot{\mathbf{y}} - \mathrm{tr}\,\mathbf{y}^2 - (\mathrm{tr}\,\mathbf{y})^2 \text{ is constant whenever } t\mapsto\mathbf{y} \text{ satisfies (5.3).} \tag{6.1}$$

LEMMA 6.1. *For any solution $I\ni t\mapsto\mathbf{y}\in\mathrm{I\!E}$ of (5.3) defined on $\mathrm{I\!R}$, and not identically equal to zero, one must have $\mathrm{s}<0$ in (6.1).*

PROOF. Under the assumption that $\mathrm{s}\ge 0$, (6.1) gives $2\,\mathrm{tr}\,\dot{\mathbf{y}}\ge\mathrm{tr}\,\mathbf{y}^2+(\mathrm{tr}\,\mathbf{y})^2$ for our solution $\mathrm{I\!R}\ni t\mapsto\mathbf{y}\in\mathrm{I\!E}$, and so $\mathrm{tr}\,\mathbf{y}$ is nondecreasing and nonconstant. Fixing $t'\in\mathrm{I\!R}$ such that $\mathrm{tr}\,\mathbf{y}(t')\ne 0$, we define a constant $c>0$ by $(n-1)c^2=[\mathrm{tr}\,\mathbf{y}(t')]^2$. Depending on whether $\mathrm{tr}\,\mathbf{y}(t')$ is positive or negative, monotonicity of $\mathrm{tr}\,\mathbf{y}$ gives $(\mathrm{tr}\,\mathbf{y})^2\ge(n-1)c^2$ on $[t',\infty)$ or, respectively, on $(-\infty,t']$. The Schwarz inequality $(\mathrm{tr}\,\mathbf{x})^2\le(n-1)\,\mathrm{tr}\,\mathbf{x}^2$ now shows that $\mathrm{tr}\,\mathbf{y}^2\ge c^2$ on $[t',\infty)$, or on $(-\infty,t']$. The relation $2\,\mathrm{tr}\,\dot{\mathbf{y}}\ge\mathrm{tr}\,\mathbf{y}^2+(\mathrm{tr}\,\mathbf{y})^2$ (see above) thus yields $2\,\mathrm{tr}\,\dot{\mathbf{y}}\ge c^2+(\mathrm{tr}\,\mathbf{y})^2$, that is, $\dot\alpha\ge c^2$ on $[t',\infty)$ or $(-\infty,t']$, where $\alpha=2\tan^{-1}(\mathrm{tr}\,\mathbf{y}/c)$. Consequently, $\alpha\to\pm\infty$ as $t\to\pm\infty$ for some sign $\pm$, contrary to boundedness of α. □

REMARK 6.2. A Riemannian manifold $(I\times\mathrm{I\!R}^{n-1},g)$ arising from (5.1) – (5.3), which makes it real-analytic, may be locally isometric to a compact (and hence complete) real-analytic Riemannian manifold, in the sense of the paragraph following (1.2), even if the solution $I\ni t\mapsto\mathbf{y}\in\mathrm{I\!E}$ of (5.3) has no extension to one defined on $\mathrm{I\!R}$. This is illustrated by the trivial extension (Example 5.4), with $m>0$ additional zeros, of the solution $y_2(t)=2\tan 2t$ of Example 5.3, for $n=2$, further modified using $a=1/2$ in Remark 5.5, so as to become $t\mapsto(\tan t,0,\dots,0)$. Since the latter realizes (5.2) with $g_{22}=\cos^2\!t$, it represents, locally, a product of the standard sphere S^2 with a flat torus T^m.

