ON WEAKLY EINSTEIN KÄHLER SURFACES

ANDRZEJ DERDZINSKI[1], YUNHEE EUH[2], SINHWI KIM[2], JEONGHYEONG PARK[2]

Abstract. Riemannian four-manifolds in which the triple contraction of the curvature tensor against itself yields a functional multiple of the metric are called *weakly Einstein*. We focus on weakly Einstein Kähler surfaces. We provide several conditions characterizing those Kähler surfaces which are weakly Einstein, classify weakly Einstein Kähler surfaces having some specific additional properties, and construct new examples.

1. Introduction

Following Euh, Park and Sekigawa [14], we say that a Riemannian four-manifold is *weakly Einstein* when the three-index contraction of its curvature tensor against itself equals a functional multiple of the metric. This is the case – in dimension four only – for all Einstein manifolds. See formulae (4.4) and (4.5) in Sect. 4.

The same requirement in dimensions $n > 4$, coupled with the Einstein condition, defines what one calls *super-Einstein manifolds*, and then the above 'functional multiple' must – only if $n > 4$ – be a *constant multiple* [2, p. 165], [18, p. 358], [3, Lemma 3.3].

The present paper deals with weakly Einstein Kähler surfaces. Our Theorem 5.1 provides five conditions equivalent to the weakly Einstein property of a given Kähler surface, which allows us to quickly conclude whether a specific Kähler surface is weakly Einstein.

One calls a function $\boldsymbol{\tau}$ on a Riemannian manifold *transnormal* if the integral curves of its gradient v are reparametrized geodesics. This amounts to requiring that, locally, at points where $v \neq 0$, the norm-squared Q of v be a function of $\boldsymbol{\tau}$. When the last condition holds for both Q and the Laplacian of $\boldsymbol{\tau}$, one refers to $\boldsymbol{\tau}$ as *isoparametric*. See [4, 19, 22]. Our next four theorems (three local, one assuming compactness) classify weakly Einstein Kähler surfaces having specific additional properties.

Theorem 1.1. *Up to local isometries, Riemannian products of two real surfaces of opposite nonzero constant Gaussian curvatures are the only locally homogeneous non-Einstein weakly Einstein Kähler surfaces.*

Theorem 1.2. *All self-dual weakly Einstein Kähler surfaces are locally symmetric, and hence locally homothetic to a standard complex projective or hyperbolic space, or to a product of two real surfaces of opposite constant Gaussian curvatures.*

Theorem 1.3. *On a weakly Einstein Kähler surface, every transnormal function with a holomorphic gradient is necessarily also isoparametric.*

Theorem 1.4. *Up to biholomorphic isometries, the only compact non-Einstein weakly Einstein Kähler surfaces admitting nonconstant transnormal functions with holomorphic gradients are certain compact isometric quotients of the Riemannian product of a sphere and a hyperbolic plane with opposite constant Gaussian curvatures.*

The word 'certain' in Theorem 1.4 accounts for the requirement that both factor distributions be orientable (so as to make the quotient Kähler, and not just locally Kähler).

2020 *Mathematics Subject Classification.* 53B35, 53C55.

Key words and phrases. weakly Einstein, self-dual metrics, Kähler surface.

The isometric quotients in question are precisely all the nonflat conformally flat compact Kähler surfaces. See Remark 5.5.

In the Kähler-Einstein case the conclusion of Theorem 1.3 holds for well-known and general reasons. See the second sentence of Remark 5.6.

We do not know whether there exist compact non-Einstein weakly Einstein Kähler surfaces other than the compact isometric quotients of the Riemannian products of spheres and hyperbolic planes with opposite Gaussian curvatures, mentioned in Theorem 1.4.

What we can easily conclude, using Theorem 1.4, is that examples of this kind – if they exist – cannot have a nonzero Euler characteristic and at the same time admit groups of isometries with an infinite center and with principal orbits of dimension three. Thus, a U(2)-invariant Kähler metric on the one-point and two-point blow-ups of $\mathbb{C}\mathrm{P}^2$ is never weakly Einstein.

Theorem 1.3, although not *per se* a classification result, provides a crucial step in the proof of our Theorem 9.1, which explicitly describes the local structure of all non-Einstein weakly Einstein Kähler surfaces with nonconstant transnormal functions having holomorphic gradients. We then use Theorem 9.1 to prove Theorem 1.4.

As an added bonus, Theorem 9.1 leads to new examples of (noncompact) non-Einstein weakly Einstein Kähler surfaces, presented in Sect. 12.

For compact Riemannian n-manifolds with parallel Ricci tensor, $n \geq 4$, the weakly Einstein property (as stated above when $n = 4$) is necessary and sufficient in order that the metric be a critical point of the functional associating with metrics of unit volume the L^2 norm-squared of their curvature [2, Corollary 4.72]. See also [16]. For $n = 3$, non-Einstein weakly Einstein manifolds are characterized by having a Ricci tensor of rank one [17]. For $n \geq 4$, only partial classification results exist such as Arias-Marco and Kowalski's theorem [1] in the locally homogenous case with $n = 4$, mentioned in Sect. 3. Several results in this direction can be found in [2, Sect, 6.55–6.63]. Conformally flat weakly Einstein Riemannian manifolds were classified by García-Río et al. [17]. For the weakly Einstein condition in extrinsic geometry, see [20] and the references therein.

2. Notation and preliminaries

All manifolds, mappings, tensor fields and connections are assumed smooth. Manifolds are by definition connected. Given a Riemannian manifold (M, g), we denote by $\nabla, R, \mathrm{r}, \mathrm{e}$ and s its Levi-Civita connection, curvature tensor, Ricci tensor, Einstein tensor and scalar curvature, with the sign convention such that $\mathrm{r}(w, w') = \mathrm{tr}\,[R(w, \cdot\,)w']$ for vector fields w, w'. Thus, $\mathrm{s} = \mathrm{tr}_g\mathrm{r}$. The symbol ∇ also stands for the g-gradient.

In the underlying real space of a complex vector space V of positive finite dimension m we use the natural orientation such that $e_1, ie_1, \ldots, e_m, ie_m$ is a positive real basis for any complex basis $e_1, \ldots, e_m$ of V. (The automorphism group $\mathrm{GL}(V) \approx \mathrm{GL}(m, \mathbb{C})$ is connected, since every automorphism has, in some basis, a triangular matrix, which is joined to Id by an obvious curve of nonsingular triangular matrices.) Thus,

(2.1) every almost-complex manifold is canonically oriented.

Remark 2.1. A Riemannian manifold (M, g) with $\mathrm{se} = 0$ that is, one in which sr is a functional multiple of g, necessarily has $\mathrm{s} = 0$ identically, or $\mathrm{e} = 0$ everywhere. (This needs to be justified as we are not assuming real-analyticity.) First, we may assume that $\dim M > 2$. (For surfaces, $\mathrm{se} = \mathrm{e} = 0$.) Now $\mathrm{s}\,d\mathrm{s} = 0$, since a point with $\mathrm{s} \neq 0$ has a connected neighborhood U on which $\mathrm{e} = 0$, so that, by Schur's lemma, s is constant and $d\mathrm{s} = 0$ on U. The ensuing constancy of s^2 gives $\mathrm{s} = 0$ identically or $\mathrm{s} \neq 0$ everywhere.

Remark 2.2. If $d < c$ and a differentiable function ψ of the variable $\alpha \in [d, c)$ has a positive derivative $\psi'(\alpha)$ at every α at which $\psi(\alpha) \geq \lambda$, for some given $\lambda \in \mathrm{I\!R}$, then either $\psi < \lambda$ everywhere in $[d, c)$ or, for the least $\alpha_1 \in [d, c)$ at which $\psi(\alpha_1) \geq \lambda$, the restriction of ψ to (α_1, c) is greater than λ and strictly increasing; consequently, ψ has at c a limit lying in $(\lambda, \infty]$.

In fact, our claim will follow immediately once we verify that $\psi \geq \lambda$ on (α_1, c). Assuming this not to be the case, we let α_2 be the infimum of those $\alpha \in (\alpha_1, c)$ at which $\psi(\alpha) < \lambda$. Thus, $\alpha_2 > \alpha_1$, since at α_1 one has $\psi(\alpha_1) \geq \lambda$ and $\psi'(\alpha_1) > 0$. It follows now that $\psi(\alpha_2) = \lambda$, and hence $\psi'(\alpha_2) > 0$, giving $\psi(\alpha) > \lambda$ for all $\alpha > \alpha_2$ close to α_2, contrary to the definition of α_2.

3. Proof of Theorem 1.1

Recall from the Introduction that a Riemannian four-manifold is said to be weakly Einstein when the three-index contraction of its curvature tensor R against itself equals some function ϕ times the metric g (in coordinates: $R_{ikpq}R_j{}^{kpq} = \phi g_{ij}$).

Arias-Marco and Kowalski [1] showed that a non-Einstein locally homogeneous weakly Einstein four-manifold must be locally isometric either to a Riemannian product of surfaces with opposite nonzero constant Gaussian curvatures, or to what they call an EPS space: one of the examples constructed by Euh, Park, and Sekigawa [15, Example 3.7].

This proves Theorem 1.1, since an EPS space (M, g) is not Kähler.

We verify the last claim as follows. In [15, Example 3.7], M is a Lie group with left-invariant g-orthonormal vector fields $e_1, \ldots, e_4$ satisfying some Lie-bracket relations that involve constants $a \neq 0$ and b. According to [15, formula (3.14)], the only nonzero components of the curvature tensor R in this frame are those algebraically related to $R_{1212} = R_{1313} = R_{1414} = R_{3434} = -a^2$ and $R_{2323} = R_{2424} = a^2$. (The sign of R in [15] is the opposite of ours.) Consequently, the frame diagonalizes the Ricci tensor r, with the eigenvalues $\mathrm{r}_{11} = -3a^2$, $\mathrm{r}_{22} = a^2$ and $\mathrm{r}_{33} = \mathrm{r}_{44} = -a^2$.

Thus, g is not a Kähler metric: if it were, the complex structure, leaving r invariant, would cause r to have two double eigenvalues, contrary to the above formulae.

