

# Non-Walker Self-Dual Neutral Einstein Four-Manifolds of Petrov Type III

By *Andrzej Derdzinski*

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*ABSTRACT.* The local structure of the manifolds named in the title is described. Although curvature homogeneous, they are not, in general, locally homogeneous. Not all of them are Ricci-flat, which answers an existence question about type III Jordan-Osserman metrics, raised by Díaz-Ramos, García-Río and Vázquez-Lorenzo (2006).

## 1. Introduction

The main result of this paper, Theorem 22.1, describes the local structure of all non-Walker self-dual oriented Einstein four-manifolds  $(M, g)$  of the neutral metric signature  $--++$  which are of Petrov type III, in the sense that so is the self-dual Weyl tensor at every point of  $M$ . Such  $(M, g)$ , also referred to as the *non-Walker, type III four-dimensional Jordan-Osserman manifolds* [7, Remark 2.1], are known to be curvature homogeneous, cf. Remark 5.3.

According to Theorem 22.1, all  $(M, g)$  with the stated properties are, locally, parametrized by arbitrary solutions to equations (18.4), which, by Remark 18.4, are equivalent to the system (18.8) of four first-order quasi-linear partial differential equations imposed on eight unknown real-valued functions of two real variables.

A description of all solutions to (18.4) is given in Section 25. It applies, however, only to the dense open subset of  $\mathbb{R}^2$  formed by points which are in *general position*, in the sense that, on some neighborhood of the point in question, each of several specific vector-valued functions associated with the solution is either identically zero, or nonzero everywhere.

The precise meaning of the adjective ‘non-Walker’ used above is that the metrics  $g$  are assumed to represent the *strictly non-Walker case*, in which a certain natural 1-form  $\beta$  is nonzero everywhere. By contrast, the *Walker case*, defined in Section 6 by requiring  $\beta$  to vanish identically, is equivalent to the existence of a two-dimensional null parallel distribution compatible with the orientation. In the Walker case, Díaz-Ramos, García-Río and Vázquez-Lorenzo have already found a canonical coordinate form of such metrics [7, Theorem 3.1(ii.3)]. For a coordinate-free version of their result, see [5, Theorem 13.1].

The Walker case implies Ricci-flatness of  $(M, g)$ , which is why Díaz-Ramos, García-Río and Vázquez-Lorenzo asked in [7, Remark 3.5] whether a type III self-dual neutral Einstein four-manifold can have nonzero scalar curvature. Theorem 22.2 of this paper shows, by means of explicit examples, that the answer is ‘yes’ while, at the same time, there also exist Ricci-

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flat strictly non-Walker self-dual neutral four-manifolds, and, whether Ricci-flat or not, such manifolds need not, in general, be locally homogeneous.

Besides [7], a few other papers contain results in this direction. Blažić, Bokan and Rakić [1] found a characterization of type III self-dual neutral Einstein four-manifolds in terms of a system of first-order differential equations imposed on the Levi-Civita connection forms in a suitable local trivialization of the tangent bundle. Brans [2] described all *Lorentzian* Einstein metrics of Petrov type III in dimension four. Curvature-homogeneous Einstein four-manifolds of any metric signature, with a curvature operator that is complex-diagonalizable, are known to be locally homogeneous, and have been fully classified [4]. In particular, curvature-homogeneous four-dimensional *Riemannian* Einstein manifolds, which obviously satisfy the diagonalizability condition, are all locally symmetric [8, p. 476, Corollary 7.2].

For more on Osserman metrics and curvature-homogeneity, see [9], [10] and [3].

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## 2. An outline of the argument

For any type III self-dual neutral oriented Einstein four-manifold  $(M, g)$ , it is shown in Lemmas 5.1(a)–(b) and 5.2(i) that, locally,  $M$  admits a natural identification with the total space of an affine plane bundle over a surface, endowed with a distinguished “nonlinear connection” in the form of a horizontal distribution  $\mathcal{H}$ , transverse to the vertical distribution  $\mathcal{V}$  of the bundle. Both  $\mathcal{V}$  and  $\mathcal{H}$  consist of  $g$ -null vectors.

In the next step,  $\mathcal{H}$  is ignored. What is kept in the picture consists of some natural differential forms (such as  $\beta$ , mentioned in the Introduction) along with the vertical distribution  $\mathcal{V}$  on  $M$ , the family  $\mathcal{D}$  of the standard flat torsionfree connections of the affine leaves of  $\mathcal{V}$ , and a partial version  $h$  of the original metric  $g$ . Specifically,  $h$  “remembers” only how to evaluate inner products in which one of the vectors is vertical. The data just listed form a *basic octuple* defined, as an abstract object, in Section 8.

