Special Ricci-Hessian equations on Kähler manifolds

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Abstract. Special Ricci-Hessian equations on Kähler manifolds (M, g), as defined by Maschler [Ann. Global Anal. Geom. 34 (2008), 367–380] involve functions τ on M and state that, for some function α of the real variable τ, the sum of $\alpha\nabla d\tau$ and the Ricci tensor equals a functional multiple of the metric g, while $\alpha\nabla d\tau$ itself is assumed to be nonzero almost everywhere. Three well-known obvious "standard" cases are provided by (non-Einstein) gradient Kähler-Ricci solitons, conformally-Einstein Kähler metrics, and special Kähler-Ricci potentials. We show that, outside of these three cases, such an equation can only occur in complex dimension two and, at generic points, it must then represent one of three types, for which, up to normalizations, $\alpha = 2\cot\tau$, or $\alpha = 2\coth\tau$, or $\alpha = 2\tanh\tau$. We also use the Cartan-Kähler theorem to prove that these three types are actually realized in a "nonstandard" way.

Introduction

Following Maschler [**17**, p. 367], one says that functions τ, α, σ on a Riemannian manifold (M, g) with the Ricci tensor r satisfy a *Ricci-Hessian equation* if

(0.1) $$\alpha\nabla d\tau + \mathrm{r} = \sigma g \quad \text{for some function } \sigma : M \to \mathbb{R},$$

∇ being the Levi-Civita connection of g. We call equation (0.1) *special* when

(0.2) $$\alpha\nabla d\tau \neq 0 \text{ on a dense set, } \dim M = n > 2, \text{ and } \alpha \text{ is a } C^\infty \text{ function of } \tau.$$

Conditions (0.1) – (0.2) are satisfied in several situations that have been studied – see below – raising a natural question: *Which functions $\tau \mapsto \alpha$ can be realized in this way?* The present paper provides an answer in the Kähler case, outside of the classes that are already well understood. See Theorems D and E.

There are three well-known classes of examples leading to (0.1) – (0.2).

(I) Non-Einstein gradient Ricci almost-solitons [**20, 1**], including (non-Einstein) gradient Ricci solitons [**15**]. Here α is a nonzero constant.

(II) Conformally-Einstein metrics g, with $\tau > 0$ and $\alpha = (n-2)/\tau$, the Einstein metric being $\hat{g} = g/\tau^2$. Cf. [**11**, formula (6.2)].

(III) Special Kähler-Ricci potentials τ on Kähler manifolds, at points where r is not a multiple of g. See [**11**, Remark 7.4].

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A *special Kähler-Ricci potential* [**11**, Sect. 7] on a Kähler manifold (M,g) with the complex-structure tensor J is any nonconstant function τ on M having a real-holomorphic gradient $v=\nabla\tau$ for which, at points where $v\ne 0$, all nonzero vectors orthogonal to v and Jv are eigenvectors of both $\nabla d\tau$ and r. Such triples (M,g,τ) are completely understood, both locally [**11**] and in the compact case [**12**].

The classes (I) – (III) are far from disjoint: for instance [**11**, Corollary 9.3], in the Kähler category, if $n>4$, (II) is a special case of (III).

We are interested in M,g,τ,α,σ satisfying (0.1) – (0.2) and such that

$$2\mathrm{r}(v,\,\cdot\,)=-dY \text{ for } v=\nabla\tau \text{ and } Y=\Delta\tau. \tag{0.3}$$

Here, and throughout the paper, we use the notational conventions

$$v=\nabla\tau,\qquad Q=g(v,v),\qquad Y=\Delta\tau,\qquad n=\dim M \tag{0.4}$$

whenever (M,g) is a Riemannian manifold and $\tau:M\to\mathrm{I\!R}$. As we point out near at the end of Section 1, with J denoting the complex-structure tensor,

(0.5) for Kähler metrics g, conditions (0.1) – (0.2) imply (0.3), and the gradient $v=\nabla\tau$ is a real-holomorphic vector field or, equivalently, Jv is a real-holomorphic g-Killing field.

Assuming (0.1) – (0.2), we may treat the derivatives $\alpha'=d\alpha/d\tau$ and α'' both as functions of the real variable τ and as functions $M\to\mathrm{I\!R}$. In Sections 2 and 3 we prove the following two results, as well as Theorem D, stated below.

Theorem A. *Under the hypotheses* (0.1) – (0.3)*, at points where* $\alpha''+\alpha\alpha'\ne 0$ *and* $d\tau\ne 0$*, both* $Q=g(\nabla\tau,\nabla\tau)$ *and* $Y=\Delta\tau$ *are, locally, functions of* τ*.*

Theorem B. *Let functions* τ,α,σ *satisfy a special Ricci-Hessian equation* (0.1)*, with* (0.2)*, on a Kähler manifold* (M,g) *of real dimension* $n\ge 4$*. If* $\alpha\,d\alpha$ *and* $d\tau$ *are nonzero at all points of an open submanifold* U *of* M*, and*

(i) $n>4$*, or*
(ii) $n=4$ *and* $d\sigma\wedge d\tau=0$ *identically in* U *or, finally,*
(iii) $dQ\wedge d\tau=0$ *everywhere in* U*, where* $Q=g(\nabla\tau,\nabla\tau)$*,*

then $\tau:U\to\mathrm{I\!R}$ *is a special Kähler-Ricci potential on the Kähler manifold* (U,g)*.*

With v,Q,Y as in (0.4), a function τ on a Riemannian manifold (M,g) has $dQ\wedge d\tau=0$ if and only if Q is locally, at points where $d\tau\ne 0$, a function of τ. This amounts to requiring the integral curves of v to be reparametrized geodesics (since, due to formula (1.2) below, the latter condition means that $\nabla_{\!v}v$ is a functional multiple of v). Such functions τ, called *transnormal*, have been studied extensively [**21, 18, 3**], and are referred to as *isoparametric* when, in addition, $dY\wedge d\tau=0$.

Theorem B renders the transnormal case $dQ\wedge d\tau=0$, as well as real dimensions $n>4$, rather uninteresting in the context of special Ricci-Hessian equations (0.1) – (0.2) on Kähler manifolds, since at $d\alpha$-generic points (see the end of Section 1) one then ends up with examples (I) or (III) above, cf. Remark 3.3, of which the former is the subject of a large existing literature, and the latter, as mentioned earlier, has been completely described. This is why our next two results focus exclusively on the real dimension four and functions τ with $dQ\wedge d\tau$ not identically zero.

Remark C. Equation (0.1), with (0.2), remains satisfied after τ and the function $\tau\mapsto\alpha=\alpha(\tau)$ have been subjected to an *affine modification* in the sense of being replaced with $\hat\tau$ and $\hat\tau\mapsto\hat\alpha(\hat\tau)$ given by $\hat\tau=p+\tau/c$ and $\hat\alpha(\hat\tau)=c\alpha(c\hat\tau-cp)$ for real constants $c\ne 0$ and p.

Theorem D. *If the special Ricci-Hessian equation* (0.1) *and* (0.2) *both hold for functions* τ, α, σ *on a Kähler manifold* (M, g) *of real dimension four, while* $dQ \wedge d\tau \neq 0$ *everywhere in an open connected set* $U \subseteq M$, *then the function* α *of the variable* τ *and its derivative* $\alpha' = d\alpha/d\tau$ *satisfy, on* U, *the equation*

$$(0.6) \qquad \alpha'' + \alpha\alpha' = 0, \text{ that is, } 2\alpha' + \alpha^2 = 4\varepsilon \text{ with a constant } \varepsilon \in \mathbb{R}.$$

In addition, for Q *and* Y *as in* (0.4), *and the scalar curvature* s, *the functions*

$$(0.7) \qquad 2\theta = \alpha\mathrm{s} + 4\varepsilon Y \text{ and } \kappa = \theta\psi + \alpha^{-1}Y - Q \text{ are both constant,}$$

ψ *being given by* $4\varepsilon\psi = \tau - 2/\alpha$, *if* $\varepsilon \neq 0$, *or* $3\psi = 2/\alpha^3$, *when* $\varepsilon = 0$. *Furthermore,* σ *in* (0.1) *and the function* F *of the variable* τ *characterized by*

$$(0.8) \qquad 4\varepsilon F = \theta(2 - \tau\alpha) + 4\varepsilon\kappa\alpha \text{ for } \varepsilon \neq 0, \text{ and } F = \kappa\alpha - 2\theta/(3\alpha^2) \text{ if } \varepsilon = 0,$$

and thus depending on the real constants θ, κ, *satisfy the conditions*

$$(0.9) \qquad \text{a) } Y - Q\alpha = F, \quad \text{b) } 2\sigma = -(Q\alpha' + F'), \quad \text{c) } \Delta\alpha = F\alpha' = -F''.$$

Finally, up to affine modifications – see Remark C *– the pair* $(\alpha(\tau), \varepsilon)$ *is one of the following five*: $(2, 1)$, $(2/\tau, 0)$, $(2\tanh\tau, 1)$, $(2\coth\tau, 1)$, $(2\cot\tau, -1)$.

Theorem E. *Each of the five options listed in Theorem* D, *namely,*

$$(2, 1), \quad (2/\tau, 0), \quad (2\tanh\tau, 1), \quad (2\coth\tau, 1), \quad (2\cot\tau, -1),$$

is realized by a special Ricci-Hessian equation (0.1) – (0.2) *on a real-analytic Kähler manifold* (M, g) *of real dimension four such that, with* $v = \nabla\tau$ *and* $Q = g(v, v)$, *one has* $dQ \wedge d\tau \neq 0$ *somewhere in* M *and* Jv *lies in a two-dimensional Abelian Lie algebra of Killing fields.*

For $(2, 1)$ *and* $(2/\tau, 0)$ *one can choose* (M, g) *to be compact and biholomorphic to the two-point blow-up of* $\mathbb{C}\mathrm{P}^2$, *with* g *which is the Wang-Zhu toric Kähler-Ricci soliton* [**22**, Theorem 1.1] *or, respectively, the Chen-LeBrun-Weber conformally-Einstein Kähler metric* [**6**, Theorem A].