4. Algebraic curvature tensors

For an algebraic curvature tensor R in a Euclidean space $\mathcal{T}$ of dimension $n \geq 4$, denoting by g the inner product, by $W, \mathrm{r}, \mathrm{e}$ the Weyl, Ricci and Einstein tensors of R, and by s its scalar curvature, with $\mathrm{r}_{ij} = g^{pq}R_{ipjq}$, $\mathrm{s} = g^{pq}\mathrm{r}_{pq}$ and $\mathrm{e} = \mathrm{r} - \mathrm{s}g/n$, we have

$$(4.1)\qquad \begin{aligned} \text{a)}\quad W_{ijpq} &= R_{ijpq} - \frac{1}{n-2}(g_{ip}\mathrm{r}_{jq} + g_{jq}\mathrm{r}_{ip} - g_{jp}\mathrm{r}_{iq} - g_{iq}\mathrm{r}_{jp}) \\ &\quad + \frac{\mathrm{s}}{(n-1)(n-2)}(g_{ip}g_{jq} - g_{jp}g_{iq}). \\ \text{b)}\quad W_{ijpq} &= R_{ijpq} - \frac{1}{n-2}(g_{ip}\mathrm{h}_{jq} + g_{jq}\mathrm{h}_{ip} - g_{jp}\mathrm{h}_{iq} - g_{iq}\mathrm{h}_{jp}), \end{aligned}$$

where the more concise version involves the Schouten tensor $\mathrm{h} = \mathrm{r} - \mathrm{s}g/(2n-2)$, We use here components relative to any basis of $\mathcal{T}$, with index raising and lowering via g, and summation over repeated indices. See [2, Sect. 1.108].

The *triple contraction* of *any such* R is the symmetric 2-tensor $\mathrm{trc}R$ given by

$$(4.2)\qquad [\mathrm{trc}R]_{ij} = R_{ikpq}R_j{}^{kpq}.$$

Using (4.1-b) and, respectively, (4.1-a), one easily verifies that

$$(4.3)\qquad \begin{aligned} &\text{i)}\quad (n-2)^2(\mathrm{trc}R - \mathrm{trc}\,W) + 2[2\mathrm{s}\mathrm{r} - 2(n-2)R\mathrm{r} - n\mathrm{r}^2] \text{ is a multiple of } g, \\ &\text{ii)}\quad (n-2)(R\mathrm{r} - W\mathrm{r}) + 2\mathrm{r}^2 - n\mathrm{s}\mathrm{r}/(n-1) \text{ is a multiple of } g. \end{aligned}$$

Here $b = \mathrm{r}^2$ has the components $b_{ij} = \mathrm{r}_j^k\mathrm{r}_{ik}$, and an algebraic curvature tensor R acts on arbitrary $(0,2)$ tensors b by $[Rb]_{ij} = R_{ipjq}b^{pq}$, which preserves (skew)symmetry of

b, and – due to the Bianchi identity – becomes $2[R\alpha]_{ij} = R_{ijpq}\alpha^{pq}$ when $b = \alpha$ is skew-symmetric. We will repeatedly refer to tensors (or, tensor fields on a manifold with a metric g) as multiples (or, functional multiples) of g, without the need to specify the factor, since it is trivially found by contraction.

For instance, given a Riemannian manifold (M, g) with the curvature tensor R,

(4.4) (M, g) is called *weakly Einstein* when $\mathrm{trc}\,R$ is a functional multiple of g.

It is well known – see, e.g., [9, p. 413] – that $\mathrm{trc}\,W$ is a multiple of g when $n = 4$. Thus, (4.3-i) with $n = 4$ implies that, as pointed out by Euh, Park and Sekigawa [14],

(4.5) in dimension four, $\mathrm{trc}\,R - 2R\mathrm{r} + \mathrm{sr} - 2\mathrm{r}^2$ is a multiple of g, so that, for Riemannian 4-manifolds, Einstein implies weakly Einstein.

Setting $n = 4$ in (4.3-ii) and adding the resulting expression to the one in (4.5),

(4.6) we see that $\mathrm{trc}\,R - 2W\mathrm{r} - \mathrm{sr}/3$ is a multiple of g when $n = 4$,

which easily yields [17, Theorem 2(i)]: a four-dimensional non-Einstein conformally flat Riemannian manifold is weakly Einstein if and only if its scalar curvature vanishes. (This involves a technical detail, discussed in Remark 2.1.) As the contractions of W vanish, which makes $W\mathrm{r}$ obviously equal to $W\mathrm{e}$, (4.6) implies that, in dimension four,

(4.7) (M, g) is weakly Einstein if and only if $6W\mathrm{e} = -\mathrm{se}$,

$\mathrm{e} = \mathrm{r} - \mathrm{s}g/4$ being again the Einstein tensor. The additional assumption

(4.8) that e has the spectrum of the form $(a, a, -a, -a)$, which is obviously satisfied by Kähler surface metrics,

leads to the following consequence:

(4.9) $\mathrm{sr} - 2\mathrm{r}^2$ is a multiple of g.

In fact, by (4.8), e^2 is a multiple of g, while $2\mathrm{e}^2 = 2\mathrm{r}^2 - \mathrm{sr} + \mathrm{s}^2 g/8$.

Therefore, from (4.5) – (4.9) it trivially follows that

(4.10) $\mathrm{trc}\,R - 2R\mathrm{r}$ is a multiple of g if $n = 4$ and (4.8) holds.

5. Kähler surfaces

A twice-covariant tensor field a on an almost-complex manifold (M, J) gives rise to two more such tensor fields,

(5.1) $aJ = a(J\,\cdot\,,\,\cdot\,)$ and $Ja = -a(\,\cdot\,,\,J\,\cdot\,)$,

as well as the *commutator* $[a, J] = -[J, a] = aJ - Ja$. Obviously,

(5.2) a is J-invariant, that is, $a(J\,\cdot, J\,\cdot\,) = a$, if and only if $aJ = Ja$.

The tensor field a is said to be *Hermitian* if it is symmetric at every point and J-invariant. Clearly, if a is Hermitian, aJ is a 2-form and $(aJ)J = J(Ja) = -a$. By a *Hermitian metric* on an almost-complex manifold (M, J) we mean a Riemannian metric g on M which is a Hermitian tensor ($gJ = Jg$), which amounts to skew-adjointness of J at every point or – equivalently – the requirement that J act in every tangent space as a linear isometry. If g is Hermitian, and one identifies bundle morphisms $A, B : TM \longrightarrow TM$ with $a = g(A\,\cdot\,,\,\cdot\,)$ and $b = g(B\,\cdot\,,\,\cdot\,)$, then the operation $a \longmapsto b = Ja$ (or, $a \longmapsto b = aJ$), defined by (5.1) for twice-covariant tensor fields a,

(5.3) corresponds to the ordinary composition $B = JA$ or, $B = AJ$.

As an example, in a Kähler manifold (M,g,J) of real dimension n, both g and the Ricci tensor r are Hermitian, and hence so is the Einstein (traceless Ricci) tensor $\mathrm{e}=\mathrm{r}-\mathrm{s}g/n$, giving rise to the Kähler, Ricci and Einstein 2-forms $\omega=gJ$, $\rho=\mathrm{r}J$ and $\eta=\mathrm{e}J$, with

$$\nabla\omega=0,\qquad d\rho=0,\qquad \eta=\rho-\mathrm{s}\omega/n. \tag{5.4}$$

In any complex manifold M, the operator $i\partial\overline{\partial}$ sends every C^∞ function $f:M\longrightarrow\mathrm{I\!R}$ to the exact 2-form $i\partial\overline{\partial}f$ given by

$$2i\partial\overline{\partial}f\;=\;-d[(df)J]. \tag{5.5}$$

Here the 1-form $(df)J$ equals, at any point $x\in M$, the composite of $J_x:T_xM\longrightarrow T_xM$ followed by $df_x:T_xM\longrightarrow\mathrm{I\!R}$. For any torsionfree connection ∇ on the complex manifold M such that $\nabla J=0$, (5.5) is easily seen to become

$$2i\partial\overline{\partial}f=bJ\,+\,Jb,\quad\text{where } b=\nabla df. \tag{5.6}$$

The Weyl tensor acts on 2-forms α in a Riemannian manifold of dimension $n\ge 4$, with the scalar curvature s, via the Weitzenböck formula [9, p. 409], [10, p. 458], immediate from the Ricci identity:

$$W\alpha=\frac{1}{2}\left[\delta(\nabla\alpha-d\alpha)-d\delta\alpha\right]+\frac{n-4}{2(n-2)}\{\mathrm{r},\alpha\}+\frac{\mathrm{s}}{(n-1)(n-2)}\alpha, \tag{5.7}$$

$\{\,,\}$ being the anticommutator, r the Ricci tensor. In local coordinates,

$$\begin{aligned}W_{ijpq}\alpha^{pq}&=-\,\alpha_{pi,j}{}^{p}-\alpha_{jp,i}{}^{p}-\alpha_{pj,}{}^{p}{}_{i}+\alpha_{pi,}{}^{p}{}_{j}\\&\quad+\frac{n-4}{n-2}(\mathrm{r}_j^p\alpha_{ip}+\mathrm{r}_i^p\alpha_{pj})+\frac{2\mathrm{s}}{(n-1)(n-2)}\alpha_{ij}.\end{aligned} \tag{5.8}$$

When $n=4$, (5.4) and (5.8) yield

$$\text{a) } W\omega\;=\;\frac{\mathrm{s}}{6}\omega,\qquad \text{b) } 2W\eta\;=\;\Delta\rho\;-\;i\partial\overline{\partial}\mathrm{s}\;+\;\frac{\mathrm{s}}{3}\eta. \tag{5.9}$$

Here Δ sends a twice-covariant tensor field a to Δa given by $[\Delta a]_{pq}=a_{pq,k}{}^{k}$.

Theorem 5.1. *For a Kähler surface (M,g,J) with the Weyl, Ricci, Einstein tensors W,r,e, Ricci and Einstein forms ρ,η, and the scalar curvature s, the following six conditions are mutually equivalent.*

(a) *(M,g,J) is weakly Einstein.*
(b) *$R\mathrm{r}$ equals a functional multiple of g.*
(c) $6W\mathrm{e}\;=\;-\mathrm{s}\mathrm{e}$.
(d) $3W\eta\;=\;-\mathrm{s}\eta$.
(e) $\Delta\rho\;-\;i\partial\overline{\partial}\mathrm{s}\;=\;-\mathrm{s}\eta$.
(f) $2\Delta\mathrm{r}\;-\;\nabla d\mathrm{s}\;+\;J[\nabla d\mathrm{s}]J\;=\;-2\mathrm{s}\mathrm{e}$.

We prove Theorem 5.1 in the next section.

Remark 5.2. Since $R\mathrm{r}$ behaves "multiplicatively" under Riemannian products, and equals K^2g for a surface metric with Gaussian curvature K, a Riemannian product of two surfaces, being locally Kähler, is – according to (a), (b) above – weakly Einstein if and only if the two Gaussian curvatures are constant and equal or mutually opposite. The latter case constitutes the example found by Euh, Park, and Sekigawa [15].