The original metric  $g$  may be reconstructed from its associated basic octuple through a choice of a suitable horizontal distribution  $\mathcal{H}$ , declared to be  $g$ -null. The desired properties of  $g$  (being a self-dual Einstein metric of Petrov type III such that the given  $\mathcal{V}$  and  $\mathcal{H}$  correspond to it as in Lemma 5.1(a)) can be rephrased as a system of four curvature conditions, appearing in Theorem 12.2, which are differential equations with the unknown  $\mathcal{H}$ .

What makes basic octuples convenient to use is the fact that they all represent a unique local diffeomorphic type (Theorem 9.2), and so choosing to work with just one of them leads to no loss of generality. Secondly, the horizontal distributions for a fixed basic octuple form an affine space (and, in fact, constitute arbitrary sections of a certain affine bundle). The discussion of the four curvature conditions may thus be simplified by selecting one horizontal distribution  $\mathcal{H}$  to serve as the origin, and expressing other horizontal distributions as sums  $\tilde{\mathcal{H}} = \mathcal{H} + F$ , where  $F$  is a section of a specific vector bundle over  $M$ .

A simplification of this kind is provided by *two-plane systems*, introduced in Section 9. Specifically,  $M$  is replaced with the product  $\Sigma \times \Pi_+$  of an affine plane  $\Sigma$  and a (vector) half-plane  $\Pi_+$ , so that the  $\Pi_+$  and  $\Sigma$  factor distributions serve as  $\mathcal{V}$  and the “origin”  $\mathcal{H}$ . Of the four curvature conditions in Theorem 12.2, imposed on  $\tilde{\mathcal{H}} = \mathcal{H} + F$ , with  $F$  as above,

three then turn out to be (nonhomogeneous) linear, of first or second order, and involve only derivatives in  $\Pi_+$  directions. Therefore, they can be solved explicitly in each fibre  $\{y\} \times \Pi_+$ , and the solutions form a nine-dimensional affine space, which is the same for all  $y$ .

The sections  $F$  just mentioned may thus be viewed as functions on nonempty open subsets  $U$  of  $\Sigma$ , valued in the affine 9-space. For such functions  $F$ , the fourth curvature condition in Theorem 12.2 is equivalent to the system (18.4) of quasi-linear first-order differential equations with the unknown function  $(q, \lambda, \mu) : U \rightarrow V$ , related to  $F$ , and taking values in a specific eight-dimensional vector space  $V$ . That solutions to (18.4) exist is obvious (Example 18.3). A description of all solutions is achieved by interpreting a function  $U \rightarrow V$  as a pair  $(q, \bar{\nabla})$  consisting of a section  $q$  of a (vector) plane bundle  $\mathcal{P}$  over the surface  $U$  and a unimodular connection  $\bar{\nabla}$  in  $\mathcal{P}$ . The system (18.4) then amounts to the algebraic condition (25.3.i) on the curvature of  $\bar{\nabla}$  coupled with the covariant-derivative equation (25.3.ii) on  $q$ . As a result, (18.4) is easily solved with the aid of gauge transformations and the method of characteristics.

Solutions to (18.4) play a central role in Theorem 22.1, which presents a construction giving rise, locally and up to isometries, to all strictly non-Walker type III self-dual neutral Einstein metrics in dimension four, and only to such metrics. The construction is explicit enough to yield easy answers to the most obvious questions: for instance, metrics with the properties just listed need not be Ricci-flat or locally homogeneous (Theorem 22.2). Theorem 22.1 does not, however, provide a complete local classification of these metrics, since it fails to describe a local moduli space, that is, to determine when two choices of the parameters  $(q, \bar{\nabla})$  mentioned above lead to two four-manifolds which are locally isometric to each other.

### 3. Preliminaries

All manifolds, bundles, their sections and subbundles, as well as connections and mappings, including bundle morphisms, are assumed to be of class  $C^\infty$ . A manifold is by definition connected; a bundle morphism may operate only between two bundles with the same base manifold, and acts by identity on the base.