In contrast with the final clause of Theorem E, we do not know whether the remaining three options, $(2\tanh\tau, 1)$, $(2\coth\tau, 1)$ and $(2\cot\tau, -1)$, may be realized on a *compact* Kähler surface. An analytic-continuation phenomenon described below (Section 12) may hint at plausibility of trying to obtain such compact examples via small deformations of the Wang-Zhu or Chen-LeBrun-Weber metric, combined with suitable affine modifications.

For the pairs $(2, 1)$ and $(2/\tau, 0)$ in Theorem D, the constancy conclusions of (0.7) are well known: [**7**, p. 201], [**9**, p. 417, Prop. 3(i) and p. 419, formula (40)].

The paper is organized as follows. Section 1 contains the preliminaries. Consequences of special Ricci-Hessian equations, leading to proofs of Theorems A, B and D, are presented in the next two sections. Sections 4 through 11 are devoted to proving Theorem E: we rephrase it as solvability of the system (5.1) of quasi-linear first-order partial differential equations, subject to the additional conditions (5.2), which allows us to derive our claim from the Cartan-Kähler theorem for exterior differential systems.

1. Preliminaries

All manifolds and Riemannian metrics are assumed to be of class C^∞. By definition, a manifold is connected. We use the symbol δ for divergence.

On a manifold with a torsion-free connection ∇, the Ricci tensor r satisfies the Bochner identity $\mathrm{r}(\,\cdot\,,v) = \delta\nabla v - d[\delta v]$, where v is any vector field. Its coordinate form $R_{jk}v^k = v^k{}_{,jk} - v^k{}_{,kj}$ arises via contraction from the Ricci identity $v^l{}_{,jk} - v^l{}_{,kj} = R_{jkq}{}^l v^q$. (We use the sign convention for R such that $R_{jk} = R_{jqk}{}^q$.) Applied to the gradient v of a function τ on a Riemannian manifold, this yields

$$\delta[\nabla d\tau] = \mathrm{r}(v,\,\cdot\,) + dY, \qquad \text{with } v = \nabla\tau \text{ and } Y = \Delta\tau. \tag{1.1}$$

On the other hand, given a function τ on a Riemannian manifold,

$$2[\nabla d\tau](v,\,\cdot\,) = dQ, \qquad \text{where } v = \nabla\tau \text{ and } Q = g(v,v), \tag{1.2}$$

as one sees noting that, in local coordinates, $(\tau_{,k}\tau^{,k})_{,j} = 2\tau_{,kj}\tau^{,k}$. We can rewrite the relations (1.1) – (1.2) using the interior product $\imath_v$, and then they read

$$\text{a) } \delta[\nabla d\tau] = \imath_v\mathrm{r} + dY, \qquad \text{b) } 2\imath_v[\nabla d\tau] = dQ, \qquad \text{with (0.4).} \tag{1.3}$$

Finally, for the Ricci tensor r and scalar curvature s of any Riemannian metric,

$$2\delta\mathrm{r} = d\mathrm{s}, \tag{1.4}$$

which is known as the Bianchi identity for the Ricci tensor. Its coordinate form $2g^{kl}R_{jk,l} = s_{,j}$ is immediate if one transvects with (“multiplies” by) g^{kl} the equality $R_{jkl}{}^q{}_{,q} = R_{jl,k} - R_{kl,j}$ obtained by contracting the second Bianchi identity.

The *harmonic-flow condition* for a vector field v on a Riemannian manifold (M,g), meaning that the flow of v consists of (local) harmonic diffeomorphisms, is known [**19**] to be equivalent to the equation

$$g(\Delta v,\,\cdot\,) = -\mathrm{r}(v,\,\cdot\,) \tag{1.5}$$

the vector field Δv having the local components $[\Delta v]^j = v^{j,k}{}_k$. See also [**14**, Theorem 3.1]. When $v = \nabla\tau$ is the gradient of a function $\tau : M \to \mathbb{R}$,

(1.6) the harmonic-flow condition (1.5) amounts to (0.3).

In fact, by (1.1), $2\mathrm{r}(v,\,\cdot\,) + dY = \delta[\nabla d\tau] + \mathrm{r}(v,\,\cdot\,) = g(\Delta v,\,\cdot\,) + \mathrm{r}(v,\,\cdot\,)$, as $[\Delta v]_j = v_{j,k}{}^k = \tau_{,jk}{}^k = \tau_{,kj}{}^k = \tau^{,kj}{}_k = (\delta[\nabla d\tau])_j$.

On the other hand – see, e.g., [**11**, Lemma 5.2] – on a Kähler manifold (M,g),

(1.7) conditions (0.1)–(0.2) imply real-holomorphicity of the gradient $v = \nabla\tau$, while Jv is then a holomorphic Killing field, due to the resulting Hermitian symmetry of $\nabla d\tau$.

Since holomorphic mappings between Kähler manifolds are harmonic, every holomorphic vector field on a Kähler manifold satisfies (1.5), cf. [**14**, Remark 3.2]. Now (0.5) follows from (1.6). In other words, as observed by Calabi [**5**], on Kähler manifolds one has

(1.8) equation (0.3), with (0.4), for all real-holomorphic gradients $v = \nabla\tau$.

Given a tensor field θ on a manifold M, we say that a point $x \in M$ is *θ-generic* if x has a neighborhood on which either $\theta = 0$ identically, or $\theta \neq 0$ everywhere. Such points clearly form a dense open subset of M.

2. Ricci-Hessian equations

As a consequence of (0.1), for the scalar curvature s, with (0.4),

$$n\sigma = Y\alpha + \mathrm{s}, \quad \text{where} \quad n = \dim M. \tag{2.1}$$

Applying $2\imath_v$ or 2δ to (0.1), we obtain, from (1.3) – (1.4) and (0.4),

$$\begin{aligned} &\text{i)} \quad \alpha\, dQ + 2\mathrm{r}(v,\,\cdot\,) = 2\sigma\, d\tau, \\ &\text{ii)} \quad 2[\nabla d\tau](\nabla\alpha,\,\cdot\,) + 2\alpha[\mathrm{r}(v,\,\cdot\,) + dY] + d\mathrm{s} = 2d\sigma. \end{aligned} \tag{2.2}$$

In the case where (0.1) – (0.2) hold along with (0.3), one may rewrite (2.2) as

$$\begin{aligned} &\text{i)} \quad \alpha\, dQ - dY = 2\sigma\, d\tau, \\ &\text{ii)} \quad 2[\nabla d\tau](\nabla\alpha,\,\cdot\,) + \alpha\, dY + d\mathrm{s} = 2d\sigma, \end{aligned} \tag{2.3}$$

which, in view of (1.2) and (2.1), amounts to nothing else than

$$\begin{aligned} &\text{i)} \quad d(Q\alpha - Y) = (Q\alpha' + 2\sigma)\, d\tau, \\ &\text{ii)} \quad d[Q\alpha' + (n-2)\sigma] = (Q\alpha'' + Y\alpha')\, d\tau, \end{aligned} \tag{2.4}$$

as the assumption, in (0.2), that α is a C^∞ function of τ allows us to write

$$d\alpha = \alpha' d\tau, \quad \nabla\alpha = \alpha' v, \quad 2[\nabla d\tau](\nabla\alpha,\,\cdot\,) = \alpha' dQ, \quad \text{where} \quad \alpha' = d\alpha/d\tau, \tag{2.5}$$

since (1.2) gives $2[\nabla d\tau](\nabla\alpha,\,\cdot\,) = 2\alpha'[\nabla d\tau](v,\,\cdot\,) = \alpha' dQ$. Due to (2.4), conditions (0.1) – (0.3) imply that, locally, at points at which $d\tau \neq 0$,

(2.6) $Q\alpha - Y$ and $Q\alpha' + (n-2)\sigma$ are functions of τ, with the respective τ-derivatives $Q\alpha' + 2\sigma$ and $Q\alpha'' + Y\alpha'$ which, consequently, must themselves be functions of τ.

Proof of Theorem A. At the points in question, using (2.6) to equate both $Q\alpha - Y$ and $Q\alpha'' + Y\alpha'$ to some specific functions of τ, we obtain a system of two linear equations with the nonzero determinant $\alpha'' + \alpha\alpha'$, imposed on the unknowns Q, Y, and our assertion follows since $\alpha'' + \alpha\alpha'$ is also a function of τ. □

Assuming only (0.1), for $n = \dim M$, with the aid of (2.1) we rewrite (2.2) as

$$\begin{aligned} &n[\alpha\, dQ + 2\mathrm{r}(v,\,\cdot\,)] - 2(Y\alpha + \mathrm{s})\, d\tau = 0, \\ &2n\{[\nabla d\tau](\nabla\alpha,\,\cdot\,) + \alpha\mathrm{r}(v,\,\cdot\,)\} + 2[(n-1)\alpha\, dY - Y d\alpha] + (n-2)\, d\mathrm{s} = 0, \end{aligned}$$

If (0.3) holds as well, replacing $2\mathrm{r}(v,\,\cdot\,)$ here with $-dY$ we obtain $n(\alpha\, dQ - dY) - 2(Y\alpha + \mathrm{s})\, d\tau = 0$ and $2n[\nabla d\tau](\nabla\alpha,\,\cdot\,) + (n-2)(\alpha\, dY + d\mathrm{s}) - 2Y d\alpha = 0$. Thus, when (0.1) – (0.3) are all satisfied, (2.5) gives

$$\begin{aligned} &\text{a)} \quad n(\alpha\, dQ - dY) - 2(Y\alpha + \mathrm{s})\, d\tau = 0, \\ &\text{b)} \quad n\alpha' dQ + (n-2)(\alpha\, dY + d\mathrm{s}) - 2Y\alpha' d\tau = 0. \end{aligned} \tag{2.7}$$

3. Ricci-Hessian equations on Kähler manifolds

The goal of this section is to prove Theorems B and D.