Remark 5.3. The equivalence of (a) through (c) above really amounts to a statement about algebraic curvature tensors, once we replace the Kähler condition by (4.8) and (a) by $\mathrm{trc}R$ is a multiple of g. We could include (d) here as well if, instead of just (4.8), we invoked one further property of Kähler-type algebraic curvature tensors, namely, (5.9-a). For more details, see Section 6.

Remark 5.4. As shown by Cartan [7], see also [10, Theorem 14.7], local-homothety types of locally symmetric Riemannian four-manifolds form seven disjoint classes: *flat, spherical, hyperbolic, complex projective, complex hyperbolic* and, finally, *nonflat* $2+2$ *and* $1+3$ *Riemannian products* (with Einstein factors). Of these, the first five – due to their being Einstein, cf. (4.5) – are weakly Einstein. The sixth one is weakly Einstein when the two Gaussian curvatures are equal (the Einstein case) or opposite (Remark 5.2). Remark 5.2 also shows that the remaining products listed above are not weakly Einstein, the $1+3$ case, with $W=0$ and $\mathrm{s}\neq 0$, being immediate from the sentence following (4.6).

Remark 5.5. As shown by Tanno [21], up to local isometries, the only conformally flat Kähler surfaces are the Riemannian products of real surfaces with opposite constant Gaussian curvatures.

Remark 5.6. As observed by Calabi [6], cf. also [2, Sect. 2.140], on any Kähler manifold, for a function $\boldsymbol{\tau}$ with a real holomorphic gradient $v=\nabla\boldsymbol{\tau}$, one has $2\mathrm{r}(v,\,\cdot\,)=-d\Delta\boldsymbol{\tau}$. In the Kähler-Einstein case this implies that, locally, at points where $v\neq 0$, the Laplacian of $\boldsymbol{\tau}$ is a function of $\boldsymbol{\tau}$.

6. Proof of Theorem 5.1

That (a) is equivalent to both (b) and (c), is immediate: the former from (4.4) and (4.10), the latter due to (4.7). We now proceed to show that (c) holds if and only if (d) does, still using a purely algebraic argument (cf. Remark 5.3), and adopting the notation of the lines preceding (4.1), with $\mathcal{T}$ standing for the tangent space of M at a given point. The inner product g provides the identifications

$$\text{(6.1)}\qquad \text{a) } \mathcal{T}=\mathcal{T}^*,\qquad \text{b) } \mathcal{T}^{\wedge 2}=\mathfrak{so}(\mathcal{T})=[\mathcal{T}^*]^{\wedge 2},$$

so that $u\wedge v$, for $u,v\in\mathcal{T}$, becomes the endomorphism of $\mathcal{T}$ given by

$$\text{(6.2)}\qquad w\longmapsto(u\wedge v)w=\langle u,w\rangle v-\langle v,w\rangle u$$

We assume that $\mathrm{e}\neq 0$, and choose in $\mathcal{T}$ an orthonormal basis of the form u,Ju,v,Jv, diagonalizing e with some eigenvalues $(a,a,-a,-a)$. Thus, by (6.2), $J=u\wedge Ju+v\wedge Jv$, and so, for the basis $\xi^1,\dots,\xi^4$ of $\mathcal{T}^*$ dual to u,Ju,v,Jv,

$$\text{(6.3)}\quad \omega=\xi^1\wedge\xi^2+\xi^3\wedge\xi^4,\quad \mathrm{e}=a(\xi^1\otimes\xi^1+\xi^2\otimes\xi^2-\xi^3\otimes\xi^3-\xi^4\otimes\xi^4),\quad \eta=a(\xi^1\wedge\xi^2-\xi^3\wedge\xi^4),$$

with $a\neq 0$. Here $\omega=g(J\,\cdot\,,\,\cdot\,)$ and $\eta=\mathrm{e}(J\,\cdot\,,\,\cdot\,)$, while, for 1-forms $\xi,\zeta\in\mathcal{T}^*$, we set

$$\text{(6.4)}\qquad [\xi\otimes\zeta]_{pq}=\xi_p\zeta_q,\qquad [\xi\wedge\zeta]_{pq}=\xi_p\zeta_q-\xi_q\zeta_p.$$

Due to (5.9-a) and (6.3) with $a\neq 0$, we have $3W\eta=-\mathrm{s}\eta$ if and only if

$$\text{(6.5)}\quad 12W(\xi^1\wedge\xi^2)=-\mathrm{s}\,\xi^1\wedge\xi^2+3\mathrm{s}\,\xi^3\wedge\xi^4,\quad 12W(\xi^3\wedge\xi^4)=3\mathrm{s}\,\xi^1\wedge\xi^2-\mathrm{s}\,\xi^3\wedge\xi^4.$$

As stated in the lines following (4.3), any algebraic curvature tensor R acts on bivectors via $2[R\alpha]_{ij}=R_{ijpq}\alpha^{pq}$, so that $R(w\wedge w')=R(w,w')$. Thus, (6.5) is nothing else than

$$\text{(6.6)}\qquad \begin{array}{l} W_{1212}=W_{3434}=-\mathrm{s}/12\ \text{ and }\ W_{1234}=\mathrm{s}/4,\ \text{ while}\\ W_{12ij}=W_{34ij}=0\ \text{ unless }\ \{i,j\}\ \text{ is }\ \{1,2\}\ \text{ or }\ \{3,4\}.\end{array}$$

In terms of components relative to our basis u,Ju,v,Jv, the well-known fact [10, p. 647] that W commutes with the Hodge star amounts to

$$\text{(6.7)}\quad W_{ijkl}=W_{pqrs}\ \text{ if }\ (i,j,p,q)\ \text{ and }\ (k,l,r,s)\ \text{ are even permutations of }\ (1,2,3,4).$$

Since $[W\mathrm{e}]_{ij}=W_{ipjq}\mathrm{e}^{pq}$, and $Wg=0$, (6.3) and (6.7) give $[W\mathrm{e}]_{11}=[W\mathrm{e}]_{22}=2aW_{1212}$, $[W\mathrm{e}]_{12}=[W\mathrm{e}]_{34}=0$, $[W\mathrm{e}]_{13}=2aW_{1232}$, $[W\mathrm{e}]_{14}=2aW_{1242}$, $[W\mathrm{e}]_{23}=2aW_{1213}$, $[W\mathrm{e}]_{24}=2aW_{1214}$, $[W\mathrm{e}]_{33}=[W\mathrm{e}]_{44}=-2aW_{3434}$. As $\mathrm{e}_{11}=\mathrm{e}_{22}=a=-\mathrm{e}_{33}=-\mathrm{e}_{44}$ and $\mathrm{e}_{ij}=0$

otherwise, this description of We, combined with (6.6), proves that (c) is equivalent to (d). More precisely, the above equalities amount to (6.7) except the formula for W_{1234} which, however, then follows as (5.9-a) and (6.3) yield $W_{1234} = -W_{1212} + \mathrm{s}/6 = \mathrm{s}/4$.

The equivalence of (d) and (e) is in turn immediate from (5.9-b).

Finally, (f) is – due to (5.6) – precisely the result of applying J to (e).

7. Proof of Theorem 1.2

On any Kähler surface, with the orientation as in (2.1),

$$\text{the Einstein form } \eta \text{ is always anti-self-dual.} \tag{7.1}$$

This is clear since 'anti-self-dual' is well known [10, Corollary 37.3, Proposition 37.5] to mean the same as *orthogonal to* ω *and commuting with* ω, in the sense of the identification of 2-forms with skew-adjoint endomorphisms provided by (6.1-b), while J-invariance of r amounts to its commuting with J.

Let a Kähler surface (M, g) be weakly Einstein and self-dual.

Theorem 5.1(d) gives $3W\eta = -\mathrm{s}\eta$ while, from (7.1), $W\eta = 0$ due to the self-duality assumption. Thus, $\mathrm{se} = 0$ everywhere. By Remark 2.1, one of s, e is identically zero, and in either case s is constant. Constancy of s combined with self-duality implies local symmetry [5, Proposition 9.3], [9, Lemma 7].

The final clause of Theorem 1.2 is immediate from Remarks 5.4 and 5.5.

8. Proof of Theorem 1.3

In [11, Sect. 5] one fixes a nonuple $I, a, \Sigma, h, \mathcal{L}, (\,,\,), \mathcal{H}, \gamma, Q$ consisting of

(i) a nontrivial closed interval $I = [\boldsymbol{\tau}_{\min}, \boldsymbol{\tau}_{\max}]$ of the variable $\boldsymbol{\tau}$,

(ii) a real number $a > 0$,

(iii) a compact Kähler manifold (Σ, h) of complex dimension 1,

(iv) a function $Q : I \longrightarrow \mathrm{I\!R}$ equal to 0 at the endpoints of I and positive on its interior I^o, such that $\dot{Q}(\boldsymbol{\tau}_{\min}) = 2a = -\dot{Q}(\boldsymbol{\tau}_{\max})$,

(v) a mapping $\gamma : I \longrightarrow \mathrm{I\!RP}^1$, with $I \subseteq \mathrm{I\!R} \subseteq \mathrm{I\!RP}^1$,

(vi) a complex line bundle $\mathcal{L}$ over Σ with a Hermitian fibre metric $(\,,\,)$,

(vii) the horizontal distribution $\mathcal{H}$ of a connection in $\mathcal{L}$ making the fibre metric $(\,,\,)$ parallel and having the curvature form $-a(\boldsymbol{\tau}_* - \gamma)^{-1}\omega^{(h)}$,

where $\boldsymbol{\tau}_* \in I$ is the midpoint, $\omega^{(h)}$ is the Kähler form of (Σ, h) and $(\,)^{\dot{}} = d/d\boldsymbol{\tau}$. One also fixes a diffeomorphism $I^o \ni \boldsymbol{\tau} \longmapsto r \in (0, \infty)$ such that $\dot{r} = ar/Q$, and uses the symbol r both for an independent variable ranging over $[0, \infty)$ and for the norm function $r : \mathcal{L} \longrightarrow [0, \infty)$ of the fibre metric $(\,,\,)$, so that our fixed diffeomorphism turns $\boldsymbol{\tau}$, and hence Q as well, into a function on the total space $\mathcal{L}$.