7. Completeness

Let $n\ge 3$. In the usual fashion, (5.3) is equivalent to the first-order system

$$\dot{\mathbf{y}}\,=\,\mathbf{p},\qquad \dot{\mathbf{p}}\,=\,(\mathrm{tr}\,\mathbf{y}+\mathbf{y})\mathbf{p}\,+\,(\mathrm{tr}\,\mathbf{y}^2)\mathbf{y}\,-\,(\mathrm{tr}\,\mathbf{y})\mathbf{y}^2. \tag{7.1}$$

Solutions $t\mapsto\mathbf{y}$ of (5.3) thus correspond to integral curves $t\mapsto(\mathbf{y},\mathbf{p})$ of the vector field v on $\mathrm{I\!E}\times\mathrm{I\!E}$ represented by (7.1), and expressed as

$$(\mathbf{y},\mathbf{p})\,\mapsto\,v_{(\mathbf{y},\mathbf{p})}\,=\,(\mathbf{p},(\mathrm{tr}\,\mathbf{y}+\mathbf{y})\mathbf{p}+(\mathrm{tr}\,\mathbf{y}^2)\mathbf{y}-(\mathrm{tr}\,\mathbf{y})\mathbf{y}^2) \tag{7.2}$$

when identified with a mapping $\mathrm{I\!E}\times\mathrm{I\!E}\to\mathrm{I\!E}\times\mathrm{I\!E}$. This v has an obvious curve $\mathrm{I\!R}\ni q\mapsto q(\mathbf{1},\mathbf{0})$ of zeros, where $\mathbf{1}\in\mathrm{I\!E}$ is the identity. Evaluating the differentials of $v:\mathrm{I\!E}\times\mathrm{I\!E}\to\mathrm{I\!E}\times\mathrm{I\!E}$ at $q(\mathbf{1},\mathbf{0})$, and of the function $\mathrm{I\!E}\times\mathrm{I\!E}\ni(\mathbf{y},\mathbf{p})\mapsto\mathrm{s}=2\,\mathrm{tr}\,\mathbf{p}-\mathrm{tr}\,\mathbf{y}^2-(\mathrm{tr}\,\mathbf{y})^2\in\mathrm{I\!R}$, cf. (6.1), at any $(\mathbf{y},\mathbf{p})\in\mathrm{I\!E}\times\mathrm{I\!E}$, we obtain $dv_{q(\mathbf{1},\mathbf{0})}(\hat{\mathbf{y}},\hat{\mathbf{p}})=(\hat{\mathbf{p}},\,nq\hat{\mathbf{p}}+q^2\,\mathrm{tr}\,\hat{\mathbf{y}}-(n-1)q^2\hat{\mathbf{y}})$ and $d\mathrm{s}_{(\mathbf{y},\mathbf{p})}(\hat{\mathbf{y}},\hat{\mathbf{p}})=2[\mathrm{tr}\,\hat{\mathbf{p}}-\mathrm{tr}\,\mathbf{y}\hat{\mathbf{y}}-(\mathrm{tr}\,\mathbf{y})\mathrm{tr}\,\hat{\mathbf{y}}]$. When $q\ne 0$, the linear endomorphism $dv_{q(\mathbf{1},\mathbf{0})}$ of $\mathrm{I\!E}\times\mathrm{I\!E}$ is diagonalizable, with the eigenvalues $0,nq,(n-1)q,q$ of multiplicities $1,1,n-2,n-2$, the eigenspace for each of the four eigenvalues λ consisting of all $(\hat{\mathbf{y}},\hat{\mathbf{p}})$ such that $\hat{\mathbf{p}}=\lambda\hat{\mathbf{y}}$ and either $\hat{\mathbf{y}}$ equals a multiple of the identity (for $\lambda\in\{0,nq\}$), or $\mathrm{tr}\,\hat{\mathbf{y}}=0$ (if $\lambda\in\{(n-1)q,q\}$).