In any complex manifold, $d\omega = 0$ and $\omega(J\,\cdot\,,\,\cdot\,)$ is symmetric if $\omega = i\partial\overline{\partial}\tau$, that is, if $2\omega = -d[J^*d\tau]$ for a real-valued function τ, with the 1-form $J^*d\tau = (d\tau)J$ which sends any tangent vector field v to $d_{Jv}\tau$. Our exterior-derivative and exterior-product conventions, for 1-forms ξ, ξ' and vector fields u, v, are

$$\begin{aligned} [d\xi](u,v) &= d_u[\xi(v)] - d_v[\xi(u)] - \xi([u,v]), \\ [\xi \wedge \xi'](u,v) &= \xi(u)\xi'(v) - \xi(v)\xi'(u). \end{aligned} \tag{3.1}$$

For a torsionfree connection ∇, (3.1) gives $[d\xi](u,v) = [\nabla_u\xi](v) - [\nabla_v\xi](u)$, so that, if in addition $\nabla J = 0$, on an almost-complex manifold,

$$2i\partial\overline{\partial}\tau \,=\, [\nabla d\tau](J\,\cdot\,,\,\cdot\,) \,-\, [\nabla d\tau](\,\cdot\,,J\,\cdot\,). \tag{3.2}$$

In the case of a Kähler metric g on a complex manifold M, (0.1) implies that

$$i\alpha\partial\overline{\partial}\tau \,+\, \rho \,=\, \sigma\omega, \tag{3.3}$$

ω,ρ being the Kähler and Ricci forms, with both terms on the right-hand side of (3.2) equal, as Hermitian symmetry of $\nabla d\tau$ follows from those of ρ and ω.

Remark 3.1. As an obvious consequence of the last line in (1.7), if g is a Kähler metric, conditions (0.1) – (0.2) are *equivalent* to (3.3) along with (0.2) and real-holomorphicity of the gradient $v = \nabla\tau$.

Remark 3.2. For the Kähler form ω of a Kähler manifold (M,g) of real dimension $n \geq 4$, the operator $\zeta \mapsto \zeta\wedge\omega$ acting on differential q-forms is injective if $q = 2$ and $n > 4$, or $q = 1$. Namely, the contraction of $\zeta\wedge\omega$ against ω yields a nonzero constant times $(n-4)\zeta + 2\langle\omega,\zeta\rangle\omega$ (if $q = 2$), or times $(n-2)\zeta$ (if $q = 1$). In the former case, ζ with $\zeta\wedge\omega = 0$ is thus a multiple of ω, and hence 0.

Remark 3.3. Whenever (3.3) with a *constant* α holds on a Kähler manifold of real dimension $n \geq 4$, constancy of σ follows (from Remark 3.2, as $d\sigma\wedge\omega = 0$).

We have the following result, due to Maschler [**17**, Proposition 3.3].

Lemma 3.4. *Condition* (0.1) *on a Kähler manifold* (M,g) *of real dimension* $n > 4$ *implies that* $d\sigma\wedge d\alpha = 0$. *Equivalently, wherever* $d\alpha$ *is nonzero,* σ *must, locally, be a function of* α.

Proof. By (3.3), $0 = d\rho = d\sigma\wedge\omega - d\alpha\wedge i\,\partial\overline{\partial}\tau$, so that $d\alpha\wedge d\sigma\wedge\omega = 0$, and our assertion is immediate from Remark 3.2. □

Proof of Theorem B. In all three cases (i) – (iii), $d\sigma\wedge d\tau = 0$. For (i), this follows from Lemma 3.4 while, when $dQ\wedge d\tau = 0$ on U, we see that, in view of the equality $\alpha' dQ + \alpha\, dY = d(2\sigma - \mathrm{s})$ arising from (2.3.ii) and (2.5), Y and $2\sigma - \mathrm{s}$ are, locally, functions of τ, and hence so is σ, as a consequence of (2.1) with $n \geq 4$. Now [**11**, Corollary 9.2] yields our claim. □

Proof of Theorem D. As $dQ\wedge d\tau \neq 0$ everywhere in U, Theorem A implies (0.6) and, consequently, also the final clause about the five possible pairs.

Next, in (0.6), $4\,d\theta = 2d[\alpha\mathrm{s} + (2\alpha' + \alpha^2)Y] = 2[\alpha\, d\mathrm{s} + \mathrm{s}\, d\alpha + (2\alpha' + \alpha^2)dY]$ which, as $n = 4$, equals, in view of (2.5),

$$\alpha[n\alpha' dQ + (n-2)(\alpha\, dY + d\mathrm{s}) - 2Y\alpha' d\tau] \,-\, \alpha'[n(\alpha\, dQ - dY) - 2(Y\alpha + \mathrm{s})\, d\tau],$$

and hence vanishes due to (2.7). On the other hand, the function ψ of τ defined in the theorem is an antiderivative of $1/\alpha^2$, meaning that

$$\psi' \,=\, 1/\alpha^2. \tag{3.4}$$

Namely, by (0.6). $4\varepsilon\psi' = 1 + 2\alpha'/\alpha^2 = (2\alpha' + \alpha^2)/\alpha^2 = 4\varepsilon/\alpha^2$ if $\varepsilon \neq 0$, and $3\psi' = -6\alpha'/\alpha^4 = 3/\alpha^2$ when $\varepsilon = 0$, as $2\alpha' = -\alpha^2$.

Furthermore, $d(\theta\psi + \alpha^{-1}Y - Q) = 0$. In fact, $d\alpha = \alpha' d\tau$ in (2.5), and similarly for ψ, so that, from (3.4), $d(\theta\psi) = \theta\, d\psi = \theta\psi' d\tau = \theta\alpha^{-2}d\tau$, and $\alpha d(\alpha^{-1}Y) = dY - \alpha^{-1}Y\alpha' d\tau$. Also, $2(\theta - Y\alpha') = (Y\alpha + \mathrm{s})\alpha$ from (0.6) – (0.7). These relations yield $-4\alpha\, d(\theta\psi+\alpha^{-1}Y-Q)=4[(\alpha\, dQ-dY)-(\theta-Y\alpha')\alpha^{-1}d\tau]=n(\alpha\, dQ-dY)-2(Y\alpha+\mathrm{s})\,d\tau$, with $n=4$, which equals 0 by (2.7.a).

Finally, (0.8) and the second relation in (0.7) easily give (0.9.a). Thus, by (0.4) and (2.5), $(Q\alpha'+F')Q=(Q\alpha'+F')\,d_v\tau=Q\,d_v\alpha+d_vF$ which, due to (0.9.a), equals $d_vY-\alpha\, d_vQ$. At the same time, $-\imath_v$ applied to (2.3.i) yields $d_vY-\alpha\, d_vQ=-2Q\sigma$. We thus get (0.9.b). To obtain (0.9.c), note that, from (0.4), $\Delta\alpha=Q\alpha''+Y\alpha'$ which, by (0.6) and (0.9.a), equals $(Y-Q\alpha)\alpha'=F\alpha'=-F''$, where the last equality trivially follows from (0.8) $\square$

Theorem D has a partial converse: if a nonconstant function τ with real-holomorphic gradient $v=\nabla\tau$ on a Kähler surface (M,g) and a function α of the real variable τ satisfy (0.6) and (0.7), then they must also satisfy the Ricci-Hessian equation (0.1) with σ given by (2.1) for $n=4$.

In fact, $b(v,\,\cdot\,)=0$, where b denotes the traceless Hermitian symmetric 2-tensor field $\alpha\nabla d\tau+\mathrm{r}-\sigma g$. Namely, (0.3) – (0.5) and (1.3.b) yield $4b(v,\,\cdot\,)=2\alpha\, dQ-2\,dY-4\sigma\, d\tau$ which, due to (2.1) and (2.5), equals $2\alpha\, dQ-2\,dY-(Y\alpha+\mathrm{s})\,d\tau$, and so $-4\alpha b(v,\,\cdot\,)=2\alpha^2d(\theta\psi+\alpha^{-1}Y-Q)+(\alpha\mathrm{s}+4\varepsilon Y-2\theta)\,d\tau$. (Note that, by (0.6) and (3.4), $4\varepsilon=2\alpha'+\alpha^2$ and $2\alpha^2d(\theta\psi)=2\theta\, d\tau$.) Thus, $b=0$, since b corresponds, via g, to a *complex-linear* bundle morphism $TM\to TM$.

4. The local Kähler potentials

This and the following seven sections are devoted to proving Theorem E.