For the section v of the vertical distribution $\mathcal{V}$ on $\mathcal{L}$ which, restricted to each fibre, equals our a times the radial (identity) vector field, d_v acts on functions of $\boldsymbol{\tau}$ as $Q\,d/d\boldsymbol{\tau}$, and v equals the g-gradient of $\boldsymbol{\tau}$ for the Kähler-surface metric g on $\mathcal{L}$ with

$$\begin{aligned} &g(v, v) = g(u, u) = Q, \qquad g(v, u) = g(v, w) = g(u, w) = 0,\\ &g(w, w') = (\boldsymbol{\tau}_* - \gamma)^{-1}(\boldsymbol{\tau} - \gamma)h(w, w'), \end{aligned} \tag{8.1}$$

w, w', w'' always denoting horizontal lifts (that is, projectable horizontal vector fields), and u the vector field on $\mathcal{L}$ defined by $u = iv$ (multiplication by i in each fibre). See [11, pp. 1648–1649]. Then g is a Kähler metric for the almost-complex structure J obtained by requiring that the vertical subbundle $\mathcal{V}$ of TM and $\mathcal{H}$ be J-invariant and the restriction of J to $\mathcal{V}$, or to $\mathcal{H}$, coincide with the complex structure of the fibres or, respectively, with the pullback of the complex structure of Σ.

Let M be the $\mathbb{C}\mathrm{P}^1$ bundle over $\varSigma$ resulting from the projective compactification of $\mathcal{L}$. According to [11, Theorem 5.3] and the text preceding it in [11], the above construction gives rise to a compact Kähler surface (M,g) with the nonconstant transnormal function $\boldsymbol{\tau}$ having the holomorphic gradient v, and $\boldsymbol{\tau}$ is isoparametric if and only if γ is constant, while, conversely, any compact Kähler surface carrying a non-isoparametric transnormal function with a holomorphic gradient arises from this construction, for a nonconstant γ.

The second paragraph of [11, Remark 5.2] points out that we can relax conditions (iii) and (iv), while keeping (ii) and (v) – (vii), so that $\varSigma$ need not be compact, and Q is defined and positive on an open interval. The construction then yields

$$\text{a quadruple } M, g, J, \boldsymbol{\tau} \text{ with the same properties} \tag{8.2}$$

except compactness of M, where M now is any connected component of the open set in $\mathcal{L}\smallsetminus\varSigma$ defined by requiring that $\boldsymbol{\tau}\neq\gamma\neq\boldsymbol{\tau}_*$ and that the values of the norm function r lie in the resulting new range.

The classification result of [11, Theorem 5.3] remains valid, *mutatis mutandis*, in this more general setting: any point at which $d\boldsymbol{\tau}\wedge d\Delta\boldsymbol{\tau}\neq 0$, for a transnormal function $\boldsymbol{\tau}$ with a holomorphic gradient on a Kähler surface, has a neighborhood biholomorphically isometric to a noncompact Kähler surface obtained as described in the last paragraph, with nonconstant γ. To see this, note that, instead of creating the data (i) – (vii) "globally" as in [11, Sect. 11], we may invoke the arguments of [12, Sect. 7], since they all remain valid if $\boldsymbol{\tau}$ is just assumed transnormal, rather than isoparametric.

We now proceed to show that the case of nonconstant γ just mentioned cannot occur when the resulting Kähler surface is weakly Einstein, by assuming the weakly Einstein property with nonconstant γ and deriving a contradiction.

This will clearly prove Theorem 1.3.

To simplify some expressions later, let us note that

$$\begin{aligned} h(w,w')D\gamma-h(D\gamma,w')w&=-h(JD\gamma,w)Jw',\\ h(D\gamma,w')Jw-h(Jw,w')D\gamma&=h(D\gamma,w)Jw',\\ h(D\gamma,w')w-h(D\gamma,w)w'&=-h(Jw,w')JD\gamma,\\ h(D\gamma,w')Jw-h(D\gamma,w)Jw'&=h(Jw,w')D\gamma. \end{aligned}\tag{8.3}$$

In fact, the second equality in (8.3) holds trivially when w,w' are linearly dependent, and its two sides yield the same h-inner product with w, as well as with w'. The first (or, third, or fourth) equality arises from the second one by replacing w with Jw (or, applying $-J$ to both sides or, respectively, moving two terms to the other side).

The third line of (8.3) gives $h(D\gamma,D\gamma)w-h(D\gamma,w)D\gamma=h(JD\gamma,w)JD\gamma$ if one sets $w'=D\gamma$. Applying $h(\,\cdot\,,w)$ we get

$$h(D\gamma,D\gamma)h(w,w)\;-\;[h(D\gamma,w)]^2\;=\;[h(JD\gamma,w)]^2.\tag{8.4}$$

This will explain a seemingly strange presence of $[h(JD\gamma,w)]^2$ in a later discussion, instead of the (expected) left-hand side of (8.4).

For the Levi-Civita connection of g by ∇,

$$\begin{aligned} &\nabla_{\!v}v=-\nabla_{\!u}u=\psi v,\qquad \nabla_{\!v}u=\nabla_{\!u}v=\psi u,\\ &\nabla_{\!v}w=\nabla_{\!w}v=\phi w,\qquad \nabla_{\!u}w=\nabla_{\!w}u=\phi Jw,\\ &\nabla_{\!w}w'=D_ww'+S[h(D\gamma,w)w'+h(JD\gamma,w)Jw']\\ &\qquad\quad-[2(\boldsymbol{\tau}_*-\gamma)]^{-1}[h(w,w')v+h(Jw,w')u]. \end{aligned}\tag{8.5}$$

Here D stands both for the Levi-Civita connection of the base-surface metric h and for the h-gradient, while ψ,ϕ,S are the functions given by

$$2\psi=\dot{Q},\qquad 2\phi=Q/(\boldsymbol{\tau}-\gamma),\qquad 2S=(\boldsymbol{\tau}_*-\gamma)^{-1}-(\boldsymbol{\tau}-\gamma)^{-1},\tag{8.6}$$

where $(\ )^{\cdot} = d/d\boldsymbol{\tau}$. We have used the first line of (8.3) to replace the expression $S[h(D\gamma,w)w' + h(D\gamma,w')w - h(w,w')D\gamma]$ appearing (with a slightly different notation) in the description of ∇ [11, p. 1648], by $S[h(D\gamma,w)w' + h(JD\gamma,w)Jw']$.

Applying the equality $2\mathrm{r}(v,\,\cdot\,) = -d\Delta\boldsymbol{\tau}$ in Remark 5.6 to $\boldsymbol{\tau}$ and v appearing above, that is, in [11, formula (5.2)], and noting that $\Delta\boldsymbol{\tau} = 2(\psi+\phi)$, while d_v acts on functions of $\boldsymbol{\tau}$ as $Q\,d/d\boldsymbol{\tau}$, we get

$$\begin{aligned}&2(\boldsymbol{\tau}-\gamma)^2\mathrm{r}(v,v) = 2(\boldsymbol{\tau}-\gamma)^2\mathrm{r}(u,u) = QP,\\ &\text{for } P = Q - (\boldsymbol{\tau}-\gamma)\dot{Q} - (\boldsymbol{\tau}-\gamma)^2\ddot{Q},\\ &\mathrm{r}(v,u) = 0, \quad 2(\boldsymbol{\tau}-\gamma)^2\mathrm{r}(v,w) = -Qh(D\gamma,w),\\ &2(\boldsymbol{\tau}-\gamma)^2\mathrm{r}(u,w) = -Qh(JD\gamma,w),\end{aligned} \tag{8.7}$$

using J-invariance of r to derive the equations involving u from those for v. This shortcut describes all components of r except $\mathrm{r}(w,w')$ for two horizontal vectors w,w'.

With the sign convention $R(v,w)u = \nabla_{[v,w]}u + \nabla_w\nabla_v u - \nabla_v\nabla_w u$, (8.5) and the equality $-Q\dot{\phi} = 2(\phi-\psi)\phi$, immediate from (8.6), yield the following equalities, describing all components of the $(0,4)$ curvature tensor except those involving four horizontal vectors:

$$\begin{aligned}&2R(v,u)v = -Q\ddot{Q}u, \quad 2R(v,u)u = Q\ddot{Q}v, \quad R(v,u)w = 2(\phi-\psi)\phi Jw,\\ &R(v,w)v = (\phi-\psi)\phi w, \quad R(v,w)u = (\phi-\psi)\phi Jw,\\ &2R(v,w)w' = (\boldsymbol{\tau}_*-\gamma)^{-1}(\psi-\phi)[h(w,w')v + h(Jw,w')u] - (\boldsymbol{\tau}-\gamma)^{-2}Qh(JD\gamma,w)Jw',\\ &R(u,w)v = (\psi-\phi)\phi Jw, \quad R(u,w)u = (\phi-\psi)\phi w,\\ &2R(u,w)w' = (\boldsymbol{\tau}_*-\gamma)^{-1}(\phi-\psi)[h(Jw,w')v - h(w,w')u] + (\boldsymbol{\tau}-\gamma)^{-2}Qh(D\gamma,w)Jw',\\ &2R(w,w')v = -(\boldsymbol{\tau}-\gamma)^{-2}Qh(Jw,w')JD\gamma + 2(\boldsymbol{\tau}_*-\gamma)^{-1}(\phi-\psi)h(Jw,w')u,\\ &2R(w,w')u = (\boldsymbol{\tau}-\gamma)^{-2}Qh(Jw,w')D\gamma + 2(\boldsymbol{\tau}_*-\gamma)^{-1}(\psi-\phi)h(Jw,w')v.\end{aligned}$$

where we simplified $2R(w,w')v$ using the third line of (8.3) and then derived the expression for $2R(w,w')u$ from the fact that $R(w,w')u = J[R(w,w')v]$.