On the other hand, s has no critical points in $\mathrm{I\!E}\times\mathrm{I\!E}$, and v is tangent to the level sets of s. The latter sets are codimension-one real-analytic submanifolds of

$\mathbb{E}\times\mathbb{E}$, and those among them intersecting the curve $\mathbb{R}\ni q\mapsto q(\mathbf{1},\mathbf{0})$ correspond, by (6.1), to $\mathrm{s}=-n(n-1)q^2$, that is, to all nonpositive values of s. If we fix $q\neq 0$, the tangent space at $z=q(\mathbf{1},\mathbf{0})$ of the hypersurface N given by $\mathrm{s}=-n(n-1)q^2$, equal to the kernel of $d\mathrm{s}_{q(\mathbf{1},\mathbf{0})}$, coincides, due to dimensional reasons, with the span of the eigenspaces of $dv_{q(\mathbf{1},\mathbf{0})}$ for the three nonzero eigenvalues $nq,(n-1)q,q$. (See the preceding paragraph and the above formula for $d\mathrm{s}_{(\mathbf{y},\mathbf{p})}(\hat{\mathbf{y}},\hat{\mathbf{p}})$.) From (2.1) it now follows that ∂w_z, for the vector field w on N arising as the restriction of v, is diagonalizable, with positive (or, negative) eigenvalues. Thus, as $z=q(\mathbf{1},\mathbf{0})$,

$$\text{our } z,N,w \text{ and } \varepsilon=-\operatorname{sgn} q \text{ satisfy the hypothesis of Lemma 2.3.} \tag{7.3}$$

Remark 7.1. Whenever $c\in\mathbb{R}\smallsetminus\{0\}$, the assignment $(\mathbf{y},\mathbf{p})\mapsto(c\mathbf{y},c^2\mathbf{p})$ is a diffeomorphism $F_c:\mathbb{E}\times\mathbb{E}\to\mathbb{E}\times\mathbb{E}$, sending our vector field v to v/c, and pulling the function s back to $c^2\mathrm{s}$. Using our N given by $\mathrm{s}=-n(n-1)q^2$ we obtain a diffeomorphism $(0,\infty)\times N\ni(c,x)\mapsto F(c,x)=F_c(x)$ onto the open set in $\mathbb{E}\times\mathbb{E}$ on which $\mathrm{s}<0$, as one sees defining its inverse by $F^{-1}(x')=(c,x)$, if $\mathrm{s}(x')<0$, with c,x such that $n(n-1)(cq)^2=-\mathrm{s}(x')$ and $x=F(1/c,x')$.

In the next theorem, we fix an integer $n\geq 3$, again denoting by $\mathbb{E}$ the space of all diagonal $(n-1)\times(n-1)$ matrices, and by $\mathbf{1}\in\mathbb{E}$ the identity.

Theorem 7.2. *For any $(\xi,\zeta)\in\mathbb{R}\times(0,\infty)$, every maximal solution $t\mapsto\mathbf{y}$ of (5.3) with $(\mathbf{y}(0),\dot{\mathbf{y}}(0))$ sufficiently close to $(\xi\mathbf{1},-\zeta\mathbf{1})$ in $\mathbb{E}\times\mathbb{E}$ has the domain $\mathbb{R}$, and the metric g on $\mathbb{R}^n$ defined by (5.1) – (5.2) is complete.*

Proof. The solution $\mathbb{R}\ni t\mapsto\mathbf{y}_{1,0}(t)=-2\tanh nt$ (times the identity $\mathbf{1}$) of Example 5.3 leads, via Remark 5.5, to further solutions $t\mapsto\mathbf{y}_{a,b}(t)=a\mathbf{y}_{1,0}(at+b)$, where $a,b\in\mathbb{R}$ and $a\neq 0$. Suitably chosen and fixed such a,b clearly realize, at $t=0$, any prescribed initial data $(\xi\mathbf{1},-\zeta\mathbf{1})=(\mathbf{y}_{a,b}(0),\dot{\mathbf{y}}_{a,b}(0))\in\mathbb{R}\times(0,\infty)$. Setting $x_{a,b}(t)=(\mathbf{y}_{a,b}(t),\dot{\mathbf{y}}_{a,b}(t))$ and $z_\pm=\mp 2|a|(\mathbf{1},\mathbf{0})$ we get $x_{a,b}(t)\to z_\pm$ as $t\to\pm\infty$. In the discussion preceding (7.3), applied to $q=\mp 2|a|$, both choices of the sign $\pm$ lead to the same N, given by $\mathrm{s}=-n(n-1)q^2$, and the same w, while $z_+,z_-\in N$ are two different zeros of w. Using (7.3) we now choose neighborhoods $U_\pm$ of $z_\pm$ in N satisfying the assertion of Lemma 2.3 for $x(t)$ equal to our $x_{a,b}(t)$, and $t'_\pm\in\mathbb{R}$ with $x_{a,b}(t'_\pm)\in U_\pm$. Since $z_\pm=\mp 2|a|(\mathbf{1},\mathbf{0})$, we may also require that