In an open set $M\subseteq\mathbb{R}^4$ with the Cartesian coordinates x,x',u,u' arranged into the complex coordinates $(x+ix',u+iu')$ for the complex plane $\mathbb{C}^2=\mathbb{R}^4$ carrying the standard complex structure J, one has $J^*dx=-dx'$ and $J^*du=-du'$, so that, if a C^∞ function f on M only depends on x and u, the relation $2i\partial\bar\partial f=-d[J^*df]$ yields, with subscripts denoting partial differentiations, $2i\partial\bar\partial f=f_{xx}\,dx\wedge dx'+f_{xu}(dx\wedge du'+du\wedge dx')+f_{uu}\,du\wedge du'$, since $df=f_x\,dx+f_u\,du$. Furthermore, we set

$$v=\partial_x \quad\text{and}\quad w=\partial_u \quad \text{(the real coordinate vector fields).} \tag{4.1}$$

For the Kähler metric g on M having the Kähler form $\omega=2i\partial\bar\partial\phi$, where the function $\phi:M\to\mathbb{R}$ is assumed to depend on x and u only, 2ϕ is a Kähler potential [**2**, p. 85] of g, and the above formula for $2i\partial\bar\partial f$, with $f=\phi$, becomes

$$\omega=\phi_{xx}\,dx\wedge dx'+\phi_{xu}(dx\wedge du'+du\wedge dx')+\phi_{uu}\,du\wedge du'. \tag{4.2}$$

Generally, for a skew-Hermitian 2-form $\zeta=Q\,dx\wedge dx'+S(dx\wedge du'+du\wedge dx')+B\,du\wedge du'$ and the Hermitian symmetric 2-tensor field a with $\zeta=a(J\,\cdot\,,\,\cdot\,)$ one has $a=Q(dx\otimes dx+dx'\otimes dx')+S(dx\otimes du+du\otimes dx+dx'\otimes du'+du'\otimes dx')+B(du\otimes du+du'\otimes du')$, due to (3.1), and so the components of a relative to the coordinates (x,x',u,u') form the matrix

$$\begin{bmatrix} Q & 0 & S & 0\\ 0 & Q & 0 & S\\ S & 0 & B & 0\\ 0 & S & 0 & B\end{bmatrix},\quad \text{with the determinant } (QB-S^2)^2. \tag{4.3}$$

When $a=g$, (4.2) woth $\zeta=\omega$ gives $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$. Thus,

$$\phi_{xx}>0 \text{ and } \mathit{\Pi}>0, \text{ for } \mathit{\Pi}=\phi_{xx}\phi_{uu}-\phi_{xu}^2, \tag{4.4}$$

which amounts to Sylvester's criterion for positive definiteness of g, namely, positivity of the upper left subdeterminants of (4.3). From now on we set

$$(4.5)\qquad \begin{array}{l}(\tau,\lambda,Q,S,B)=(\phi_x,\phi_u,\phi_{xx},\phi_{xu},\phi_{uu}),\ \text{ so that}\\ Q>0\ \text{ and }\ \varPi=QB-S^2>0\ \text{ due to (4.4).}\end{array}$$

With div,Δ denoting the g-divergence and g-Laplacian, for τ,λ,Q in (4.5),

(a) the functions τ,λ have the holomorphic g-gradients $v=\partial_x$ and $w=\partial_u$,
(b) the other coordinate fields Jv and Jw are holomorphic g-Killing fields,
(c) $Q=g(v,v)$ and $\Delta\tau=\mathrm{div}\,v=[\log\varPi]_x$, while $\Delta\lambda=\mathrm{div}\,w=[\log\varPi]_u$.

Namely, (4.1) and (4.3) yield (a). Also, (b) follows since ϕ only depends on x and u. Finally, (4.3) has the determinant $\varPi^2$, and so $\varPi\,dx\wedge dx'\wedge du\wedge du'$ is the volume form of g, on which $\pounds_v,\pounds_w$ act – see (4.1) – via partial differentiations ∂_x, ∂_u of the $\varPi$ factor. Thus, $\mathrm{div}\,v=[\log\varPi]_x$ and $\mathrm{div}\,w=[\log\varPi]_u$, cf. [**16**, p. 281]. On the other hand, by (a), (4.1) and (4.5), $g(v,v)=d_v\tau=\partial_x\tau=\partial_x\phi_x=\phi_{xx}=Q$.

For our $(\tau,\lambda)=(\phi_x,\phi_u)$, the mapping $(x,u)\mapsto(\tau,\lambda)$ is locally diffeomorphic due to (4.4), which makes $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$, locally, a triple of real-valued functions of the new variables τ,λ. Thus, from the chain rule, in matrix form

$$(4.6)\qquad \begin{array}{l}\text{i)}\ \ \begin{bmatrix}\partial_x & \partial_u\end{bmatrix}=\begin{bmatrix}\partial_\tau & \partial_\lambda\end{bmatrix}\begin{bmatrix}Q & S\\ S & B\end{bmatrix},\quad \text{ii)}\ \begin{bmatrix}d\tau\\ d\lambda\end{bmatrix}=\begin{bmatrix}Q & S\\ S & B\end{bmatrix}\begin{bmatrix}dx\\ du\end{bmatrix},\ \text{ and so}\\ \text{iii)}\ \ (QB-S^2)\begin{bmatrix}\partial_\tau & \partial_\lambda\end{bmatrix}=\begin{bmatrix}\partial_x & \partial_u\end{bmatrix}\begin{bmatrix}B & -S\\ -S & Q\end{bmatrix},\\ \text{iv)}\ \ (QB-S^2)\begin{bmatrix}dx\\ du\end{bmatrix}=\begin{bmatrix}B & -S\\ -S & Q\end{bmatrix}\begin{bmatrix}d\tau\\ d\lambda\end{bmatrix}.\end{array}$$

With subscripts still denoting partial differentiations, the obvious integrability conditions $Q_u-S_x=S_u-B_x=0$ and (4.5) give, due to (4.6.i),

$$(4.7)\quad SQ_\tau+BQ_\lambda=QS_\tau+SS_\lambda,\quad SS_\tau+BS_\lambda=QB_\tau+SB_\lambda,\quad Q>0,\quad QB>S^2.$$

Conversely, if functions Q,S,B of the variables τ,λ satisfy (4.7), then, locally,

(d) $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$ for a function ϕ, with (4.4), of the variables x,u related to τ,λ via $(\tau,\lambda)=(\phi_x,\phi_u)$, and Q,S,B determine each of ϕ,x,u uniquely up to additive constants.

In fact, the equalities in (4.7) state precisely that the vector fields $Q\partial_\tau+S\partial_\lambda$ and $S\partial_\tau+B\partial_\lambda$ commute or, equivalently, the 1-forms $(QB-S^2)^{-1}(B\,d\tau-S\,d\lambda)$ and $(QB-S^2)^{-1}(Q\,d\lambda-S\,d\tau)$, dual to them, are closed, and we may declare these vector fields (or, 1-forms) to be ∂_x, ∂_u or, respectively, dx, du. Now that, locally, x,u are defined, up to additive constants, we obtain ϕ by solving the system $(\phi_x,\phi_u)=(\tau,\lambda)$, where τ,λ are treated as functions of x,u via the resulting locally diffeomorphic coordinate change $(\tau,\lambda)\mapsto(x,u)$. Closedness of $\tau\,dx+\lambda\,du$ and the equality $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$ are obvious: our choice of dx and du gives (4.6.iv), and hence (4.6.ii), so that $d\tau\wedge dx+d\lambda\wedge du=0$.

The g-Laplacians of τ and λ can also be expressed as

(e) $\Delta\tau=Q_\tau+S_\lambda$ and $\Delta\lambda=S_\tau+B_\lambda$, while
(f) $\varPi_x=(Q_\tau+S_\lambda)\varPi$ and $\varPi_u=(S_\tau+B_\lambda)\varPi$, for $\varPi=QB-S^2$.

To see this, first note that, by (4.6.i), $(QB-S^2)_x=Q(QB-S^2)_\tau+S(QB-S^2)_\lambda=Q(QB_\tau+SB_\lambda-SS_\tau)-S(QS_\tau+SS_\lambda-BQ_\lambda)-S^2S_\lambda+QBQ_\tau$. Using (4.7) to replace the two three-term sums in parentheses by BS_λ and SQ_τ, we thus obtain the first part of (f). For the second one we similarly rewrite $(QB-S^2)_u=S(QB-S^2)_\tau+$

$B(QB-S^2)_\lambda$ as $S(QB_\tau - BS_\lambda - SS_\tau) - S^2S_\tau + B(BQ_\lambda + SQ_\tau - SS_\lambda) + QBB_\lambda$, and use analogous three-term replacements based on (4.7). Now (e) follows from (c), (4.5) and (f).

THEOREM 4.1. *In* $\mathbb{C}^2$ *with the complex coordinates* $(x+ix', u+iu')$*, given an open subset* M *and a function* $\phi : M \to \mathrm{I\!R}$ *of the real variables* x, u*, having the property* (4.4)*, let* g *be the Kähler metric on* M *with the Kähler potential* 2ϕ*. The following two conditions are equivalent.*

(i) *The special Ricci-Hessian equation* (0.1) – (0.2) *holds on* M *for* $\tau = \phi_x$*, and* $dQ \wedge d\tau \neq 0$ *everywhere in* M*, with* $Q = g(\nabla\tau, \nabla\tau)$*. Thus, by Theorem* D*, one has* (0.6) *and* (0.9.a)*, where* $Y = \Delta\tau$ *and* F *is the function of* τ *characterized by* (0.8)*, for the constants* θ, κ *in* (0.7)*.*

(ii) *The triple* $(Q, S, B) = (\phi_{xx}, \phi_{xu}, \phi_{uu})$ *of functions of the new variables* $(\tau, \lambda) = (\phi_x, \phi_u)$ *satisfies* (4.7) *with* $Q_\lambda \neq 0$ *everywhere, as well as the equations* $S_\tau + B_\lambda = S\alpha + G$*,* $Q_\tau + S_\lambda = Q\alpha + F$*,* $G_\tau = -S\alpha'$*,* $G_\lambda = Q\alpha' + F'$ *for some function* G *and* $(\,)' = d/d\tau$*.*

PROOF. By (0.9.b), equation (0.1) is, in case (i), equivalent to

$$2\alpha\nabla d\tau \,+\, 2\mathrm{r} \,=\, -(Q\alpha' + F')g, \tag{4.8}$$

where all the terms are Hermitian symmetric 2-tensor fields, and hence correspond, via g, to *complex-linear* bundle morphisms $TM \to TM$. Thus, (4.8) amounts to

(f) equalities of the images of both sides in (4.8) under $\imath_v$ and $\imath_w$.