For dimensional reasons, a "horizontal" component $R(w,w',w'',w''')$ must be given by

$$2(\boldsymbol{\tau}_*-\gamma)(\boldsymbol{\tau}-\gamma)R(w,w',w'',w''') = Z[h(w,w'')g(w',w''') - h(w',w'')g(w,w''')] \tag{8.8}$$

for some function Z not depending on w,w',w'',w'''. Before determining what Z is, we now characterize the weakly-Einstein case as a condition imposed on Z. First,

$$\begin{aligned}2(\boldsymbol{\tau}_*-\gamma)(\boldsymbol{\tau}-\gamma)R(w,w')w'' = {}& Z[h(w,w'')w' - h(w',w'')w]\\ &+ h(Jw,w')h(JD\gamma,w'')v - h(Jw,w')h(D\gamma,w'')u,\end{aligned} \tag{8.9}$$

which now easily implies that, with $\widetilde{Z} = Z + 2(\boldsymbol{\tau}-\gamma)(\phi-\psi)$,

$$2(\boldsymbol{\tau}_*-\gamma)(\boldsymbol{\tau}-\gamma)\mathrm{r}(w,w') = \widetilde{Z}h(w,w').$$

The Ricci endomorphism of TM acts by

$$\begin{aligned}&2(\boldsymbol{\tau}-\gamma)^3\mathrm{r}v = (\boldsymbol{\tau}-\gamma)Pv - (\boldsymbol{\tau}_*-\gamma)QD\gamma, \quad \mathrm{r}u = J\mathrm{r}v,\\ &2(\boldsymbol{\tau}-\gamma)^2\mathrm{r}w = \widetilde{Z}w - h(D\gamma,w)v - h(JD\gamma,w)u.\end{aligned} \tag{8.10}$$

The value assigned by $R\mathrm{r}$ to each of the six pairs

$$(v,v),\ (v,u),\ (v,w),\ (u,u),\ (u,w),\ (w,w')$$

of vector fields equals the trace of the composition in which the Ricci endomorphism, with (8.10), is followed by

$$R(v,\,\cdot\,)v,\ R(v,\,\cdot\,)u,\ R(v,\,\cdot\,)w,\ R(u,\,\cdot\,)u,\ R(u,\,\cdot\,)w,\ R(w,\,\cdot\,)w'. \tag{8.11}$$

Due to Hermitian symmetry of $R\mathrm{r}$ (that is, its J-invariance) we only need to consider three of the six pairs: (v,v), (v,w) and (w,w'). First,

$$\begin{array}{l}R(v,\,\cdot\,)v \quad\text{and}\quad R(u,\,\cdot\,)u \quad\text{send the triple}\quad (v,u,w)\quad\text{to}\\ (0,-Q\ddot{Q}u/2,(\phi-\psi)\phi w)\quad\text{and}\quad(-Q\ddot{Q}v/2,0,(\phi-\psi)\phi w).\end{array}\tag{8.12}$$

Hence, with $\widetilde{Z}=Z+2(\boldsymbol{\tau}-\gamma)(\phi-\psi)$ as before,

$$4(\boldsymbol{\tau}-\gamma)^2[R\mathrm{r}](v,v)=4(\phi-\psi)\phi\widetilde{Z}-Q\ddot{Q}P.$$

Similarly,

$$4(\boldsymbol{\tau}-\gamma)^4[R\mathrm{r}](v,w)=[4(\boldsymbol{\tau}-\gamma)^2(\psi-\phi)\phi-Q\widetilde{Z}]h(D\gamma,w),$$

and $4(\boldsymbol{\tau}_*-\gamma)(\boldsymbol{\tau}-\gamma)^4[R\mathrm{r}](w,w')$ equals $h(w,w')$ times

$$(\boldsymbol{\tau}-\gamma)Z\widetilde{Z}+2(\boldsymbol{\tau}_*-\gamma)Qh(D\gamma,D\gamma)+2(\boldsymbol{\tau}-\gamma)^2(\phi-\psi)P\tag{8.13}$$

or, equivalently, $4(\boldsymbol{\tau}-\gamma)^5[R\mathrm{r}](w,w')$ equals $g(w,w')$ times (8.13). In the case where γ is nonconstant, for $R\mathrm{r}$ to be a functional multiple of g it is necessary and sufficient that $4(\boldsymbol{\tau}-\gamma)^2(\psi-\phi)\phi=Q\widetilde{Z}$ and that $(\boldsymbol{\tau}-\gamma)^3[4(\phi-\psi)\phi\widetilde{Z}-Q\ddot{Q}P]$ be equal to Q times (8.13). Since $2(\boldsymbol{\tau}-\gamma)\phi=Q$, the first condition amounts to

$$\widetilde{Z}=2(\boldsymbol{\tau}-\gamma)(\psi-\phi),\quad\text{that is,}\ Z=4(\boldsymbol{\tau}-\gamma)(\psi-\phi).\tag{8.14}$$

Assuming (8.14), and noting that

$$-(\boldsymbol{\tau}-\gamma)^2\ddot{Q}=P-Q+(\boldsymbol{\tau}-\gamma)\dot{Q}=P+2(\boldsymbol{\tau}-\gamma)(\psi-\phi),$$

we rewrite the second condition as

$$2(\boldsymbol{\tau}_*-\gamma)Qh(D\gamma,D\gamma)=(\boldsymbol{\tau}-\gamma)[P+6(\boldsymbol{\tau}-\gamma)(\psi-\phi)][P-2(\boldsymbol{\tau}-\gamma)(\psi-\phi)].\tag{8.15}$$

Since $P=Q-(\boldsymbol{\tau}-\gamma)\dot{Q}-(\boldsymbol{\tau}-\gamma)^2\ddot{Q}$ while, by (8.6), $2\psi=\dot{Q}$ and $2(\boldsymbol{\tau}-\gamma)\phi=Q$, the two factors in square brackets on the right-hand side of (8.15) are equal to

$$-(\boldsymbol{\tau}-\gamma)^2\ddot{Q}\;+\;2[(\boldsymbol{\tau}-\gamma)\dot{Q}-Q]\quad\text{and}\quad-(\boldsymbol{\tau}-\gamma)^2\ddot{Q}\;-\;2[(\boldsymbol{\tau}-\gamma)\dot{Q}-Q],$$

so that their product is $(\boldsymbol{\tau}-\gamma)^4\ddot{Q}^2-4[(\boldsymbol{\tau}-\gamma)\dot{Q}-Q]^2$ and we may rewrite the right-hand side of (8.15), divided by Q, as

$$\begin{array}{l}-Q^{-1}\ddot{Q}^2\gamma^5+5\boldsymbol{\tau}Q^{-1}\ddot{Q}^2\gamma^4+2Q^{-1}[2\dot{Q}^2-5\boldsymbol{\tau}^2\ddot{Q}^2]\gamma^3+2[4\dot{Q}+6\boldsymbol{\tau}Q^{-1}\dot{Q}^2+5\boldsymbol{\tau}^3Q^{-1}\ddot{Q}^2]\gamma^2\\ \quad+[4Q-16\boldsymbol{\tau}\dot{Q}+12\boldsymbol{\tau}^2Q^{-1}\dot{Q}^2-5\boldsymbol{\tau}^4Q^{-1}\ddot{Q}^2]\gamma-4\boldsymbol{\tau}Q+8\boldsymbol{\tau}^2\dot{Q}-4\boldsymbol{\tau}^3Q^{-1}\dot{Q}^2+\boldsymbol{\tau}^5Q^{-1}\ddot{Q}^2.\end{array}$$

This is a quintic polynomial in γ, with coefficients that are functions of $\boldsymbol{\tau}$, which equals – according to (8.15) – a function on base-surface Σ. Applying d_v to the latter function we get 0 and, since d_v acts on functions of $\boldsymbol{\tau}$ as $Q\,d/d\boldsymbol{\tau}$, we conclude that *the coefficients of the above quintic polynomial are constant functions of* $\boldsymbol{\tau}$. However, constancy of both $Q^{-1}\ddot{Q}$ and $\boldsymbol{\tau}Q^{-1}\ddot{Q}$ means that $\ddot{Q}=0$, and hence $\dot{Q}$ is constant. Looking at the coefficients of γ^3 and γ we now see that Q must be constant, and hence zero.

This contradiction proves that γ is constant. In other words, our $\boldsymbol{\tau}$, besides being transnormal, is also isoparametric.

9. The local-structure theorem

By a *special Kähler-Ricci potential* [12, Sect. 7] on a Kähler manifold (M,g,J) one means any nonconstant function $\boldsymbol{\tau}$ on M having a real-holomorphic gradient $v=\nabla\boldsymbol{\tau}$ for which, at points where $v\neq 0$, all nonzero vectors orthogonal to v and Jv are eigenvectors of both $\nabla d\boldsymbol{\tau}$ and the Ricci tensor r. Such quadruples $(M,g,J,\boldsymbol{\tau})$ have been completely described, both locally [12] and in the compact case [13].

In the case of Kähler surfaces, $\boldsymbol{\tau}$ as above is nothing else than a nonconstant isoparametric function with a holomorphic gradient. In fact, generally, on any Riemannian manifold, for $v = \nabla\boldsymbol{\tau}$ and $Q = g(v,v)$ one has $dQ = 2[\nabla d\boldsymbol{\tau}](v,\,\cdot\,)$ while, if v is real holomorphic on a Kähler manifold, $2\mathrm{r}(v,\,\cdot\,) = -d\Delta\boldsymbol{\tau}$ (see Remark 5.6).

The construction in [12] is a special case of [11, pp. 1648–1649], with constant γ. The nonzero constant $\boldsymbol{\tau}_* - \gamma$ in (8.1) is replaced with $\varepsilon/2$, where $\varepsilon = \pm 1$ (with no loss of generality, since the base-surface metric h can be rescaled). Now, from (8.1), as in [12, pp. 791–792 and Sect. 16]

$$\begin{aligned} &g(v,v) = g(u,u) = Q, \qquad g(v,u) = g(v,w) = g(u,w) = 0,\\ &g(w,w') = 2\varepsilon(\boldsymbol{\tau}-\gamma)h(w,w'), \end{aligned} \tag{9.1}$$

w, w' still denoting horizontal lifts (projectable horizontal vector fields). In terms of the vertical and horizontal distributions $\mathcal{V} = \mathrm{Span}(v,u)$ and $\mathcal{H} = \mathcal{V}^\perp$,

$$\begin{aligned} &\mathrm{r} = \mu g \text{ on } \mathcal{V}, \qquad \mathrm{r} = \lambda g \text{ on } \mathcal{H}, \qquad \mathrm{r}(\mathcal{V},\mathcal{H}) = \{0\}, \quad \text{where}\\ &\mu = -\dot{Y}/2, \quad \lambda = [2\varepsilon(\boldsymbol{\tau}-\gamma)]^{-1}(K - \varepsilon Y), \text{ for } Y = 2(\psi+\phi) \end{aligned} \tag{9.2}$$

(so that $Y = \Delta\boldsymbol{\tau}$) and K is the Gaussian curvature of h. See (8.7) and [12, formula (7.4), the lines following (8.1), and (b) in Sect. 16], where our γ is denoted by c. Thus, once we identify v with $g(v,\,\cdot\,)$, and similarly for u,

$$\begin{aligned} \mathrm{r} \;&=\; \lambda g \,+\, (\mu-\lambda)Q^{-1}(v\otimes v \,+\, u\otimes u), \quad \text{and hence}\\ R\mathrm{r} \;&=\; \lambda\mathrm{r} \,+\, (\mu-\lambda)Q^{-1}[R(v,\,\cdot\,,v,\,\cdot\,) + R(u,\,\cdot\,,u,\,\cdot\,)]. \end{aligned} \tag{9.3}$$

From (8.12) and (9.2), $R\mathrm{r}$ treated as an endomorphism of TM sends v, u, w to

$$[\lambda\mu + (\lambda-\mu)\ddot{Q}/2]v, \quad [\lambda\mu + (\lambda-\mu)\ddot{Q}/2]u, \quad [\lambda^2 + (\boldsymbol{\tau}-\gamma)^{-1}(\mu-\lambda)(\phi-\psi)]w,$$

ψ, ϕ being – as in (8.6) – the functions given by

$$2\psi = \dot{Q}, \qquad 2\phi = Q/(\boldsymbol{\tau}-\gamma) \tag{9.4}$$

Thus, $R\mathrm{r}$ is a functional multiple of the identity (cf. Theorem 5.1) if and only if

$$(\boldsymbol{\tau}-\gamma)^2\ddot{Q} \,+\, 2Q \;=\; \varepsilon K(\boldsymbol{\tau}-\gamma), \tag{9.5}$$

as long as we exclude the Einstein case by assuming that $\mu \neq \lambda$. (See Remark 9.2 below.)