$$\mp y_j>|a| \text{ whenever } (y_2,\ldots,y_n,p_2,\ldots,p_n)\in U_\pm \text{ and } j\in\{2,\ldots,n\}. \tag{7.4}$$

By continuity, $x(t'_\pm)\in U_\pm$ for some neighborhood U_0 of $x_{a,b}(0)$ in N and all integral curves $t\mapsto x(t)\in N$ of w with $x(0)\in U_0$. The image of $(0,\infty)\times U_0$ under the diffeomorphism F of Remark 7.1 is now a neighborhood of $x_{a,b}(0)=(\xi\mathbf{1},-\zeta\mathbf{1})$ in $\mathbb{E}\times\mathbb{E}$, the existence of which constitutes our assertion: according to Remark 7.1, this F-image equals the union of $F_c(U_0)$ over $c>0$, and each F_c maps N diffeomorphically onto the s-preimage of the value $-n(n-1)(cq)^2$, while the push-forward, under $F_c:N\to F_c(N)$, of w obtained by restricting v to N, is the restriction of v/c to $F_c(N)$. However, the discussion preceding (7.3), and (7.3) itself, apply to every $q\neq 0$, and the use of v/c rather than v makes no difference (Remark 2.4). Now (7.4) combined with Remark 5.1 yields completeness of g. ∎

Our next result shows that the examples arising from Lemma 5.2(e) are not generally Ricci-parallel, or locally reducible, or (when $n\geq 4$) conformally flat.

Theorem 7.3. *The local-isometry types of Riemannian n-manifolds satisfying* (0.1) – (0.4) *form a set with a nonempty interior in the $(2n-3)$-dimensional moduli space of Remark* 5.7.

Proof. According to Theorem 5.6, the local-isometry types of all n-dimensional (M,g) with (0.1) – (0.3) arise from (5.1) when one chooses a maximal Ricci-generic solution $I\ni t\mapsto \mathbf{y}\in \mathbb{E}$ of (5.3), and then fixes a smooth curve $I\ni t\mapsto (g_{22}(t),\dots,g_{nn}(t))\in(0,\infty)^{n-1}$ satisfying (5.2). Restricting our discussion to the case where $0\in I$, and then parametrizing such solutions (allowed, this time, not to be Ricci-generic) by their initial data at $t=0$, we identify them with points of a specific Euclidean space, and completeness of g is guaranteed by Theorem 7.2 once one assumes (as we do from now on) that the initial data range over a certain nonempty open subset of the latter space. Now, as in Remark 5.7, if $n\ge 3$, we can make the Ricci eigenvalue functions $\mu_1(0),\dots,\mu_n(0)$ of Lemma 5.2 mutually distinct (which leads to Ricci-genericity) just by ensuring that $(\mu_2(0),\dots,\mu_n(0))$ does not lie within a specific finite union of hyperplanes in $\mathbb{E}$. However, rather than using any prescribed $\mathbf{y}(0)$, cf. Remark 5.7, let us require $y_1(0),\dots,y_n(0)$ to be all nonzero. This amounts to imposing on the solution $t\mapsto\mathbf{y}$ of (5.3) a *further open condition* implying (see the proof of Lemma 5.2) that $R_{j1,j}(0)\ne 0$, and so g is not Ricci-parallel. In the proof of Lemma 5.2 we also saw that $\Gamma_{jj}^1(0)\ne 0$ and, consequently, g cannot be locally reducible. (If it were, the Ricci eigenvector fields $\partial_1,\dots,\partial_n$ of Lemma 5.2, with distinct eigenvalue functions $\mu_1,\dots,\mu_n$, would each be tangent to one or the other parallel factor distribution, giving $\Gamma_{jj}^1=0$ with some $j=2,\dots,n$.) For $k\ne j$, one easily verifies that $g^{jj}g^{kk}R_{jkjk}=-y_jy_k$. Therefore, if W denotes the Weyl tensor, $(n-1)(n-2)g^{jj}g^{kk}W_{jkjk}=\,2\,\mathrm{tr}\,\dot{\mathbf{y}}\,-\,\mathrm{tr}\,\mathbf{y}^2-(\mathrm{tr}\,\mathbf{y})^2+(n-1)[(y_j+y_k)\,\mathrm{tr}\,\mathbf{y}-(n-2)y_jy_k-\dot y_j-\dot y_k]$, where $\dot y_j$ appears with the coefficient $3-n$. An enhanced version of the last open condition thus precludes conformal flatness of our examples when $n\ge 4$. □