The equality of the $\imath_v$-images is, by (1.3.b) and (0.3), the result of applying d, via (2.5), to the relation (0.9.a) in (i): $Y - Q\alpha = F$. This last relation and (e), with $Y = \Delta\tau$ due to (0.4), show that (i) implies the equality $Q_\tau + S_\lambda = Q\alpha + F$ in (ii). Defining G to be $S_\tau + B_\lambda - S\alpha$ we get $S_\tau + B_\lambda = S\alpha + G$. On the other hand, the equality of the $\imath_w$-images in (4.8) reads

$$\alpha\, dS \,-\, d\Delta\lambda \,=\, -(Q\alpha' + F')\, d\lambda. \tag{4.9}$$

In fact, the first term equals $\alpha\, dS$ since, for the two commuting gradients $v = \nabla\tau$ and $w = \nabla\lambda$, one has $2\nabla_w d\tau = d[g(v,w)]$ or, in local coordinates, $2w^k v_{,jk} = w^k v_{,jk} + v^k w_{,jk} = (v^k w_k)_{,j}$, and $S = \phi_{xu} = g(v,w)$ by (4.3) and (4.1). The second term is $-d\Delta\lambda$ due to (a) and (1.8). By (e), $G = S_\tau + B_\lambda - S\alpha = \Delta\lambda - S\alpha$, and so (4.9) becomes $\alpha\, dS - d(G + S\alpha) = -(Q\alpha' + F')\, d\lambda$, that is, according to (2.5),

$$dG \,=\, (Q\alpha' + F')\, d\lambda \,-\, S\, d\alpha \,=\, (Q\alpha' + F')\, d\lambda \,-\, S\alpha' d\tau$$

or, in other words, $G_\tau = -S\alpha'$ and $G_\lambda = Q\alpha' + F'$. Consequently, (i) implies (ii), since (4.7) arises as the integrability conditions $Q_u - S_x = S_u - B_x = 0$ combined with (4.4), and the equality $dQ = Q_\tau d\tau + Q_\lambda d\lambda$ yields $dQ \wedge d\tau = -Q_\lambda d\tau \wedge d\lambda$.

Conversely, assuming (ii), we get (i) from (f). Namely, as we saw above, the equality of the $\imath_v$-images in (4.8) arises by applying d to $Y - Q\alpha = F$, that is – cf. (e) – to $Q_\tau + S_\lambda = Q\alpha + F$. Also, (4.9) follows from (ii) and (e): $\alpha\, dS - d\Delta\lambda = \alpha\, dS - d(S_\tau + B_\lambda) = \alpha\, dS - d(S\alpha + G) = -dG - S\, d\alpha = -dG - S\alpha' d\tau = G_\tau d\tau - dG = -G_\lambda d\lambda = -(Q\alpha' + F')\, d\lambda$. This completes the proof. □

Note that (e), (f), Theorem 4.1(ii) and (4.1) give

$$\Delta\tau = Q\alpha + F, \quad \Delta\lambda = S\alpha + G, \quad d_v\varPi = \varPi\Delta\tau, \quad d_w\varPi = \varPi\Delta\lambda.$$

5. Some linear algebra

We now proceed to discuss the first-order system equivalent, as we saw in the last section (Theorem 4.1), to the Kähler-potential problem, the solution of which amounts to proving Theorem E. The main result established here, Theorem 5.3, will lead – in Section 8 – to a unique-extension property of integral lines, which results in applicability of the Cartan-Kähler theorem to our situation.

Theorem 4.1 reduces constructing local examples of special Ricci-Hessian equations (0.1) – (0.2) with $dQ\wedge d\tau\neq 0$, on Kähler surfaces, which is a fourth-order problem in the Kähler potential 2ϕ, to the following system of quasi-linear first-order partial differential equations, with subscripts representing partial derivatives:

$$(5.1)\qquad \begin{array}{rl} \text{a)} & Q_\tau+S_\lambda-Q\alpha-F=0,\\ \text{b)} & S_\tau+B_\lambda-S\alpha-G=0,\\ \text{c)} & QB_\tau+SB_\lambda-SS_\tau-BS_\lambda=0,\\ \text{d)} & SQ_\tau+BQ_\lambda-QS_\tau-SS_\lambda=0,\\ \text{e)} & G_\tau+S\alpha'=0,\\ \text{f)} & G_\lambda-Q\alpha'-F'=0,\end{array}$$

on which one imposes the additional conditions

$$(5.2)\qquad QB>S^2,\qquad Q\,>\,0,\qquad Q_\lambda\neq 0\quad\text{everywhere.}$$

Generally, if Q,S,B are real-valued functions of the real variables τ,λ and subscripts denote partial differentiations, writing $d[(QB–S^2)^{-1}(B\,d\tau–S\,d\lambda)]=\varPhi\,d\tau\wedge d\lambda$ and $d[(QB-S^2)^{-1}(S\,d\tau-Q\,d\lambda)]=\varPsi\,d\tau\wedge d\lambda$ at points where $QB\neq S^2$, one easily verifies that

$$(5.3)\qquad \begin{bmatrix}\varPhi\\ \varPsi\end{bmatrix}=\begin{bmatrix}S & B\\ Q & S\end{bmatrix}\begin{bmatrix}QB_\tau+SB_\lambda-SS_\tau-BS_\lambda\\ SQ_\tau+BQ_\lambda-QS_\tau-SS_\lambda\end{bmatrix}.$$

For the remainder of this section we treat the letter symbols in (5.1) – (5.2) as real variables, so as to derive some consequences of conditions (5.1) – (5.2) just by using linear algebra.

Lemma 5.1. *If* $(Q,S,B,G,Q_\tau,S_\tau,B_\tau,G_\tau,Q_\lambda,S_\lambda,B_\lambda,G_\lambda)\in\mathrm{I\!R}^{12}$ *satisfies the conditions* (5.1.a) – (5.1.d)*, with some* $(\alpha,F)\in\mathrm{I\!R}^2$*, then*

$$(5.4)\qquad \begin{array}{rl}\text{i)} & QB_\tau+BQ_\tau-2SS_\tau-(QB-S^2)\alpha-BF+SG=0,\\ \text{ii)} & QB_\lambda+BQ_\lambda-2SS_\lambda-QG+SF=0.\end{array}$$

Proof. The left-hand side of (5.4.i) is, obviously, $(QB_\tau+SB_\lambda-SS_\tau-BS_\lambda)+(Q_\tau+S_\lambda-Q\alpha-F)B-(S_\tau+B_\lambda-S\alpha-G)S$, and each sum in parentheses vanishes due to (5.1). Similarly, (5.4.ii) has the left-hand side $(SQ_\tau+BQ_\lambda-QS_\tau-SS_\lambda)+(S_\tau+B_\lambda-S\alpha-G)Q-(Q_\tau+S_\lambda-Q\alpha-F)S=0+0+0$. □

With subscripts denoting partial differentiations, (5.1.e-f) and (5.4) read

$$(5.5)\qquad \begin{array}{rl}\text{i)} & dG\,=\,-S\alpha'd\tau\,+\,(Q\alpha'+F')\,d\lambda,\\ \text{ii)} & d\varPi=(\varPi\alpha+BF-SG)\,d\tau\,+\,(QG-SF)\,d\lambda\ \text{ for }\ \varPi=QB-S^2.\end{array}$$

Since $d\varPi=\varPi[(Q_\tau+S_\lambda)\,dx+(S_\tau+B_\lambda)\,du]$ due to (f), one can also obtain (5.5.ii) from (4.6.iv), with $\varPi=QB-S^2$, and (5.1.a-b).