One immediate conclusion is that K, the Gaussian curvature of the base-surface metric h, must be constant: $d_w\boldsymbol{\tau} = 0$ and Q is a function of $\boldsymbol{\tau}$, so d_w applied to (9.5) yields $d_wK = 0$. Solving (9.5), we see that

$$\begin{aligned} &Q \text{ as a function of } \boldsymbol{\tau} \text{ equals } \varepsilon K(\boldsymbol{\tau}-\gamma)/2 \text{ plus a linear combination of}\\ &|\boldsymbol{\tau}-\gamma|^{1/2}\cos[\sqrt{7}\,(\log|\boldsymbol{\tau}-\gamma|)/2] \;\text{ and }\; |\boldsymbol{\tau}-\gamma|^{1/2}\sin[\sqrt{7}\,(\log|\boldsymbol{\tau}-\gamma|)/2]. \end{aligned} \tag{9.6}$$

Theorem 9.1. *Defining the quadruple* $(M, g, J, \boldsymbol{\tau})$ *by the local version* (8.2) *of the construction in Section* 8*, for* $Q > 0$ *as in* (9.6)*, we obtain a nonconstant isoparametric function* $\boldsymbol{\tau}$ *with a real-holomorphic gradient on a weakly Einstein Kähler surface.*

Conversely, up to local biholomorphic isometries, any nonconstant transnormal function with a real-holomorphic gradient on a weakly Einstein Kähler surface arises in the manner just described.

Proof. The first part of the theorem is immediate from the preceding discussion and Theorem 5.1(b). The final clause is, according to the second paragraph of this section, a special case of [12, Theorem 18.1], since 'transnormal' in our situation implies 'isoparametric' as a consequence of Theorem 1.3. □

Remark 9.2. We arrived at (9.5) after dividing an intermediate equality by $\lambda-\mu$ ("excluding the Einstein case"). By (9.2) and (9.4), $2(\boldsymbol{\tau}-\gamma)(\lambda-\mu) = (\boldsymbol{\tau}-\gamma)^2\ddot{Q} - 2Q + \varepsilon K(\boldsymbol{\tau}-\gamma)$, so that g is an Einstein metric on any nonempty open set on which

$$(\boldsymbol{\tau}-\gamma)^2\ddot{Q} \,-\, 2Q \;=\; -\varepsilon K(\boldsymbol{\tau}-\gamma). \tag{9.7}$$

Dividing by $\lambda-\mu$ in the non-Einstein case is allowed, without assuming real-analyticity, even when such a nonempty open set exists. In fact, for some open interval I' of the variable $\boldsymbol{\tau}$, one then has (9.7) on a proper subset P of I' with a nonempty interior (which is a disjoint union of a countable family of open intervals), and on $I'\smallsetminus P$, which is also such a union, (9.5) holds. If $I''\subseteq P$, or $I''\subseteq I'\smallsetminus P$, is one of those countably many subintervals, then an endpoint $\boldsymbol{\tau}_0\in I'$ of I'' is a cluster point for both countable unions. The Taylor series of Q at $\boldsymbol{\tau}_0$ thus satisfies the series versions of both equations (9.5) and (9.7), which determines it uniquely – see below – while Q is a solution to (9.7) or, respectively, (9.5), on the half-open interval $I''\cup\{\boldsymbol{\tau}_0\}$. Namely, treated as a formal power series, our Taylor series, satisfying both (9.5) and (9.7), must correspond to

$$Q\,=\,\varepsilon K(\boldsymbol{\tau}-\gamma)/2, \tag{9.8}$$

and consequently have $Q(\boldsymbol{\tau}_0)=\varepsilon K(\boldsymbol{\tau}_0-\gamma)/2$ and $\dot Q(\boldsymbol{\tau}_0)=\varepsilon K/2$. Our function Q is thus given by (9.8), due to uniqueness of a solution to (9.7), or (9.5), on $I''\cup\{\boldsymbol{\tau}_0\}$ with the above initial data. We therefore have (9.8) on an open dense set, and hence everywhere, in I'. In other words, (9.5) *always follows in the weakly-Einstein non-Einstein case*, since otherwise, as we just saw, $Q=\varepsilon K(\boldsymbol{\tau}-\gamma)/2$, implying the Einstein property – namely, the weakly-Einstein metric resulting from the construction based on (9.5), and mentioned in Theorem 9.1, is Einstein if and only if $Q=\varepsilon K(\boldsymbol{\tau}-\gamma)/2$.

Remark 9.3. The weakly-Einstein metric arising in Theorem 9.1 are not locally homogenous case except in the Einstein case (when $Q=\varepsilon K(\boldsymbol{\tau}-\gamma)/2$, cf. Remark 9.2). In fact, for the eigenvalue function μ of the Ricci tensor, (9.2) gives $-2\mu(\boldsymbol{\tau}-\gamma)^2=(\boldsymbol{\tau}-\gamma)^2\ddot Q+(\boldsymbol{\tau}-\gamma)\dot Q-Q$ which, by (9.5), equals $(\boldsymbol{\tau}-\gamma)\dot Q+\varepsilon K(\boldsymbol{\tau}-\gamma)-3Q$. Solving the resulting equation $(\boldsymbol{\tau}-\gamma)\dot Q-3Q=-\varepsilon K(\boldsymbol{\tau}-\gamma)-2\mu(\boldsymbol{\tau}-\gamma)^2$ under the assumption that μ is constant, we get $Q=a(\boldsymbol{\tau}-\gamma)^3+\mu(\boldsymbol{\tau}-\gamma)^2+\varepsilon K(\boldsymbol{\tau}-\gamma)/2$ with a constant a, which is not of the form (9.6) except when $a=\mu=0$ and $Q=\varepsilon K(\boldsymbol{\tau}-\gamma)/2$.

Remark 9.4. In [12, sixth line of Sect. 8] there is a second case, with the factor $2\varepsilon(\boldsymbol{\tau}-\gamma)$ in (9.1) replaced by 1. We leave it out of our discussion, since this is precisely the case of product metrics [12, Corollary 13.2 and (c) in Sect. 16]. See also Remark 5.2.

10. Nonrealizable boundary conditions

The following lemma is a crucial step in the proof of Theorem 1.4.

Lemma 10.1. *For* $F:\mathrm{I\!R}\longrightarrow\mathrm{I\!R}$ *given by* $F(\alpha)=e^{-\alpha\cot c}\sin\alpha$, *where* $c=\tan^{-1}\!\sqrt{7}$, *there do not exist* $\alpha,\beta\in\mathrm{I\!R}$ *such that* $\alpha<\beta$ *and* $F(\alpha)-F(\beta)=F'(\alpha)+F'(\beta)=0$, *while* F *is nonzero everywhere in the open interval* (α,β).

Proof. Since F is positive/negative on the interval $(n\pi,(n+1)\pi)$ for each even/odd integer n, such α,β, if they existed, would both lie in one of the closed intervals $I_n=[n\pi,(n+1)\pi]$. The identity $F(\alpha+\pi)=-e^{-\pi\cot c}F(\alpha)$ shows that the graph of F is the same, up to vertical rescaling, on all I_n, and so it suffices to consider the case $n=1$, that is, prove the nonexistence of $\alpha,\beta\in[0,\pi]$ with the stated properties.

Our F assumes in $[0,\pi]$ the minimum value 0, just at the endpoints, and the maximum value $F(c)$, only at c, as induction on $q\ge 0$ gives, for $F^{(q)}=d^qF/d\alpha^q$,

$$\begin{array}{ll}\text{i)} & F^{(q)}(\alpha)\sin^q c\,=\,(-1)^q e^{-\alpha\cot c}\sin(\alpha-qc),\\ \text{ii)} & F'>0\ \text{ on } [0,c),\ \text{while}\ F'<0\ \text{ on } (c,\pi].\end{array} \tag{10.1}$$

Numerically, $c\approx 1.209429$, and so $c\in(0,\pi/2)$. By (10.1-ii), F maps both $[0,c)$ and $(c,\pi]$ diffeomorphically onto $[0,F(c))$. Thus, for every $\alpha\in[0,\pi]\smallsetminus\{c\}$ there exists a unique $\beta = \beta(\alpha) \in [0,\pi] \smallsetminus \{c\}$ with $\beta \neq \alpha$ and $F(\beta) = F(\alpha)$. Now

$$\text{setting } \beta(c) = c \text{ we obtain a decreasing diffeomorphism } [0,\pi] \ni \alpha \longmapsto \beta(\alpha) \in [0,\pi] \text{ with } \beta'(c) = -1. \tag{10.2}$$

Namely, smoothness at c in (10.2) follows from the Morse lemma: (c,c) is a nondegenerate critical point with the index 0 and value 0 for the function on $\mathrm{I\!R}^2$ sending (α,β) to $F(\alpha) - F(\beta)$, and so the zeros of this function near (c,c) form two smooth curves intersecting transvesally at (c,c). The two curves are the graphs of the identity function and the function $\alpha \longmapsto \beta(\alpha)$ in (10.2). Hence $\beta'(c)$ exists and equals -1, as the invariance of the graph in (10.2) under the switch of α with β causes the vector $(1,-1)$ to be tangent to the graph at (c,c), and thus ensuring that the graph does not pass through (c,c) vertically. Now (10.1-i) gives $F''(\alpha)\sin^2 c = e^{-\alpha\cot c}\sin(\alpha - 2c)$ and $F'''(\alpha)\sin^3 c = e^{-\alpha\cot c}\sin(3c - \alpha)$, so that,

$$\begin{aligned}&\text{as } 3c - \pi \approx 0.4867 \text{ lies in } (0,\pi/4), \text{ and hence } 0 < 3c - \pi < c,\\ &F'' \text{ decreases on } [0, 3c-\pi] \text{ from } -2\cot c \text{ to } F''(3c-\pi), \text{ increases on } [3c-\pi, 2c] \text{ from } F''(3c-\pi) \text{ to } 0, \text{ and increases on } [2c,\pi].\end{aligned} \tag{10.3}$$

The identity $F(\alpha) = F(\beta(\alpha))$ and the chain rule give

$$\beta'(\alpha) = \frac{F'(\alpha)}{F'(\beta(\alpha))} \text{ whenever } \alpha \in [0,c), \tag{10.4}$$

and so the assertion of the lemma amounts to

$$\beta'(\alpha) \neq -1 \text{ for all } \alpha \in [0,c). \tag{10.5}$$

We will now obtain (10.5), and complete the proof, by showing that

$$\beta' < -1 \text{ on the interval } [0,c). \tag{10.6}$$

Let $\alpha_0 = \beta(2c)$. By (10.2), $\alpha_0 \in (0,c)$. Also,

$$\beta'(0) < -1 \text{ and } \beta'(\alpha_0) < -1. \tag{10.7}$$

In fact, as $\beta(0) = \pi$, (10.4) and (10.1-i) with $q = 1$ yield $\beta'(0) = -e^{\pi\cot c} < -1$ while, with the approximate values $\alpha_0 \approx 0.3017$ and $\beta(\alpha_0) = 2c \approx 2.418858$ provided by *Mathematica*, (10.4) and (10.1-i) for $q = 1$ give $\beta'(\alpha_0) \approx -1.8755$.