Appendix: Warped products with harmonic curvature

For the reader's convenience, we gather here some facts that are well known [**7**] and easily verified. The repeated indices are always summed over. In (1.4) we set $m=\dim\overline{M}$ and $p=\dim\varSigma$, assuming that $mp\ge 1$ and $\phi:\overline{M}\to(0,\infty)$ is nonconstant. Thus, $\dim M=n$ with $n=m+p\ge 2$. We use product coordinates x^λ in M, consisting of local coordinates x^i for $\overline{M}$ and x^a for $\varSigma$, declaring

(A.1) $$\lambda,\mu,\nu\in\{1,\dots,n\},\;\; i,j,k\in\{1,\dots,m\},\;\; a,b,c\in\{m+1,\dots,n\}$$

to be our index ranges. Therefore, $\bar g_{ij}$ as well as $\theta=\log\phi$ depend only on the variables x^k, and η_{ab} only on x^c, that is, $\partial_a\bar g_{ij}=\partial_a\theta=\partial_i\eta_{ab}=0$. Furthermore,

(A.2) $$g_{ij}=\bar g_{ij},\;\; g_{ia}=g_{ai}=0,\;\; g_{ab}=e^{2\theta}\eta_{ab}.$$

For the Christoffel symbols $\Gamma_{\lambda\mu}^\nu$, $\bar\Gamma_{ij}^k$, H_{ab}^c of $g,\bar g,\eta$, their Ricci-tensor components $R_{\lambda\mu}$, $\bar R_{ij}$, P_{ab}, and the components $\overline{\nabla}_{\!i}\overline{\nabla}_{\!j}\theta$ of the $\bar g$-Hessian of θ, one has

$$\begin{aligned}&g^{ij}=\bar g^{ij},\;\; g^{ia}=g^{ai}=0,\;\; g^{ab}=e^{-2\theta}\eta^{ab},\;\; \Gamma_{ij}^k=\bar\Gamma_{ij}^k,\;\; \Gamma_{ia}^k=\Gamma_{ij}^a=0,\;\; \Gamma_{ia}^b=\delta_a^b\theta_{,i},\\ &\Gamma_{ab}^i=-e^{2\theta}\eta_{ab}\theta^{,i},\;\; \Gamma_{ab}^c=H_{ab}^c,\;\; R_{ij}=\bar R_{ij}-p[\overline{\nabla}_{\!i}\overline{\nabla}_{\!j}\theta+\theta_{,i}\theta_{,j}],\end{aligned}$$

while, in terms of the $\bar g$-Laplacian $\overline{\Delta}$,

(A.3) $$R_{ia}=0,\qquad R_{ab}=P_{ab}-p^{-1}e^{(2-p)\theta}[\overline{\Delta}e^{p\theta}]\eta_{ab}.$$