Lemma 5.2. *Let* $(\dot\tau,\dot\lambda,Q,S,B,G)\in\mathrm{I\!R}^6$ *have* $(\dot\tau,\dot\lambda)\neq(0,0)$ *and* $QB>S^2$*.*

(a) $QB>0$ *and* $\varPsi\neq 0$*, where* $\varPsi=B\dot\tau^2-2S\dot\tau\dot\lambda+Q\dot\lambda^2$*.*

(b) $(0,0,0,0,0,0)$ *is the only vector* $(Q_\tau, S_\tau, B_\tau, Q_\lambda, S_\lambda, B_\lambda) \in \mathrm{I\!R}^6$ *with*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -S & Q & 0 & -B & S \\ S & -Q & 0 & B & -S & 0 \\ \dot\tau & 0 & 0 & \dot\lambda & 0 & 0 \\ 0 & \dot\tau & 0 & 0 & \dot\lambda & 0 \\ 0 & 0 & \dot\tau & 0 & 0 & \dot\lambda \end{bmatrix} \begin{bmatrix} Q_\tau \\ S_\tau \\ B_\tau \\ Q_\lambda \\ S_\lambda \\ B_\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) *The first four rows of the above* 7×6 *matrix are linearly independent.*

Proof. Assertion (a) follows as $\varPsi$ is positive or negative definite in $(\dot\tau, \dot\lambda)$. Next, one easily verifies that, for $\varPsi$ as in (a),

$$(B\dot\tau - S\dot\lambda)\dot\lambda \boldsymbol{r}_1 + \varPsi \boldsymbol{r}_2 + \dot\lambda^2 \boldsymbol{r}_4 - B\dot\lambda \boldsymbol{r}_5 + (2S\dot\lambda - B\dot\tau)\boldsymbol{r}_6 = (0,0,0,0,0,\varPsi).$$

Depending on whether $\dot\lambda \neq 0$ (or, $\dot\lambda = 0$ and hence $\dot\tau \neq 0$), the 6×6 matrix with the rows $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3 + S\boldsymbol{r}_2$, $\boldsymbol{r}_4 - S\boldsymbol{r}_1 + Q\boldsymbol{r}_2$ followed by $\boldsymbol{r}_6 - \dot\tau \boldsymbol{r}_2$ (or, respectively, by $\boldsymbol{r}_5 - \dot\tau \boldsymbol{r}_1$), and then by $(0,0,0,0,0,\varPsi)$ displayed above, is upper triangular, with the diagonal entries $1, 1, Q, B, \dot\lambda, \varPsi$ or, respectively, $1, 1, Q, B, -\dot\tau, \varPsi$, all of them nonzero by (a). These six new rows thus form a basis of $\mathrm{I\!R}^6$ (so that the matrix has rank 6), while the first four of them are linear combinations of the first four original rows, which proves both (b) and (c). □

Theorem 5.3. *Given vectors* $(\dot\tau, \dot\lambda, Q, S, B, G) \in \mathrm{I\!R}^6$ *and* $(\alpha, F, \alpha', F') \in \mathrm{I\!R}^4$ *with* $(\dot\tau, \dot\lambda) \neq (0,0)$ *and* $QB > S^2$, *the set of all*

$$(Q_\tau, S_\tau, B_\tau, G_\tau, Q_\lambda, S_\lambda, B_\lambda, G_\lambda) \in \mathrm{I\!R}^8 \tag{5.6}$$

satisfying (5.1) *forms a two-dimensional affine subspace* $\mathcal{A}$ *of* $\mathrm{I\!R}^8$. *Furthermore, the restriction to* $\mathcal{A}$ *of the linear operator* $\varPhi : \mathrm{I\!R}^8 \to \mathrm{I\!R}^4$ *sending* (5.6) *to*

$$(\dot Q, \dot S, \dot B, \dot G) = (Q_\tau \dot\tau + Q_\lambda \dot\lambda,\ S_\tau \dot\tau + S_\lambda \dot\lambda,\ B_\tau \dot\tau + B_\lambda \dot\lambda,\ G_\tau \dot\tau + G_\lambda \dot\lambda) \tag{5.7}$$

is an affine isomorphism $\varPhi : \mathcal{A} \to \mathcal{L}$ *onto the two-dimensional affine subspace* $\mathcal{L}$ *of* $\mathrm{I\!R}^4$ *consisting of all* $(\dot Q, \dot S, \dot B, \dot G)$ *such that*

$$\begin{aligned} \dot G &= -S\alpha'\dot\tau + (Q\alpha' + F')\dot\lambda, \\ Q\dot B + B\dot Q - 2S\dot S &= [(QB - S^2)\alpha + BF - SG]\dot\tau + (QG - SF)\dot\lambda. \end{aligned} \tag{5.8}$$

Proof. That $\mathcal{A} \subseteq \mathrm{I\!R}^8$, or $\mathcal{L} \subseteq \mathrm{I\!R}^4$, if nonempty, is an affine subspace, clearly follows since (5.1), or (5.8), is a system of nonhomogeneous linear equations imposed on (5.6) or, respectively, $(\dot Q, \dot S, \dot B, \dot G)$. Also, $\mathcal{A}$ is nonempty, and two-dimensional, being the preimage of $(Q\alpha + F, S\alpha + G, 0, 0, -S\alpha', Q\alpha' + F')$ under the obvious linear operator $\mathrm{I\!R}^8 \to \mathrm{I\!R}^6$. Namely, this operator is surjective: it equals the direct sum of the identity operator $\mathrm{I\!R}^2 \to \mathrm{I\!R}^2$, acting on (G_τ, G_λ), and an operator $\mathrm{I\!R}^6 \to \mathrm{I\!R}^4$ which has rank 4 due to Lemma 5.2(c). Next, $\varPhi$ maps $\mathcal{A}$ into $\mathcal{L}$, which one sees adding (5.1.e), or (5.4.i), times $\dot\tau$ to (5.1.f), or (5.4.ii), times $\dot\lambda$, and then using (5.7). Thus, $\mathcal{L}$ is nonempty, and $\dim \mathcal{L} = 2$, the matrix of the homogeneous system associated with (5.8) having rank 2 since Lemma 5.2(a) gives $Q \neq 0$. Let $\mathcal{A}' \subseteq \mathrm{I\!R}^8$ and $\mathcal{L}' \subseteq \mathrm{I\!R}^4$ now be the vector subspaces parallel to $\mathcal{A}$ and $\mathcal{L}$. Our assertion will thus follow once we establish injectivity of $\varPhi : \mathcal{A} \to \mathcal{L}$, that is, injectivity of its linear part $\varPhi : \mathcal{A}' \to \mathcal{L}'$. Equivalently, we need to show that zero is the only vector (5.6) lying in $\mathcal{A}'$ and having the $\varPhi$-image $(0,0,0,0)$ or, in other words, the only solution to the matrix equation in Lemma 5.2, with $(\dot{G}, G_\tau, G_\lambda) = (0,0,0)$. This, however, is precisely what Lemma 5.2(b) states. □

6. The existence of solutions

After the preceding foray into linear algebra, we now return to treating (5.1) as a system of quasi-linear first-order partial differential equations with four unknown real-valued functions Q, S, B, G of the real variables τ, λ, subject to the additional conditions (5.2).

Subscripts again denote partial differentiations, while α and F are functions of the variable τ, also depending on three fixed real constants $\varepsilon, \theta, \kappa$, so that

$$\text{(6.1)}\qquad \begin{array}{l} 2\alpha' + \alpha^2 = 4\varepsilon, \text{ where } (\ )' = d/d\tau, \\ 4\varepsilon F = \theta(2 - \tau\alpha) + 4\varepsilon\kappa\alpha \ \text{ if } \varepsilon \neq 0, \\ F = \kappa\alpha - 2\theta/(3\alpha^2) \ \text{ when } \varepsilon = 0, \end{array}$$

In addition, it is natural to assume here that

(6.2) τ ranges over the domain of α (which is also the domain of F).

Consequently, α and F satisfy the ordinary differential equations

$$\text{(6.3)}\qquad \alpha'' + \alpha\alpha' = 0, \qquad F'' = -F\alpha'.$$

Theorem 6.1. *For any fixed* α, F *as in* (6.1) – (6.2)*, real-analytic solutions* $\boldsymbol{Z} = (Q, S, B, G)$ *to* (5.1) – (5.2) *exist, locally, on a neighborhood of any* $(\tau, \lambda) \in \mathrm{I\!R}^2$ *with the property* (6.2).

More precisely, one obtains a locally-unique such solution $\boldsymbol{Z}$ *by prescribing* $\boldsymbol{Z}$ *and the partial derivatives* $\boldsymbol{Z}_\tau, \boldsymbol{Z}_\lambda$ *real-analytically along an arbitrary real-analytic embedded curve* $t \mapsto (\tau, \lambda) \in \mathrm{I\!R}^2$*, so as to satisfy* (5.1), (5.2), (6.2)*, and the condition* $\dot{\boldsymbol{Z}} = \dot{\tau}\boldsymbol{Z}_\tau + \dot{\lambda}\boldsymbol{Z}_\lambda$*, where* $(\ )^\cdot = d/dt$.

We prove Theorem 6.1 at the end of Section 9. As we point out in Remark 9.2, there is an infinite-dimensional freedom of choosing the data described in the second paragraph of Theorem 6.1

7. The associated exterior differential system

By an *exterior differential system* on a manifold M one means an ideal $\mathcal{I}$ in the graded algebra $\Omega^* M$, closed under exterior differentiation; its *integral manifolds* (or, *integral elements*) are those submanifolds of M (or, subspaces of its tangent spaces) on which every form in $\mathcal{I}$ vanishes [**4**, pp. 16, 65]. When such objects have dimension 1 or 2, we call them *integral curves/surfaces* or *lines/planes*.

