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Geometry of the Standard Model

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0. Introduction. The *Standard Model* of particles and interactions is the currently-accepted theory of elementary particles. It can be naturally divided into the *classical part*, a description of which is possible in the language of vector bundles over the spacetime and operations on them, and a *field quantization* procedure that transforms the classical part into a reasonable model of physical reality.

This note covers only the *classical* part of the Standard Model. Similar but more detailed expositions of this topic can be found in the following texts:

A. Derdzinski, *Geometry of the Standard Model of Elementary Particles*, Texts and Monographs in Physics, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

A. Derdzinski, *Geometry of elementary particles*, Proceedings of Symposia in Pure Mathematics **54** (1993), (edited by R. E. Greene and S.-T. Yau), Part 2, 157–171.

1. Interactions. Aside from gravity, the known kinds of particle interactions, ordered by decreasing strength, are the *strong, electromagnetic,* and *weak* forces. The latter two may be combined into the *electroweak interaction* (§5).

The *strength* of an interaction amounts to the probability of its occurrence in the given circumstances.

2. Taxonomy of particle species. (See also §3, §5.)

- known particles
  - interaction carriers: 8 species of gluons, γ (the photon), $\mathrm{W}^-, \mathrm{W}^+, \mathrm{Z}^0$
  - matter particles
    - leptons
      - leptons proper, 3 generations: $\mathrm{e}, \nu_\mathrm{e}$; μ, ν_μ; τ, ν_τ
      - antileptons, 3 generations: $\mathrm{e}^+, \overline{\nu}_\mathrm{e}$; $\mu^+, \overline{\nu}_\mu$; $\tau^+, \overline{\nu}_\tau$
    - hadrons
      - mesons (> 100)
      - baryons
        - baryons proper (> 50)
        - antibaryons (> 50)

3. Definitions. *Interaction carriers* mediate interactions, *matter* particles do not. *Leptons* can't interact strongly, *hadrons* can. *Mesons* are bosons, *baryons* are fermions (see Table 4.2). Baryons naturally form the disjoint classes of *baryons proper* and *antibaryons*, which consist of each other's antiparticles (Table 4.2). The same principle applies to leptons.

4. A physics-geometry dictionary.

Table 4.1. Particles and bundles

physics	*geometry*
a PARTICLE species	a BUNDLE ζ with some geometry over the spacetime $(\mathcal{M}, g)$; the particle is *represented by* (or *lives in*) ζ
classical STATES of the particle	SECTIONS ψ of the bundle ζ
EVOLUTION of the states	FIELD EQUATIONS imposed on ψ
a MATTER particle	a VECTOR bundle

Table 4.2. Operations

physics: operations involving matter particles	*geometry: operations on vector bundles ζ*
the GENERALIZATION of n given particle species	the DIRECT SUM $\zeta_1 + \ldots + \zeta_n$ (all n species involved live here) .
a COMPOSITE system (particle)	a natural surjective MORPHISM $\zeta_1 \ldots \zeta_n \to \zeta$ of the TENSOR PRODUCT $\zeta_1 \ldots \zeta_n$. It must be symmetric/skewsymmetric in any group of identical particles, then called *bosons/fermions*
ANTIPARTICLE formation	complex CONJUGATE $\bar{\zeta}$

Table 4.3. Interactions and gauge fields

physics: interactions	*geometry: Yang-Mills fields*
a FREE matter particle	a NATURAL VECTOR BUNDLE $\eta \to \mathcal{M}$ of first order (the *free-particle bundle* of the given species); naturality amounts to *direct observability*
an INTERACTION of some given kind	a NON-NATURAL vector bundle $\delta \to \mathcal{M}$ (the *interaction bundle*) with some geometry, mainly a G-structure
CARRIERS of the interaction	live in the AFFINE BUNDLE $\mathcal{C}(\delta)$ whose sections are the compatible connections in δ
an INTERACTING matter particle	the INTERACTING-PARTICLE BUNDLE $\alpha = \alpha(\delta, \eta)$, functorial in both δ, η and "homogeneous linear" in the free-particle bundle η (basic example: $\alpha = \delta\eta$)

Usually, neither α nor $\mathcal{C}(\delta)$ is natural. This contradicts the obvious requirement that carriers of interactions and interacting matter should be directly observable. One resolves this problem by "restoring" naturality of the bundles in question using *bound states,* or *symmetry breaking*, as described below. (Notations: N is the fibre dimension of the fixed interaction bundle δ; an integer $k > 0$ represents the product vector bundle $\mathcal{M} \times \mathbb{C}^k$.)

Table 4.4. Bound states and symmetry breaking

physics	*geometry*
BOUND STATES of n particles	MORPHISMS of $\alpha_1 \dots \alpha_n$ onto NATURAL bundles, obtained by naturally "canceling" the δ-related factors
SYMMETRY BREAKING	selection of an ADDITIONAL STRUCTURE in δ, leading to reduction of G to a subgroup
FORMAL symmetry breaking (a thought experiment)	TRIVIALIZATION of δ, so $\delta = N$ and, e.g., $\delta\eta = N\eta$ (the interacting particle comes in N separate versions), while $\mathcal{C}(\delta) = (\dim G)T^*$, i.e., the carriers appear as $\dim G$ species of matter particles living in $T^* = T^*\mathcal{M}$
SPONTANEOUS symmetry breaking (in nature, for interactions of low strength)	Example: the ELECTROWEAK MODEL (§5).