Differentiating the identity $F(\alpha) = F(\beta(\alpha))$ twice we get

$$F'(\beta(\alpha))\beta''(\alpha) = F''(\alpha) - F''(\beta(\alpha))[\beta'(\alpha)]^2. \tag{10.8}$$

One verifies numerically that $F''(0)/F''(c) = 2e^{c\cot c}\cos c \approx 1.169 > 1$, and so, by (10.3),

$$F''(\alpha) < F''(c) < F''(\beta) < 0 \text{ whenever } 0 \le \alpha < c < \beta < 2c. \tag{10.9}$$

For $\alpha \in [0,\alpha_0]$ we have $\beta(\alpha) \ge 2c$ and, by (10.3), the right-hand side of (10.8) is negative, while $F'(\beta(\alpha)) < 0$ due to (10.1-ii). Thus, $\beta''(\alpha) > 0$, so that β' is increasing on $[0,\alpha_0]$ and, by (10.7), $\beta' < -1$ on $[0,\alpha_0]$.

The remaining part of (10.6) is the inequality $\beta' < -1$ on (α_0, c). We establish it by showing that its negation leads to a contradiction. Namely, suppose that $\beta'(\alpha) \ge -1$ for some $\alpha \in (\alpha_0, c)$. *At any such* α, *negativity of* β', *due to* (10.2), *gives* $|\beta'(\alpha)| \le 1$, *and so, by* (10.9) *and* (10.8), $\beta''(\alpha) > 0$. (In fact, $\beta(\alpha) \in (c, 2c)$, cf. (10.2), so that (10.1-ii) and (10.3) imply negativity of $F'(\beta(\alpha))$, $F''(\alpha)$ and $F''(\beta(\alpha))$, and (10.9) with $1 - [\beta'(\alpha)]^2 \ge 0$ yields $F''(\alpha) - F''(\beta(\alpha)[\beta'(\alpha)]^2 < F''(\beta(\alpha)) - F''(\beta(\alpha))[\beta'(\alpha)]^2 \le 0$.) Remark 2.2 applied to $\psi = \beta'$ and $\lambda = -1$ now shows that β' has a limit at c greater than -1. This contradicts the equality $\beta'(c) = -1$ in (10.2), completing the proof. □

11. Proof of Theorem 1.4

Under our assumptions, according to Theorem 1.3, the second paragraph of Sect. 9 and [13, Theorem 16.3]. (M,g,J) arises, up to biholomorphic isometries, from the "compact" version (8.2) of the construction in Sect. 8, using some data (i) – (vii) with $\boldsymbol{\tau}_*-\gamma$ in (8.1) replaced by $\varepsilon/2$, where $\varepsilon=\pm 1$, as mentioned in the lines preceding (9.1).

The question now is, if we exclude the product-of-surfaces case (see Remark 9.4), can a function Q of the form (9.6) be positive on an open interval while, at its endpoints, $Q=0$ and the derivative $\dot Q$ has mutually opposite, nonzero values?

We will now prove Theorem 1.4 by answering this question in the negative. Let us therefore suppose that such Q exists.

Replacing the variable $\boldsymbol{\tau}\in\mathrm{I\!R}\smallsetminus\{\gamma\}$ by $\theta=(\log|\boldsymbol{\tau}-\gamma|)/2$, we get Q equal to $\delta\varepsilon Ke^{2\theta}/2$ plus a linear combination of $e^\theta\cos\sqrt{7}\theta$ and $e^\theta\sin\sqrt{7}\theta$, where $\delta=\mathrm{sgn}(\boldsymbol{\tau}-\gamma)$. As $d\theta/d\boldsymbol{\tau}=\delta e^{-2\theta}/2$, there exists an open interval of the variable θ on which Q is positive while, at the endpoints, $Q=0$ and $e^{-2\theta}dQ/d\theta$ has mutually opposite, nonzero values. We have $2e^{-\theta}Q=\delta\varepsilon Ke^\theta+p\sin\sqrt{7}(\theta-q)$ for some constants p,q, and $p\neq 0$ (as Q in neither constant, nor monotone).

Consequently, at some two distinct values of θ the function sending $\theta\in\mathrm{I\!R}$ to $pe^{-\theta}\sin\sqrt{7}(\theta-q)$ assumes the same value $-\delta\varepsilon K$, with opposite nonzero values of the derivative, and is greater than $-\delta\varepsilon K$ between them. Note that we are free to set $q=0$ by replacing the variable θ with $\theta-q$ and p with pe^{-q}. Rescaling K, we may further assume that $|p|=1$. Thus, for $F:\mathrm{I\!R}\longrightarrow\mathrm{I\!R}$ with $F(\theta)=e^{-\theta}\sin\sqrt{7}\theta$, there are

(∗) two choices of θ at which $dF/d\theta$ has opposite nonzero values, while F assumes the same value, and is different from this last value between them.

Treating F as a function of the new variable $\alpha=\sqrt{7}\theta$ and setting $c=\tan^{-1}\sqrt{7}$, we get (∗), with θ replaced by α, for $F(\alpha)=e^{-\alpha\cot c}\sin\alpha$. This contradicts Lemma 10.1, thus completing the proof of Theorem 1.4.

12. New examples of weakly Einstein Kähler surfaces

The construction mentioned in Theorem 9.1 uses any function $Q>0$ of the variable $\boldsymbol{\tau}$ having the form (9.6), that is, any positive solution of (9.5), to define a weakly Einstein Kähler surface, which – according to Remarks 9.2 and 9.3 – is neither Einstein nor locally homogenous unless $Q=\varepsilon K(\boldsymbol{\tau}-\gamma)/2$, for the constant Gaussian curvature K of the base-surface metric h.

For the reader's benefit we provide below a different, self-contained description of these examples, reflecting the fact that they have cohomogeneity one (see Remark 12.1 below), and generalizing a construction in [8], rather than following the approach of [12].

In other words, the goal of this section is to offer a more user-friendly version of the presentation given in Sect. 9.

Let us fix nonzero real constants p,q and consider a four-manifold M with vector fields $e_1,\dots,e_4$ trivializing TM and satisfying the Lie-bracket relations

$$(12.1)\qquad \begin{array}{l}[e_1,e_i]=0\ \text{ for }\ i=2,3,4,\qquad [e_2,e_4]=2pe_3,\\ [e_2,e_3]=qe_4,\quad [e_3,e_4]=qe_2.\end{array}$$

We define a metric g and an almost-complex structure J on M by

$$(12.2)\qquad g(e_1,e_1)=g(e_3,e_3)=\zeta\eta,\quad g(e_2,e_2)=g(e_4,e_4)=\zeta,\quad Je_1=e_3,\quad Je_2=e_4$$

and $g(e_i,e_k)=0$ otherwise. Here ζ,η,θ are functions of the real variable $\boldsymbol{\tau}$, with ζ,η assumed positive, and $\boldsymbol{\tau}$ also stands for a function $\boldsymbol{\tau}:M\longrightarrow\mathrm{I\!R}$ such that

$$(12.3)\qquad d_{e_1}\boldsymbol{\tau}=2\zeta\eta\theta,\quad d_{e_i}\boldsymbol{\tau}=0\ \text{ for }\ i>1,\ \text{ and so }\ \nabla\boldsymbol{\tau}=2\theta e_1,\quad g(\nabla\boldsymbol{\tau},\nabla\boldsymbol{\tau})=4\zeta\eta\theta^2.$$

Such $\boldsymbol{\tau}$ exists, locally, due to the obvious closedness of the 1-form $d\boldsymbol{\tau}$, sending e_1 to $2\zeta^2\eta^2\theta$ and the other three e_i to 0.

Writing, this time, $(\ )' = d/d\boldsymbol{\tau}$, we also assume that $\zeta'\theta = -p$, which turns out to guarantee that (M, g, J) is a Kähler manifold.

The geometric content of our discussion remains unchanged when $\boldsymbol{\tau}$ is replaced by any function χ of the real variable $\boldsymbol{\tau}$, via a diffeomorphic change of the variable (that is, with $|\chi'| > 0$). As functions on M, our ζ and η the remain the same, while the role of θ is now played by $\theta_{\mathrm{new}} = \chi'\theta$. The function $\zeta'\theta$ and the condition $\zeta'\theta = -p$ remain unaffected. We use this freedom of modifying $\boldsymbol{\tau}$ to require, without loss of generality, that $\boldsymbol{\tau} : M \longrightarrow \mathrm{I\!R} \smallsetminus \{\gamma\}$, while

$$\theta \neq 0 \quad \text{be constant and} \quad \zeta\theta = p(\gamma - \boldsymbol{\tau}) \quad \text{for some } \gamma \in \mathrm{I\!R}. \tag{12.4}$$

Namely, as $\theta \neq 0$ everywhere (due to the condition $\zeta'\theta = -p$), we may choose χ above so as to make θ_{new} constant.