If $E \subseteq T_z M$ is a p-dimensional integral element of $\mathcal{I}$, one sets [**4**, pp. 67-68]:

$$\text{(7.1)}\qquad \begin{array}{l} H(E) = \{v \in T_z M : \zeta(v, e_1, \ldots, e_p) = 0 \ \text{ for all } \zeta \in \mathcal{I} \cap \Omega^{p+1} M\} \\ \text{and } r(E) = \dim H(E) - (p+1), \ \text{for any basis } e_1, \ldots, e_p \text{ of } E, \end{array}$$

so that $H(E)$ is a vector subspace of $T_z M$, not depending on $e_1, \ldots, e_p$ since

$$\text{(7.2)}\qquad H(E) = \{v \in T_z M : \operatorname{span}(v, E) \text{ is an integral element of } \mathcal{I}\}.$$

For fixed real constants $\varepsilon, \theta, \kappa$, let the open subset $\mathcal{Y}$ of $\mathrm{I\!R}^6$ consist of all points $(\tau, \lambda, Q, S, B, G) \in \mathrm{I\!R}^6$ such that Q and $QB - S^2$ are both positive, while τ lies in the domains of α and F chosen so as to satisfy (6.1). Consider now

(7.3) the exterior differential system $\mathcal{I}$ on this $\mathcal{Y}$ generated by the two 1-forms $dG + S\alpha' d\tau - (Q\alpha' + F')\,d\lambda$ and $d(QB-S^2)-[(QB-S^2)\alpha+BF-SG]\,d\tau-(QG-SF)\,d\lambda$, the 2-form $dQ\wedge d\lambda + d\tau\wedge dS - (Q\alpha+F)\,d\tau\wedge d\lambda$, the 2-form $dS\wedge d\lambda + d\tau\wedge dB - (S\alpha+G)\,d\tau\wedge d\lambda$, their exterior derivatives, and the exterior derivatives of $(QB-S^2)^{-1}(B\,d\tau - S\,d\lambda)$ and $(QB-S^2)^{-1}(S\,d\tau - Q\,d\lambda)$.

The choice of $\mathcal{I}$ is justified as follows. We want $\mathcal{I}$ to consist of differential forms that are expected to vanish on all graphs of solutions to (5.1) with (5.2). Since such a graph is horizontal (in the sense that $d\tau\wedge d\lambda \neq 0$ on it), forms generating $\mathcal{I}$ will result from multiplying by $d\tau\wedge d\lambda$ the left-hand sides in (5.1), along with those in its consequence (5.4), as including the latter changes nothing except for enriching the differential system. Each of the terms ΞZ_τ, or ΞZ_λ, or Θ, where $Z\in\{Q,S,B,G\}$ (with various Ξ depending on Q,S,B,G, and Θ which may further depend also on α,F,α',F') then becomes $\Xi\,dZ\wedge d\lambda$, or $\Xi\,d\tau\wedge dZ$, or simply $\Theta\,d\tau\wedge d\lambda$.

The 2-forms in the fourth and fifth lines of (7.3) arise as above from (5.1.a) and (5.1.b), the exterior derivatives of the 1-forms in the last line – from (5.1.c) and (5.1.d) via (5.3). The rest of (7.3) is based on an additional principle: if our exterior differential system ends up containing $\xi\wedge d\tau$ and $\xi\wedge d\lambda$, for some 1-form ξ, we are free to include ξ among the system's generators, as the horizontal integral surfaces/planes then obviously remain unaffected. The 1-form ξ in the second, or third, line of (7.3) corresponds in this way to (5.1.e) – (5.1.f) or, respectively, (5.4).

Instead of invoking the general "additional principle" one can also justify the second and third lines in (7.3) directly from the fact that (5.5) is a consequence of (5.1), which causes the two 1-forms to vanish on all graphs of solutions to (5.1).

8. The unique-extension theorem

In $\mathbb{R}^6$ with the coordinates τ,λ,Q,S,B,G, given a subspace $E\subseteq\mathbb{R}^6$,

(8.1) we call E horizontal when $(d\tau,d\lambda):E\to\mathbb{R}^2$ is injective.

Remark 8.1. If E_1 is an integral line of $\mathcal{I}$ in (7.3) and $E_1\subseteq E_2$ for a unique horizontal integral plane E_2, then E_2 is the only integral plane containing E_1. Namely, another such plane E_2', being nonhorizontal, would intersect the kernel of $(d\tau,d\lambda)$ along a line. As $E_3\subseteq H(E_1)$ for the vector subspace $H(E)$ in (7.2) and the three-dimensional span E_3 of E_2 and E_2', all planes in E_3 containing the line E_1, other than E_2', would be horizontal integral planes, making E_2 nonunique.

Remark 8.2. The only 1-forms in $\mathcal{I}$ are, obviously, the functional combinations of those in the second and third lines of (7.3). Due to their linear independence at every point, the simultaneous kernel of these two 1-forms is a codimension-two distribution $\mathcal{D}$ on $\mathcal{Y}$, that is, a vector subbundle of $T\mathcal{Y}$, and its fibre at any $(\tau,\lambda,Q,S,B,G)\in\mathcal{Y}$ consists of all $(\dot\tau,\dot\lambda,\dot Q,\dot S,\dot B,\dot G)\in\mathbb{R}^6$ with (5.8). Hence

(8.2) vectors $(\dot\tau,\dot\lambda,\dot Q,\dot S,\dot B,\dot G)$ at (τ,λ,Q,S,B,G) spanning horizontal integral lines of $\mathcal{I}$ are characterized by (5.8) and $(\dot\tau,\dot\lambda)\neq(0,0)$,

where α,F,α',F' satisfy (6.1) – (6.2).

Theorem 8.3. *Every horizontal integral line of the system $\mathcal{I}$ defined by* (7.3) *is contained in a unique integral plane of $\mathcal{I}$, and this unique plane is also horizontal.*

Proof. Any horizontal plane in $\mathrm{I\!R}^6$ has, by (8.1), a unique basis of the form

$$(1,0,Q_\tau,S_\tau,B_\tau,G_\tau),\quad (0,1,Q_\lambda,S_\lambda,B_\lambda,G_\lambda) \tag{8.3}$$

The span of (8.3) is an *integral plane of* $\mathcal{I}$ if and only if all the 1-forms (and, 2-forms) listed in (7.3) yield the value 0 when evaluated on both vectors in (8.3) or, respectively, on the pair (8.3). Due to the two final paragraphs of Section 7, and (5.3), this is equivalent to (5.1) and (5.4), and hence (Lemma 5.1) just to (5.1).

Every vector in (8.2) is a linear combination of a unique pair (8.3) satisfying (5.1): as the coefficients of the combination must be $\dot\tau$ and $\dot\lambda$, this is immediate from Theorem 5.3. In other words, every horizontal integral line of $\mathcal{I}$ lies within a unique horizontal integral plane. Remark 8.1 now allows us to drop the last occurrence of the word 'horizontal', completing the proof. □

9. Existence of integral surfaces

The next fact – used below to derive our Theorem 6.1 – is a special case of the celebrated Cartan-Kähler theorem [**4**, pp. 81–82]. Since our phrasing differs from that of [**4**], we devote the next section to clarifying how our version amounts to adapting the one in [**4**] to our particular case.

The symbols $\mathcal{Y},\mathcal{I}$ and $\mathcal{D}$ stand here for more general objects that those in Sections 7–8. The definition (8.1) of horizontality, for integral elements, is used more generally, as well as extended, in an obvious fashion, to integral manifolds.

Theorem 9.1. *Let real-analytic functions τ,λ and 1-forms $\xi_1,\dots,\xi_q$ on a manifold $\mathcal{Y}$, where $0<q<\dim\mathcal{Y}$, have the property that $d\tau,d\lambda,\,\xi_1,\dots,\xi_q$ are linearly independent at every point. Denoting by $\mathcal{D}$ and $\mathcal{I}$ the distribution on $\mathcal{Y}$ arising as the simultaneous kernel of the 1-forms $\xi_1,\dots,\xi_q$ and, respectively, the exterior differential system on $\mathcal{Y}$ generated by $\xi_1,\dots,\xi_q$ and, possibly, some higher-degree forms, along with their exterior derivatives, let us suppose that*

(9.1) every horizontal integral line of $\mathcal{I}$, at any point of $\mathcal{Y}$, is contained in a unique integral plane of $\mathcal{I}$.

Then every horizontal real-analytic integral curve of $\mathcal{I}$ is contained, locally, in a locally-unique horizontal real-analytic integral surface. Examples of such curves are provided by unparametrized integral curves of any real-analytic vector field without zeros forming a horizontal local section of the vector bundle $\mathcal{D}$ over $\mathcal{Y}$. Also,

(9.2) integral lines of $\mathcal{I}$ are the same as lines tangent to $\mathcal{D}$.

Remark 9.2. Due to Theorem 8.3, our $\mathcal{Y}$ and $\mathcal{I}$, introduced in Section 7, satisfy the hypotheses of Theorem 9.1, with $q=2$, the coordinate functions τ,λ, and the two 1-forms in the second and third lines of (7.3). Therefore, our $\mathcal{D}$ (see Remark 8.2) then corresponds to $\mathcal{D}$ in Theorem 9.1, and hence satisfies (9.2). Horizontal integral curves of our $\mathcal{I}$ thus are, by (8.2), precisely those curves which have parametrizations $t\mapsto(\tau,\lambda,\boldsymbol{Z})=(\tau,\lambda,Q,S,B,G)\in\mathcal{Y}$ with (5.8), where $(\,)^\cdot=d/dt$, and $(\dot\tau,\dot\lambda)\neq(0,0)$ for all t. Choosing such a curve as in the sentence preceding (9.2), we obtain the additional data $\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda$ required in Theorem 6.1 by applying to $\dot{\boldsymbol{Z}}$ the inverse of the affine isomorphism $\varPhi:\mathcal{A}\to\mathcal{L}$ of Theorem 5.3.

This clearly results in an infinite-dimensional freedom of choices mentioned at the end of Section 5.

PROOF OF THEOREM 6.1. The image of the mapping $t \mapsto (\tau, \lambda, \boldsymbol{Z}) \in \mathcal{Y}$ in the second paragraph of Theorem 6.1 is a horizontal real-analytic integral curve of $\mathcal{I}$. In fact, horizontality follows since $t \mapsto (\tau, \lambda)$ is an embedding, while, as $\dot{\boldsymbol{Z}} = \dot{\tau}\boldsymbol{Z}_\tau + \dot{\lambda}\boldsymbol{Z}_\lambda$, the resulting tangent directions are integral lines of $\mathcal{I}$ as an immediate consequence of Theorem 8.3 combined with (8.2).