We treat the covariant derivatives of a vector field  $v$  and of a 1-form  $\xi$ , relative to any fixed connection  $\nabla$  on a manifold  $M$ , as a morphism  $\nabla v : TM \rightarrow TM$  and, respectively, a twice-covariant tensor field, acting on vector fields  $u, w$  by  $(\nabla v)u = \nabla_u v$  and  $(\nabla \xi)(u, w) = (\nabla_u \xi)(w)$ . For the tensor and exterior products of 1-forms  $\beta, \alpha$  on a manifold, the exterior derivative of  $\beta$ , and any tangent vector fields  $u, v$ , we have

$$(3.1) \quad \begin{array}{ll} \text{i)} & (\beta \otimes \alpha)(u, v) = \beta(u)\alpha(v), \quad \text{ii)} \quad \beta \wedge \alpha = \beta \otimes \alpha - \alpha \otimes \beta, \\ \text{iii)} & (d\beta)(u, v) = d_u[\beta(v)] - d_v[\beta(u)] - \beta([u, v]). \end{array}$$

We use the metric  $g$  of a pseudo-Riemannian manifold  $(M, g)$  to identify any vector field  $u$  on  $M$  with the 1-form  $g(u, \cdot)$ . Similarly, we identify a vector-bundle morphism  $C : TM \rightarrow TM$  with the twice-covariant tensor field  $b$  such that  $b(u, w) = g(Cu, w)$  for all vector fields  $u, w$ . In other words,  $C$  is the result of raising the second index in  $b$ . A twice-covariant tensor field  $b$  thus associates with a vector field  $u$  a new vector field  $bu$ , and with a 1-form  $\xi$  a new 1-form  $b\xi$ , characterized by

$$(3.2) \quad \text{i)} \quad bu = Cu, \quad \text{ii)} \quad b\xi = b(v, \cdot),$$

where  $C$  corresponds to  $b$  as above, and  $v$  is the vector field such that  $\xi = g(v, \cdot)$ .

By the Leibniz rule, when  $\nabla$  is the Levi-Civita connection of a pseudo-Riemannian metric  $g$  and  $u, v, w$  are tangent vector fields,  $2g(\nabla_v u, w)$  equals

$$(3.3) \quad d_v[g(u, w)] + d_u[g(v, w)] - d_w[g(v, u)] + g(u, [w, v]) + g(w, [v, u]) - g(v, [u, w]),$$

cf. [11, p. 160], where  $d_v$  is the directional derivative. Our sign convention about the curvature tensor  $R$  of any connection  $\nabla$  in a real vector bundle  $\mathcal{E}$  over a manifold  $M$  is

$$(3.4) \quad R(u, v)w = \nabla_v \nabla_u w - \nabla_u \nabla_v w + \nabla_{[u, v]} w$$

for vector fields  $u, v$  tangent to  $M$  and a section  $w$  of  $\mathcal{E}$ . Then

$$(3.5) \quad R(u, v) : \mathcal{E} \rightarrow \mathcal{E}$$

denotes the bundle morphism sending any section  $w$  to  $R(u, v)w$ . We use the symbol  $R$  also for the four-times covariant curvature tensor of a pseudo-Riemannian manifold  $(M, g)$ , with

$$(3.6) \quad R(w, w', u, v) = g(R(w, w')u, v).$$

As a consequence of (3.4) and the Leibniz rule, any vector-bundle morphism  $C : TM \rightarrow TM$  in a pseudo-Riemannian manifold  $(M, g)$  satisfies the *Ricci identity*

$$(3.7) \quad \nabla_w \nabla_v C - \nabla_v \nabla_w C + \nabla_{[v, w]} C = [R(v, w), C]$$

for tangent vector fields  $v, w$ , with  $[\cdot, \cdot]$  on the right-hand side denoting the commutator of bundle morphisms  $TM \rightarrow TM$ . (In coordinates, this reads  $C_{j,kl}^p - C_{j,lk}^p = R_{lkj}{}^s C_s^p - R_{lks}{}^p C_j^s$ .)

The four-times covariant curvature and Weyl tensors  $R$  and  $W$  of a pseudo-Riemannian Einstein four-manifold  $(M, g)$  with the scalar curvature  $12K$  are related by

$$(3.8) \quad R = W + Kg \wedge g, \quad \text{or, in coordinates, } R_{jklp} = W_{jklp} + K(g_{jl}g_{kp} - g_{kl}g_{jp}).$$

Our conventions about curvature-like tensors  $R$  acting on 2-forms  $\zeta$  and the inner product  $\langle \cdot, \cdot \rangle$  of 2-forms, are, in local coordinates,

$$(3.9) \quad \text{a) } 2(R\zeta)_{jk} = R_{jklp}\zeta^{lp}, \quad \text{b) } 2\langle \zeta, \eta \rangle = -\text{tr } \zeta\eta,$$

$\zeta\eta$  being the composite of the bundle morphisms  $TM \rightarrow TM$  corresponding to  $\zeta$  and  $\eta$  as in (3.2.i). The coordinate versions  $2\langle \zeta, \eta \rangle = \zeta_{jk}\eta^{jk}$  of (3.9.b) and  $(\beta \wedge \alpha)_{jk} = \beta_j\alpha_k - \beta_k\alpha_j$  of (3.1.ii) now give, for tangent vector fields  $u$  and  $v$ ,

$$(3.10) \quad \langle \zeta, \eta \rangle = \zeta(u, v) \quad \text{if} \quad \eta = g(u, \cdot) \wedge g(v, \cdot).$$