5. The standard model.

Table 5.1. Geometry of interactions

interaction	ELECTROMAGNETIC	ELECTROWEAK	STRONG
credits	Weyl, 1929	Glashow, Salam, Weinberg, 1961–1967	Gell-Mann, Zweig, 1964
comments	The possibility of a unified description of electromagnetism for all particles expresses the fact that the electric charge is *quantized*, i.e., occurs in multiples of a fixed amount.	The model describes one generation of (anti)leptons (§2) at a time. Choose, e.g., e, ν_e : the electron and electronic neutrino. Their free-particle bundles are: σ for e and σ_L for ν_e, where σ denotes a fixed Dirac spinor bundle, $\mathcal{M}$ is assumed orientable, and $\sigma = \sigma_L + \sigma_R$ (Weyl spinor bundles), $\sigma_R = \overline{\sigma_L}$.	Hadrons appear as composites of *quarks* and *antiquarks* (abbreviation: q, $\bar{q}$), coming in several *flavors* (species).
G, δ	$G = \mathrm{U}(1)$, $\delta = \lambda$	$G = \mathrm{U}(2)$, $\delta = \iota$	$G = \mathrm{SU}(3)$, $\delta = \rho$
what δ *is*	a complex line bundle	a complex plane bundle	a complex 3-space bundle
geometry of δ	a Hermitian fibre metric $\langle\,,\rangle$	a Hermitian fibre metric $\langle\,,\rangle$	$\langle\,,\rangle$ and a unit section Θ of $[\rho^*]^{\wedge 3}$
free-particle bundle	any η	a fixed Dirac spinor bundle σ for the whole generation e, ν_e	σ for quarks, $\overline{\sigma}$ for antiquarks
interacting particle bundle	$\alpha = \lambda^k\eta$ if particle carries k units of electron charge (with $\lambda^{-k} = \overline{\lambda^k}$)	$\alpha = \iota\sigma_L + \iota^{\wedge 2}\sigma_R$ or, if neutrinos are massive, even simpler: $\alpha = \iota\sigma$	$\alpha = \rho\sigma$ (for quarks) $\alpha = \overline{\rho\sigma}$ (antiquarks)

Table 5.2. Bound states and symmetry breaking in the standard model

inter-action	ELECTRO-MAGNETIC	ELECTROWEAK	STRONG
bound states: $\alpha_1\ldots\alpha_n$ $\downarrow$ ζ *(where both* ζ *and* $\downarrow$ *are natural)*	only if $\sum_{j=1}^{n} k_j = 0$ (electrically neutral systems, e.g., atoms), as $\lambda\overline{\lambda}=1$ and $\lambda^{k_1}\ldots\lambda^{k_n} = \lambda^{\sum_j k_j}$ under $\langle\,,\rangle$	none of interest	cancel ρ factors by $\langle\,,\rangle:\rho\overline{\rho}\to 1$, $\Theta:\rho^3\to 1$, or $\overline{\Theta}:\overline{\rho}^3\to 1$, getting q$\bar{\text{q}}$ pairs (mesons), q triples (baryons proper), $\bar{\text{q}}$ triples (antibaryons)
formal symmetry breaking	$\lambda=1, \alpha=\eta$, $\mathcal{C}(\lambda)=T^*$. Matter: same as free. Carriers: just one species, the *photon* γ.	of no interest	$\rho=3$, $\alpha=3\sigma$ or $3\overline{\sigma}$: each q,$\bar{\text{q}}$ flavor comes in 3 *colors.* As $\mathcal{C}(\rho)=8T^*$, the carriers appear as 8 species of *gluons.*
spontaneous symmetry breaking	none: too strong	Choice of a section ϕ of ι with $\lvert\phi\rvert=\text{constant}>0$ reduces U(2) to U(1). Call $\lambda=\phi^\perp$ the *electromagnetic-interaction bundle,* so $\iota = 1+\lambda$, $1=\text{Span}\,\phi$, $\iota^{\wedge 2}=\lambda$. Thus, $\alpha=\sigma_{\text{L}}+\lambda\sigma$ describes e, ν_{e} with their correct charges, and the summands of $\mathcal{C}(\iota)=\mathcal{C}(\lambda)+\lambda T^*+T^*$ represent the carriers: the *photon* γ, and the massive, matterlike *weak-interaction carriers,* $\text{W}^\pm$ (charged), and Z (neutral).	none: much too strong

6. Coupling constants and the Weinberg angle. An additional ingredient of the geometry of any interaction bundle δ is provided by a fixed *natural fibre metric* $(\,,)$ in $\mathcal{C}(\delta)$, obtained in the obvious way from a biinvariant metric on G. Since $G = \text{U}(2)$ is reducible, the freedom in choosing $(\,,)$ for the electroweak model involves not merely a scale factor (referred to as a *coupling constant*), but also an angular parameter (the *Weinberg angle*).

The latter is physically meaningful, since the decomposition of $\mathcal{C}(\iota)$ in Table 5.2 is $(\,,)$-orthogonal. In general, the coupling constant of $(\,,)$ also has a physical interpretation, namely in terms of the *strength* of the interaction described by δ.

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