The integral surface of $\mathcal{I}$ arising in Theorem 9.1, being horizontal (Theorem 8.3), forms, locally, the graph of a function $(\tau, \lambda) \mapsto \boldsymbol{Z} = (Q, S, B, G)$, which is a solution to (5.1) according to the description of $\mathcal{I}$ in (7.3) and the paragraph following (7.3). To realize the condition $Q_\lambda \neq 0$ required in (5.2) we solve (5.1), at a given point $(\tau, \lambda, Q, S, B, G) \in \mathcal{Y}$, by setting $(Q_\tau, Q_\lambda) = (0, 1)$, which uniquely determines $S_\tau, B_\tau, G_\tau, S_\lambda, B_\lambda, G_\lambda$, and then choosing the quadruple (5.7), with fixed $(\dot{\tau}, \dot{\lambda}) \neq (0, 0)$, associated with the resulting octuple (5.6). □

Under the assumptions of Theorem 9.1, let $k = \dim \mathcal{Y}$. For all p-dimensional horizontal integral elements $E = E_p$ of $\mathcal{I}$, with $p \in \{0, 1\}$, and for $r(E) = \dim H(E) - (p+1)$ in (7.1),

(9.3)
a) the integer $r(E)$ has a fixed nonnegative value, namely,
b) $\dim H(E_0) = k - q$ and $r(E_0) = k - q - 1$ when $p = 0$,
c) $\dim H(E_1) = 2$ and $r(E_1) = 0$ in the case where $p = 1$.

This is obvious from (9.2) or, respectively, (9.1).

10. Where Theorem 9.1 comes from

Here is the Cartan-Kähler theorem, cited *verbatim* from [**4**, pp. 81–82]:

Let $\mathcal{I} \subset \Omega^*(M)$ be a real analytic differential ideal. Let $P \subset M$ be a connected, p-dimensional, real analytic, Kähler-regular integral manifold of $\mathcal{I}$.

Suppose that $r = r(P)$ is a non-negative integer. Let $R \subset M$ be a real analytic submanifold of M which is of codimension r, which contains P, and which satisfies the condition that T_xR and $H(T_xR)$ are transverse in T_xM or all $x \in P$.

Then there exists a real analytic integral manifold of $\mathcal{I}$, X, which is connected and $(p+1)$-dimensional and which satisfies $P \subset X \subset R$. This manifold is unique in the sense that any other real analytic integral manifold of $\mathcal{I}$ with these properties agrees with X on an open neighborhood of P.

As we verify in the following paragraphs, the hypotheses of our Theorem 9.1 imply those listed above, for $(p, r) = (1, 0)$, the manifolds M, R above which are both replaced by our $\mathcal{Y}$, and the same ideal $\mathcal{I}$ as ours. By our $\mathcal{Y}$ and $\mathcal{I}$ we mean the "general" ones (see the three lines preceding Theorem 9.1), rather than the very special choices of $\mathcal{Y}$ and $\mathcal{I}$ made in Section 7.

Furthermore, P mentioned above is our (arbitrary) horizontal real-analytic integral curve of $\mathcal{I}$. The resulting manifold X corresponds to the horizontal real-analytic integral surface of $\mathcal{I}$ claimed to exist in Theorem 9.1.

We now proceed to explain why our horizontal integral curve must automatically be Kähler-regular [**4**, p. 81], meaning that its tangent lines are all Kähler-regular in the sense of [**4**, p. 68, Definition 1.7]. To verify this last claim, we first apply Cartan's test [**4**, p. 74, Theorem 1.11]. Namely, in the notation of [**4**, p. 74, Theorem 1.11], $n = 1$ (as we are dealing with tangent *lines*). Due to the relation $\dim H(E_0) = k - q$ in (9.3.b), and (9.2), $H(E_0)$ is of codimension q in the tangent space of $\mathcal{Y}$ containing it, the same as the codimension, in the $(2k-1)$-dimensional Grassmann manifold $\mathrm{Gr}_1\mathcal{Y}$ of lines tangent to $\mathcal{Y}$, of the $(2k-q-1)$-dimensional submanifold $V_1(\mathcal{I})$ formed by all integral lines of $\mathcal{I}$. Cartan's test thus shows that every line E_1 tangent to our horizontal integral curve is *ordinary* [**4**, p. 73, Definition 1.9]. The Kähler-regularity of E_1 now trivially follows, as r in [**4**, pp. 67-68] has the constant value 0 according to (9.3.c). This is also the value $r = r(P)$ in the italicized statement cited above from [**4**]. Cf. [**4**, pp. 81–82, the lines preceding Theorem 2.2].

11. Proof of Theorem E

Let ∇ (or, g) be a connection (or, a pseudo-Riemannian metric) on a C^∞ manifold M. We call ∇ or g *real-analytic* if, in a suitable coordinate system around every point of M, its components Γ_{jk}^l (or, g_{jk}) are real-analytic functions of the coordinates. The C^∞ structure of M then contains a unique real-analytic structure (maximal atlas) making ∇ or, g real-analytic. (The atlas consists of all coordinate systems just mentioned; their mutual transition mappings are real-analytic due to real-analyticity of affine mappings, or isometries, between manifolds with real-analytic connections/metrics, which follows since such mappings appear linear in geodesic coordinates.) Real-analyticity of a metric g obviously implies that of its Levi-Civita connection ∇ (and vice versa, since $\nabla g = 0$).

For a real-analytic (Riemannian) Kähler metric g on a complex manifold M, the unique real-analytic structure described above coincides with the one induced by the complex structure of M. In fact, local holomorphic coordinate functions, being g-harmonic, must be real-analytic relative to the former structure, as a consequence of the standard regularity theory of elliptic partial differential equations applied to the g-Laplacian Δ.

Proof of Theorem E. Combining Theorems 6.1 and 4.1, we obtain the first assertion of Theorem E.

For the second one we invoke the existence results of [**22**] and [**6**]. In both cases, $dQ \wedge d\tau \neq 0$ somewhere, and the metric is real-analytic. The former claim follows, for instance, since a compact Kähler surface with a nontrivial holomorphic gradient $\nabla\tau$ having $dQ \wedge d\tau = 0$ identically for $Q = g(\nabla\tau, \nabla\tau)$ must necessarily [**10**, Sect. 1] be biholomorphic to $\mathbb{C}\mathrm{P}^2$ or a $\mathbb{C}\mathrm{P}^1$ bundle over $\mathbb{C}\mathrm{P}^1$ (rather than the two-point blow-up of $\mathbb{C}\mathrm{P}^2$). The latter, in the case of [**22**], is due to a general reason: all Ricci solitons are real-analytic [**8**, Lemma 3.2]. So are, however, all Riemannian Einstein metrics [**13**, Theorem 5.2], and the Chen-LeBrun-Weber metric of [**6**] is conformal to an Einstein metric $\hat{g}$, while again, for a general reason [**9**, p. 417, Prop. 3(ii)], the conformal change leading from $\hat{g}$ to g has a canonical form (up to a constant factor, it is the multiplication by the cubic root of the norm-squared of the self-dual Weyl tensor). This causes g to be real-analytic as well. □

12. The analytic-continuation phenomenon

We elaborate here on the plausibility of small deformations mentioned in the lines following Theorem E, beginning with the coth-cot analytic continuation. The real-analytic function $\mathrm{I\!R} \ni y \mapsto y^{-1}\tanh y$, with the value 1 at $y = 0$, being even, has the form $\Sigma(y^2)$ for some real-analytic function Σ. Now $(\varepsilon, \tau) \mapsto \beta_\varepsilon(\tau) =$

$\tau\Sigma(\varepsilon\tau^2)$ is a real-analytic function on an open subset of $\mathrm{I\!R}^2$ and $\beta_\varepsilon(\tau)$ equals $\varepsilon^{-1/2}\tanh(\varepsilon^{1/2}\tau)$, or τ, or $|\varepsilon|^{-1/2}\tan(|\varepsilon|^{1/2}\tau)$, depending on whether $\varepsilon > 0$, or $\varepsilon = 0$, or $\varepsilon < 0$. For $\alpha_\varepsilon(\tau) = 2/\beta_\varepsilon(\tau)$ the analogous expressions are

$$2\varepsilon^{1/2}\coth(\varepsilon^{1/2}\tau) \ \ (\text{if } \varepsilon > 0), \qquad 2/\tau \ \ (\text{if } \varepsilon = 0), \qquad 2|\varepsilon|^{1/2}\cot(|\varepsilon|^{1/2}\tau) \ \ (\text{if } \varepsilon < 0).$$

All α_ε with $\varepsilon > 0$, as well as those with $\varepsilon < 0$, are thus affine (in fact, linear) modifications – see Remark C – of α_1 or, respectively, α_{-1}, and $\alpha_0(\tau) = 2/\tau$.

For a tanh-coth analytic-continuation argument we define $(t,\tau) \mapsto \alpha_t(\tau)$ by $\alpha_t(\tau) = 2(e^\tau - te^{-\tau})/(e^\tau + te^{-\tau})$. Thus, with q such that $2q = \log|t|$, if $t > 0$ (or, $t < 0$), $\alpha_t(\tau) = 2\tanh(\tau - q)$ or, respectively, $\alpha_t(\tau) = 2\coth(\tau - q)$. Again, all α_t for $t > 0$, or those with $t < 0$, are affine (this time, translational) modifications of α_1, or of α_{-1}, while $\alpha_0(\tau) = 2$.

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