The Hodge star  $*$  of an oriented pseudo-Riemannian four-manifold  $(M, g)$  of the neutral signature  $--++$ , acting in the bundle  $[T^*M]^{\wedge 2}$  of 2-forms is an involution, and so  $[T^*M]^{\wedge 2} = \Lambda^+M \oplus \Lambda^-M$ , for the  $\pm 1$ -eigenspace bundles  $\Lambda^\pm M$  of  $*$ , both of fibre dimension 3, known as the bundles of *self-dual* and *anti-self-dual* 2-forms [8, p. 641, formula 37.26]. According to [8, p. 643, formula (37.31)], with notation as in (3.9.b),

$$(3.11) \quad \zeta\eta + \eta\zeta = -\langle \zeta, \eta \rangle \text{Id} \quad \text{whenever } \zeta, \eta \text{ are sections of } \Lambda^\pm M.$$

**Remark 3.1.** A vector field  $v$  on the total space of a bundle projection  $\pi$  is called *vertical* if it is a section of the vertical distribution  $\mathcal{V} = \text{Ker } d\pi$ . As one easily verifies in suitable local coordinates, a vector field  $w$  on the total space is  $\pi$ -projectable onto some vector field on the base manifold if and only if, for every vertical vector field  $v$ , the Lie bracket  $[w, v]$  is also vertical. More generally, given an integrable distribution  $\mathcal{V}$  on a manifold  $M$ , by a  *$\mathcal{V}$ -projectable local vector field* in  $M$  we will mean any vector field  $w$  defined on an open set  $U \subset M$  and such that, whenever  $v$  is a section of  $\mathcal{V}$  defined on  $U$ , so is  $[w, v]$ .

**Remark 3.2.** For any oriented pseudo-Riemannian four-manifold  $(M, g)$  of the neutral signature  $--++$ , the bundles  $\Lambda^\pm M$  of self-dual and anti-self-dual 2-forms are canonically oriented. In fact, a positive-oriented  $(--++)$ -orthonormal basis of  $T_x M$  naturally gives rise to a basis of  $\Lambda_x^\pm M$ , which is also  $(--++)$ -orthonormal [8, formula (37.25) on p. 641], in such a way that changing the signs of the first and third vectors in the basis of  $T_x M$  leaves the resulting orientations of  $\Lambda_x^\pm M$  unchanged. Thus, both connected components of the set of positive-oriented  $(--++)$ -orthonormal bases of  $T_x M$  produce the same orientation in  $\Lambda_x^\pm M$ .

**Remark 3.3.** For an  $n$ -form  $\zeta \in [V^*]^{\wedge n}$  in a real vector space  $V$  of dimension  $n$ , any endomorphism  $\Phi$  of  $V$ , and any  $w_1, \dots, w_n \in V$ , we have  $\zeta(\Phi w_1, w_2, \dots, w_n) + \zeta(w_1, \Phi w_2, w_3, \dots, w_n) + \dots + \zeta(w_1, w_2, \dots, w_{n-1}, \Phi w_n) = (\text{tr } \Phi) \zeta(w_1, w_2, \dots, w_n)$ , as one sees using the matrix of  $\Phi$  in the basis  $w_1, \dots, w_n$ , if  $w_1, \dots, w_n$  are linearly independent, and noting that both sides vanish due to their skew-symmetry in  $w_1, \dots, w_n$ , if  $w_1, \dots, w_n$  are linearly dependent.

**Remark 3.4.** If a trilinear mapping  $(w, w', w'') \mapsto \delta(w, w', w'')$  from a two-dimensional vector space  $\Pi$  into any vector space is skew-symmetric  $w', w''$ , then  $\delta(w, w', w'')$  summed cyclically over  $w, w', w''$  yields 0. In fact, the cyclic sum then depends on  $w, w'$  and  $w''$  skew-symmetrically, so that it vanishes as  $\dim \Pi = 2$ .

#### 4. Rank versus Petrov type

We say that a traceless endomorphism of a three-dimensional pseudo-Euclidean space is of *Petrov type II* (or, III) if it is self-adjoint and sends some ordered basis  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  to  $(0, 0, \mathbf{x})$  (or, respectively, to  $(0, \mathbf{x}, \mathbf{y})$ ).