The components $R_{\lambda\mu,\nu}, \overline{\nabla}_{\!i}\bar{R}_{jk}, D_cP_{ab}$ of the covariant derivatives of the Ricci tensors of $g, \bar{g}, \eta$ satisfy, with the usual conventions $\theta_{,i} = \partial_i\theta$ and $\theta^{,i} = \bar{g}^{ij}\partial_j\theta$, the relations

$$\begin{aligned} &R_{jk,\,i} = \overline{\nabla}_{\!i}\bar{R}_{jk} - p[\overline{\nabla}_{\!i}\overline{\nabla}_{\!j}\overline{\nabla}_{\!k}\theta + \overline{\nabla}_{\!i}(\theta_{,j}\theta_{,k})], \qquad R_{ij,\,a} = R_{aj,\,i} = 0,\\ &R_{ib,\,a} = e^{2\theta}(p^{-1}e^{-p\theta}[\overline{\Delta}e^{p\theta}]\theta_{,i} + [\bar{R}_{ij} - p\overline{\nabla}_{\!i}\overline{\nabla}_{\!j}\theta - p\theta_{,i}\theta_{,j}]\theta^{,j})\eta_{ab} - \theta_{,i}P_{ab},\\ &R_{ab,\,i} = -p^{-1}e^{2\theta}(e^{-p\theta}\overline{\Delta}e^{p\theta})_{,i}\eta_{ab} - 2\theta_{,i}P_{ab}, \qquad R_{ab,\,c} = D_cP_{ab}. \end{aligned} \tag{A.4}$$

Let (a) – (e) refer to parts of Lemma 1.2, which we now proceed to prove. First,

(f) $R_{ab,\,i} = R_{ib,\,a}$ for all i, a, b as in (A.1) if and only if one has (a) and (e).

In fact, it suffices to verify (f) on the dense set $(U \cup U') \times \Sigma \subseteq M$, for the interior U of the zero set of $d\theta$ in $\overline{M}$ and the subset U' on which $d\theta \neq 0$. On U, according to (A.4), $R_{ab,\,i} = 0 = R_{ib,\,a}$ since $\overline{\Delta}e^{p\theta} = 0$. Similarly, on U', the equality $R_{ab,\,i} = R_{ib,\,a}$ amounts, by (A.4), to the condition $P_{ab} = \kappa\eta_{ab}$, for a function κ on Σ which must be constant, as it depends only on the variables x^j that are local coordinates in $\overline{M}$. Formulae (A.4) also show that κ is characterized by the relation $-\kappa e^{-2\theta}d\theta = p^{-1}(d[e^{-p\theta}\overline{\Delta}e^{p\theta}] + e^{-p\theta}[\overline{\Delta}e^{p\theta}]\,d\theta) + \bar{\mathrm{r}}(\overline{\nabla}\theta,\,\cdot\,) - p\bar{g}(\overline{\nabla}\theta,\overline{\nabla}\theta)d\theta - pd[\bar{g}(\overline{\nabla}\theta,\overline{\nabla}\theta)]/2$ which, rewritten in terms of $\phi = e^{\theta}$, becomes (e).

The equivalence of (e) and (c) is in turn obvious from (1.3). Next, by (A.4),

(g) $R_{jk,\,i} = R_{ik,\,j}$ for all i, j, k with (A.1) if and only if (b) holds,

since $\phi^{-1}\overline{\nabla}d\phi = \overline{\nabla}d\theta + d\theta \otimes d\theta$. The main claim of Lemma 1.2 is thus immediate: harmonicity of the curvature amounts to the Codazzi equation for the Ricci tensor, cf. (1.1.i), while (A.4) clearly reduces the latter to the cases (f) – (g).

Finally, (d) follows from (a) and (A.3).

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