Since $\zeta'\theta = -p$, the Levi-Civita connection ∇ of g is given by

$$\begin{array}{l}
\nabla_{\!e_1} e_1 = (\zeta\eta)'\theta e_1, \;\; \nabla_{\!e_1} e_2 = \nabla_{\!e_2} e_1 = -p\eta e_2,\\
\nabla_{\!e_1} e_3 = \nabla_{\!e_3} e_1 = (\zeta\eta)'\theta e_3, \;\; \nabla_{\!e_1} e_4 = \nabla_{\!e_4} e_1 = -p\eta e_4,\\
\nabla_{\!e_2} e_2 = p e_1, \;\; \nabla_{\!e_2} e_3 = -p\eta e_4, \;\; \nabla_{\!e_2} e_4 = p e_3,\\
\nabla_{\!e_3} e_2 = -(p\eta + q) e_4, \;\; \nabla_{\!e_3} e_3 = -(\zeta\eta)'\theta e_1,\\
\nabla_{\!e_3} e_4 = (p\eta + q) e_2,\\
\nabla_{\!e_4} e_2 = -p e_3, \;\; \nabla_{\!e_4} e_3 = p\eta e_2, \;\; \nabla_{\!e_4} e_4 = p e_1.
\end{array} \tag{12.5}$$

Also, as $\zeta'\theta = -p$, the only possibly-nonzero components of the curvature tensor R, the Ricci tensor r and the metric g are those algebraically related to

$$\begin{array}{l}
R_{1212} = R_{1234} = R_{1414} = R_{1423} = R_{2323} = R_{3434} = p\zeta^2\eta\eta'\theta,\\
R_{1313} = -2[(\zeta\eta)'\theta]'\zeta^2\eta^2\theta, \quad R_{1324} = 2p\zeta^2\eta\eta'\theta,\\
R_{2424} = -2p(2p\eta + q)\zeta,\\
r_{11} = r_{33} = 2[3p\eta' - \zeta(\eta'\theta)']\zeta\eta\theta,\\
r_{22} = r_{44} = 2p\zeta\eta'\theta - 2p(2p\eta + q),\\
g_{11} = g_{33} = \zeta\eta, \qquad g_{22} = g_{44} = \zeta.
\end{array} \tag{12.6}$$

For $\mathcal{V} = \mathrm{Span}(e_1, e_3)$ and $\mathcal{H} = \mathrm{Span}(e_3, e_4)$, we thus get

$$\mathrm{r} = \mu g \text{ on } \mathcal{V}, \qquad \mathrm{r} = \lambda g \text{ on } \mathcal{H}, \qquad \mathrm{r}(\mathcal{V}, \mathcal{H}) = \{0\}, \tag{12.7}$$

where $\mu = r_{11}/g_{11}$ and $\lambda = r_{22}/g_{22}$. Consequently, with $e_1 \otimes e_1 + e_3 \otimes e_3$ equal to $g_{11} g$ on $\mathcal{V}$ and to 0 on $\mathcal{H}$, we get, as in (9.3),

$$\mathrm{r} \,=\, \lambda g \,+\, \frac{\mu - \lambda}{g_{11}} (e_1 \otimes e_1 + e_3 \otimes e_3),$$

provided that we identify e_1 with $g(e_1, \cdot\,)$, and similarly for e_3. Therefore,

$$R\mathrm{r} \,=\, \lambda\mathrm{r} \,+\, \frac{\mu - \lambda}{g_{11}} [R(e_1, \cdot\,, e_1, \cdot\,) + R(e_3, \cdot\,, e_3, \cdot\,)]. \tag{12.8}$$

Since $R\mathrm{r}$ is J-invariant, and clearly diagonalized by our frame, g is weakly Einstein if and only if $[R\mathrm{r}]_{11}/g_{11} = [R\mathrm{r}]_{22}/g_{22}$, that is, by (12.8), with $R_{2323} = R_{1212}$ in (12.6),

$$\lambda\mu \,+\, \frac{(\mu - \lambda) R_{1313}}{g_{11}^2} \,=\, \lambda^2 \,+\, \frac{2(\mu - \lambda) R_{1212}}{g_{11} g_{22}}.$$

The Einstein case being excluded, we can subtract the two sides and – according to Remark 9.2 – divide by $\lambda-\mu$, obtaining $\lambda+R_{1313}/g_{11}^2=2R_{1212}/[g_{11}g_{22}]$. As (12.6) implies that $R_{1313}/g_{11}^2=-2[(\zeta\eta)'\theta]'\theta$ and $R_{1212}/[g_{11}g_{22}]=p\eta'\theta$, this reads

$$\lambda\,-\,2[(\zeta\eta)'\theta]'\theta\,-\,2p\eta'\theta\,=\,0$$

and, multiplied by $g_{22}=\zeta$, it becomes $r_{22}-2[(\zeta\eta)'\theta]'\zeta\theta-2p\zeta\eta'\theta=0$ or, equivalently due to (12.6), $2p\zeta\eta'\theta-2p(2p\eta+q)-2[(\zeta\eta)'\theta]'\zeta\theta-2p\zeta\eta'\theta=0$, that is

$$p(2p\eta+q)\,+\,[(\zeta\eta)'\theta]'\zeta\theta\,=\,0.$$

Since θ is constant and $\zeta\theta=p(\gamma-\boldsymbol{\tau})$ in (12.4), multiplying by $4\zeta\theta^2$ we get

$$(\boldsymbol{\tau}-\gamma)^2Q''\,+\,2Q\,=\,4q\theta(\boldsymbol{\tau}-\gamma),\tag{12.9}$$

for $Q=4\zeta\eta\theta^2$, which is precisely equation (9.5) with $K=4\varepsilon q\theta$, where $\varepsilon=\pm1$ is the signum of $\boldsymbol{\tau}-\gamma$ on our interval of the variable $\boldsymbol{\tau}$, cf. (9.1).

Remark 12.1. The Lie-bracket relations (12.1) define a direct sum Lie algebra $\mathrm{I\!R}\oplus\mathfrak{g}$, for $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathrm{I\!R})$ or $\mathfrak{su}(2)=\mathfrak{so}(3)$, so that, locally, $e_1,\dots,e_4$ are left-invariant vector fields on a direct-product Lie group $\mathrm{I\!R}\times G$ with e_1 tangent to $\mathrm{I\!R}$ and e_2,e_3,e_4 to G. Right-invariant vector fields on G, transplanted into $\mathrm{I\!R}\times G$, commute with $e_1,\dots,e_4$ and are functional combinations of e_2,e_3,e_4 which, by (12.3), makes them g-orthogonal to the gradient $\nabla\boldsymbol{\tau}$. Their flows thus preserve ζ,η and $e_1,\dots,e_4$ and, consequently, the metric g defined by (12.2), so that g has local cohomogeneity one.

Acknowledgments. The research of YE was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(Grant number RS-2023-00244736). The research of SK was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(Grant number RS-2023-00247409). The research of JP was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (RS-2024-00334956). The authors also thank the anonymous referee for comments and suggestions that greatly improved the exposition.

References

[1] T. Arias-Marco and O. Kowalski, *Classification of* 4*-dimensional homogeneous weakly Einstein manifolds*, Czechoslovak Math. J. **65**(140) (2015),21–59.

[2] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

[3] E. Boeckx and L. Vanhecke, *Unit tangent sphere bundles with constant scalar curvature*, Czechoslovak Math. J. **51**(126) (2001), 523–544.

[4] J. Bolton, *Transnormal systems*, Quart. J. Math. Oxford Ser. (2) **24** (1973), 385–395.

[5] J.-P. Bourguignon, *Les variétés de dimension* 4 *à signature non nulle dont la courbure est harmonique sont d'Einstein*, Invent. Math. **63** (1981), 263–286.

[6] E. Calabi, *Extremal Kähler metrics*, in: Yau, S.-T. (ed.), Seminar on Differential Geometry, 259–290. Annals of Math. Studies **102**, Princeton Univ. Press, Princeton, NJ, 1982.

[7] E. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **l54** (1926), 214–264.

[8] A. Derdziński, *Exemples de métriques de Kähler et d'Einstein autoduales sur le plan complexe*, Géométrie riemannienne en dimension 4, (Séminaire Arthur Besse 1978/1979), Cedic/Fernand Nathan, Paris (1981), pp. 334–346.

[9] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compos. Math. **49** (1983) 405–433.

[10] A. Derdzinski, *Einstein metrics in dimension four*, Handbook of Differential Geometry, vol. I, pp. 419–707. North-Holland, Amsterdam (2000).

[11] A. Derdzinski, *Killing potentials with geodesic gradients on Kähler surfaces*, Indiana Univ. Math. J., **61** (2012), 1643–1666.

[12] A. Derdzinski and G. Maschler, *Local classification of conformally-Einstein Kähler metrics in higher dimensions*, Proc. London Math. Soc. **87** (2003), 779–819.

[13] A. Derdzinski and G. Maschler, *Special Kähler-Ricci potentials on compact Kähler manifolds*, J. reine angew. Math. **593** (2006), 73–116.

[14] Y. Euh, J. H. Park and K. Sekigawa, *A curvature identity on a* 4*-dimensional Riemannian manifold*, Results Math. **63** (2013), 107–114.

[15] Y. Euh, J. H. Park and K. Sekigawa, *A generalization of a* 4*-dimensional Einstein manifold*, Math. Slovaca **63** (2013), 595–610.

[16] Y. Euh, J. H. Park and K. Sekigawa, *Critical metrics for quadratic functionals in the curvature on 4-dimensional manifolds*, Differen. Geom. Appl. **29** (2011), 642–646.

[17] E. García-Río, A. Haji-Badali, R. Mariño-Villar and M. E. Vázquez-Abal, *Locally conformally flat weakly-Einstein manifolds*, Arch. Math. (Basel) **111** (2018), 549–559.

[18] A. Gray and T. J. Willmore, *Mean-value theorems for Riemannian manifolds*, Proc. Roy. Soc. Edinburgh Sect. A **92** (1982), 343–364.

[19] R. Miyaoka, *Transnormal functions on a Riemannian manifold.* Differential Geom. Appl. **31** (2013), 130–139.

[20] J. Kim, Y. Nikolayevsky and J. H. Park, *Weakly Einstein Hypersurfaces in space forms*, preprint, available from https://arxiv.org/abs/2409.12766v1.

[21] S. Tanno, 4*-dimensional conformally flat Kaehler manifolds*, Tohoku Math. J. **24** (1972), 501–504.

[22] Q. M. Wang, *Isoparametric functions on Riemannian manifolds*, I. Math. Ann. **277** (1987), 639–646.

[1]Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

[2]Department of Mathematics, Sungkyunkwan University, Suwon, 16419, Korea

Email address: andrzej@math.ohio-state.edu, prettyfish@skku.edu, kimsinhwi@skku.edu,, parkj@skku.edu