The symbol  $W^\pm$  always denotes the self-dual and anti-self-dual parts of the Weyl tensor of a given oriented pseudo-Riemannian four-manifold  $(M, g)$  of the neutral signature  $--++$ . Thus,  $W^\pm$  may be viewed as an endomorphism of the bundle  $\Lambda^\pm M$  of (anti)self-dual 2-forms. As usual, we call  $(M, g)$  *self-dual* if  $W^- = 0$ , and refer to  $g$  as a *self-dual metric of Petrov type II* (or, III) if  $W^- = 0$  and  $W^+$  is of Petrov type II (or III) at every point.

**Remark 4.1.** Note that  $\text{tr } W^\pm = 0$ . See, for instance, [8, formula (38.15) on p. 650].

**Remark 4.2.** A traceless self-adjoint endomorphism  $\Phi$  of rank 1 in a pseudo-Euclidean 3-space  $V$  is necessarily of Petrov type II, and its image  $I$  is a null line.

In fact, if  $I$  were not null, we would have  $V = I \oplus I^\perp$  and  $I^\perp = \text{Ker } \Phi$ , and hence the restriction  $\Phi : I \rightarrow I$  would be traceless, contrary to its surjectivity. Now we can use  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  with  $\mathbf{x} \in I \subset I^\perp = \text{Ker } \Phi$  and  $\mathbf{y} \in \text{Ker } \Phi$ , normalizing  $\mathbf{z}$  so that  $\Phi \mathbf{z} = \mathbf{x}$ .

**Lemma 4.3.** *Let  $(M, g)$  be a pseudo-Riemannian four-manifold of the neutral signature  $--++$  such that the Weyl tensor  $W$  acting in the bundle  $[T^*M]^{\wedge 2}$  of 2-forms has constant rank  $k$ , and the restriction of the fibre metric  $\langle, \rangle$  in  $[T^*M]^{\wedge 2}$  to the subbundle  $\mathcal{E}$  forming the image of  $W$  is degenerate at every point.*

- (i) *If  $k = 1$ , then  $M$  admits an orientation for which  $g$  is self-dual of Petrov type II.*
- (ii) *If  $M$  is oriented,  $k = 2$ , and  $\mathcal{E}$  is  $\langle, \rangle$ -null at some point, then, at every point,  $\mathcal{E}$  is  $\langle, \rangle$ -null and  $W^\pm$  are both of Petrov type II.*
- (iii) *If  $k = 2$  and, at some point,  $\mathcal{E}$  is non-null relative to  $\langle, \rangle$ , then  $\mathcal{E}$  is  $\langle, \rangle$ -non-null at every point, while  $M$  admits an orientation for which  $g$  is self-dual of Petrov type III.*

**Proof.** Let  $k^\pm$  be the function assigning to each  $x \in M$  the rank of  $W^\pm$  at  $x$ . Since  $k^\pm$  is lower semicontinuous, constancy of  $k = k^+ + k^-$  implies that both  $k^\pm$  are constant. Thus, locally,  $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$  for some subbundles  $\mathcal{E}^\pm$  of  $\Lambda^\pm M$ .

If  $k = 1$ , the requirement that  $k^+ = 1$  and  $k^- = 0$  uniquely defines an orientation of  $M$ , and (i) is obvious from Remarks 4.1 and 4.2.

If  $M$  is oriented,  $k = 2$ , and  $k^+ = k^- = 1$ , Remarks 4.1 and 4.2 yield two conclusions. First, both  $W^\pm$  are of Petrov type II at each point. Secondly, both  $\mathcal{E}^\pm$  are  $\langle, \rangle$ -null at every point, and hence so is  $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ .

Finally, let  $k = 2$  and  $k^+ \neq k^-$ . For the orientation defined by requiring that  $k^+ = 2$  and  $k^- = 0$ , we have  $W^- = 0$ , while  $\mathcal{E} = \mathcal{E}^+$  cannot be  $\langle, \rangle$ -null at any point  $x$ . (If it were,  $\mathcal{E}_x$  would be a 2-dimensional subspace of  $\Lambda_x^+ M$  contained in its own 1-dimensional orthogonal complement.) Since the restriction of  $\langle, \rangle$  to the plane bundle  $\mathcal{E}$  (the image of  $W^+$ ) is assumed degenerate, Petrov's classification [8, Proposition 39.2 on p. 652] implies that, at each point,  $W^+$  is of Petrov type III. Combined with the preceding paragraph, this proves (ii) and (iii).  $\square$

## 5. The vertical-horizontal decomposition

In the following two lemmas, the meaning of  $\Lambda^+ M$  and  $W^+$  is the same as in the second paragraph of Section 4. An endomorphism  $\Psi$  of  $\Lambda^+ M$ , such as  $W^+$ , is identified with a four-times covariant tensor field (namely, a morphism  $[T^*M]^{\wedge 2} \rightarrow [T^*M]^{\wedge 2}$  vanishing on  $\Lambda^- M$ ), and  $\operatorname{div} \Psi$  is defined, in local coordinates, by  $(\operatorname{div} \Psi)_{klm} = \Psi^j{}_{klm,j}$ .

**Lemma 5.1.** *Given an oriented four-dimensional pseudo-Riemannian manifold  $(M, g)$  of the neutral metric signature  $--++$ , let  $\Psi$  be a bundle endomorphism of  $\Lambda^+ M$  with  $\operatorname{tr} \Psi = 0$  and  $\operatorname{div} \Psi = 0$ , which is of Petrov type III at every point, as defined in Section 4.*

*Then  $\Lambda^+ M$  has a  $C^\infty$  global trivialization  $(\zeta, \eta, \theta)$  with*

$$(5.1) \quad \begin{array}{ll} \text{i)} & \Psi\zeta = 0, \quad \Psi\eta = -\zeta, \quad \Psi\theta = \eta, \\ \text{ii)} & \langle \zeta, \theta \rangle = 2 = -\langle \eta, \eta \rangle, \quad \langle \zeta, \zeta \rangle = \langle \zeta, \eta \rangle = \langle \eta, \theta \rangle = \langle \theta, \theta \rangle = 0, \end{array}$$

*unique up to replacement by  $(-\zeta, -\eta, -\theta)$ , and  $\operatorname{rank} \zeta = \operatorname{rank} \theta = 2$  at each point, while*

$$(a) \quad \mathcal{V} = \operatorname{Ker} \zeta \quad \text{and} \quad \mathcal{H} = \operatorname{Ker} \theta \quad \text{are null 2-dimensional distributions on } (M, g).$$

*The trivialization  $(\zeta, \eta, \theta)$  becomes unique if one requires, in addition, that*

$$(5.2) \quad \eta v = v \quad \text{and} \quad \eta w = -w \quad \text{for sections } v \text{ of } \mathcal{V} \text{ and } w \text{ of } \mathcal{H},$$

*with  $\eta v$  as in (3.2.i). Whether (5.2) is assumed or not, we have the following conclusions.*

$$(b) \quad \mathcal{V} \text{ is integrable, its leaves are totally geodesic, and } TM = \mathcal{H} \oplus \mathcal{V}.$$

$$(c) \quad \nabla \zeta = 2\alpha \otimes \zeta + 2\beta \otimes \eta, \quad \nabla \eta = 2\gamma \otimes \zeta + 2\beta \otimes \theta \quad \text{and} \quad \nabla \theta = 2\gamma \otimes \eta - 2\alpha \otimes \theta \quad \text{for some unique 1-forms } \alpha, \beta, \gamma \text{ on } M.$$

$$(d) \quad \beta(v) = 0 \quad \text{for every section } v \text{ of } \mathcal{V}.$$

$$(e) \quad 2\Psi = \zeta \otimes \eta + \eta \otimes \zeta \quad \text{for } \Psi \text{ treated as a four-times covariant tensor field.}$$

$$(f) \quad 2\zeta\gamma + \eta\alpha + \theta\beta = 2\eta\beta + \zeta\alpha = \zeta\beta = 0, \quad \text{in the notation of (3.2.ii).}$$

**Proof.** The assumption made about  $\Psi$ , combined with Petrov's classification [8, Proposition 39.2 on p. 652], gives (5.1) at each point  $x$ , for some basis  $\zeta_x, \eta_x, \theta_x$  of  $\Lambda_x^+ M$  which is unique up to an overall sign change [8, p. 656, Remark 39.3(iv-c)]. Our trivialization  $(\zeta, \eta, \theta)$  is therefore unique, locally, up to a change of sign, and so it has a global single-valued  $C^\infty$  branch (as the bundle  $\Lambda^+ M$  is orientable, cf. Remark 3.2). According to [8, p. 645, Lemma 37.8], at

each point  $x$ , the 2-forms  $\zeta_x$  and  $\theta_x$  (written in the next five lines without the subscript  $x$ ), being nonzero, self-dual and null, can be decomposed as  $\zeta = \xi_1 \wedge \xi_2$  and  $\theta = \xi_3 \wedge \xi_4$ , with  $\xi_j = g_x(e_j, \cdot)$ ,  $j = 1, 2, 3, 4$ , for some  $e_j \in T_x M$  such that  $\mathcal{V}_x = \text{span}\{e_1, e_2\}$  and  $\mathcal{H}_x = \text{span}\{e_3, e_4\}$  are null planes. This proves (a). As  $\xi_1 \wedge \dots \wedge \xi_4 = \zeta \wedge \theta$  equals  $\langle \zeta, * \theta \rangle$  times the volume form, while  $\langle \zeta, * \theta \rangle = \langle \zeta, \theta \rangle = 2 \neq 0$ , the vectors  $e_1, e_2, e_3, e_4$  must form a basis of  $T_x M$ . Hence  $TM = \mathcal{H} \oplus \mathcal{V}$ .

By (3.11) and (5.1.ii),  $\eta\eta = \text{Id}$  and  $\eta$  anticommutes with  $\zeta$  and  $\theta$ .

We will now verify that (5.2) holds, possibly after  $\eta$  has been replaced by  $-\eta$  (which will clearly imply the uniqueness claim concerning (5.2)). To this end, let us first note that, if  $\eta_x$ , at any point  $x \in M$ , has eigenvectors  $v \in \mathcal{V}_x$  and  $u \in \mathcal{H}_x$ , then they cannot correspond to the same eigenvalue. In fact, if they did, it would follow that  $g_x(v, u) = 0$ , since each eigenspace of  $\eta_x$  is null due to skew-symmetry of  $\eta$  and its nondegeneracy. On the other hand, the relations  $\mathcal{H} = \text{Ker } \theta$  and  $TM = \mathcal{H} \oplus \mathcal{V}$  imply injectivity of  $\theta_x$  restricted to  $\mathcal{V}_x$ , and so, since  $\eta$  and  $\theta$  anticommute,  $\theta_x v \in \mathcal{H}_x^\perp = \mathcal{H}_x$  would be an eigenvector of  $\eta_x$  for the opposite of the original  $v$ -eigenvalue, and, consequently,  $u$  and  $\theta_x v$ , being linearly independent, would span  $\mathcal{H}_x$ , while  $g_x(v, \theta_x v) = 0$  as  $\theta$  is skew-symmetric. Thus,  $v$  would be orthogonal to both  $u$  and  $\theta_x v$ , so that it would lie in  $\mathcal{H}_x^\perp = \mathcal{H}_x$ , contradicting the relation  $\mathcal{H}_x \cap \mathcal{V}_x = \{0\}$ .

Since  $\eta$  anticommutes with  $\zeta$  and  $\theta$ , it leaves the distributions  $\mathcal{V} = \text{Ker } \zeta$  and  $\mathcal{H} = \text{Ker } \theta$  invariant, so that, as  $\eta\eta$  equals the identity,  $T_x M$  is, at each point  $x \in M$ , spanned by eigenvectors of  $\eta_x$  for the eigenvalues  $\pm 1$ , lying in  $\mathcal{V}_x$  and  $\mathcal{H}_x$ . Combined with the last paragraph, this proves (5.2) up to a change of sign.

The existence and uniqueness of 1-forms  $\alpha, \beta, \gamma$  with (c) is immediate from (5.1.ii) and invariance of  $\Lambda^+ M$  under parallel transports. Also, (e) follows since, by (5.1.ii) and (3.9), both sides act on  $\Lambda^+ M$  as described in (5.1.i). In view of (e) and (c),  $\nabla \Psi = 2\gamma \otimes \zeta \otimes \zeta + 2\beta \otimes \eta \otimes \eta + \alpha \otimes (\zeta \otimes \eta + \eta \otimes \zeta) + \beta \otimes (\zeta \otimes \theta + \theta \otimes \zeta)$ . Contraction now gives  $\text{div } \Psi = (2\zeta\gamma + \eta\alpha + \theta\beta) \otimes \zeta + (2\eta\beta + \zeta\alpha) \otimes \eta + \zeta\beta \otimes \theta$ . As  $\text{div } \Psi = 0$ , this implies (f). The vector field associated by  $g$  with  $\beta$  thus is a section of  $\mathcal{V} = \text{Ker } \zeta$  and, as  $\mathcal{V}$  is null, we get (d).

Finally, let  $v, v'$  be any sections of  $\mathcal{V}$ . The Leibniz rule and (c) with  $\zeta v = \zeta v' = 0$  show that  $\zeta \nabla_v v' = -(\nabla_v \zeta)v' = -2\beta(v)\eta v'$ . However,  $\beta(v) = 0$  in view of (e). Consequently,  $\nabla_v v'$  is a section of  $\mathcal{V} = \text{Ker } \zeta$ , which yields (b).  $\square$

**Lemma 5.2.** *Suppose that an oriented pseudo-Riemannian Einstein four-manifold  $(M, g)$  with the neutral metric signature  $--++$  is self-dual of Petrov type III. The assumptions of Lemma 5.1 then are satisfied by  $\Psi = W^+$ . For  $\zeta, \eta, \alpha, \beta, \gamma, \theta, \mathcal{V}$  and  $\mathcal{H}$  uniquely defined as in Lemma 5.1 with  $\Psi = W^+$ , and with  $\nabla, R$  and  $K$  denoting the Levi-Civita connection, four-times covariant curvature tensor, and  $1/12$  of the scalar curvature of  $g$ ,*

- (i) *the connection induced by  $\nabla$  on each leaf of  $\mathcal{V}$  is flat,*
- (ii)  *$2R(v, w) = \zeta(v, w)\eta + \eta(v, w)\zeta + 2K\xi \wedge \xi'$ , where  $v, w$  are any vector fields,  $\xi = g(v, \cdot)$  and  $\xi' = g(w, \cdot)$ , with  $\xi \wedge \xi'$  as in (3.1.ii),*
- (iii)  *$d_u[g(v, w)] = g(v, \nabla_u w) = g(\nabla_w u, v) = \beta(w)\theta(u, v)$  whenever  $u, v$  are sections of  $\mathcal{V}$  parallel in the direction of  $\mathcal{V}$ , while  $w$  is a  $\mathcal{V}$ -projectable local section of  $\mathcal{H}$ ,*
- (iv)  *$2d\beta + 4\beta \wedge \alpha = -K\zeta$ ,  $2d\gamma + 4\alpha \wedge \gamma = K\theta + \eta$ ,*
- (v)  *$\mathcal{V} = \mathcal{V}^\perp = \text{Ker } \zeta = \text{Im } \zeta = \text{Ker } (\eta - \text{Id})$  and  $\mathcal{H} = \mathcal{H}^\perp = \text{Ker } \theta = \text{Im } \theta = \text{Ker } (\eta + \text{Id})$ , with  $\text{Im}$  meaning ‘image’ and  $\zeta, \eta, \theta$  treated as morphisms  $TM \rightarrow TM$ , cf. (3.2.i),*
- (vi)  *$[\nabla_w u]^{\mathcal{H}} = \beta(w)\theta u$  for any vector field  $w$  and any section  $u$  of  $\mathcal{V}$ , where  $[\ ]^{\mathcal{H}}$  denotes the  $\mathcal{H}$ -component projection in  $TM = \mathcal{H} \oplus \mathcal{V}$ .*

**Proof.** The assumptions of Lemma 5.1 are satisfied by  $\Psi = W^+$ . Namely, it is the well known, cf. [8, p. 460, Lemma 5.2], that  $\operatorname{div} W = 0$  for any pseudo-Riemannian Einstein metric. Now (3.8) and Lemma 5.1(e) with  $\Psi = W^+ = W$  yield (ii). Also, by (ii),  $R(v, v')w = 0$  for any sections  $v, v', w$  of the null distribution  $\mathcal{V} = \operatorname{Ker} \zeta$ , which proves (i). Next, (v) is obvious from Lemma 5.1(a) and (5.2), since  $\zeta, \theta : TM \rightarrow TM$  are skew-adjoint at each point, while the distributions  $\mathcal{V}$  and  $\mathcal{H}$  are null and 2-dimensional. On the other hand, (iii) follows since, by (5.2), Lemma 5.1(c) and (v),  $g(\nabla_w u, v) = g(\nabla_w(\eta u), v) = g((\nabla_w \eta)u, v) + g(\eta[\nabla_w u]^{\mathcal{H}}, v) = 2g(\beta(w)\theta u, v) - g(\nabla_w u, v)$ , with  $[\ ]^{\mathcal{H}}$  as in (vi), while  $d_u[g(w, v)] = g(\nabla_u w, v)$  from the Leibniz rule, and  $g(\nabla_u w, v) = g(\nabla_w u, v)$  in view of Remark 3.1, as  $[w, v] = \nabla_w v - \nabla_v w$  and  $\mathcal{V}$  is null.

For any fixed vector fields  $v, w$ , let  $[R(v, w), \eta]$  denote the 2-form given by  $[R(v, w), \eta] = \nabla_w \nabla_v \eta - \nabla_v \nabla_w \eta + \nabla_{[v, w]} \eta$ . Evaluating  $[R(v, w), \eta]$  with the aid of Lemma 5.1(c), and using (5.1.ii) along with (3.1), we see that

$$(5.3) \quad \langle \theta, [R(v, w), \eta] \rangle = 2[2d\gamma + 4\alpha \wedge \gamma](w, v), \quad \langle \zeta, [R(v, w), \eta] \rangle = 2[2d\beta + 4\beta \wedge \alpha](w, v).$$

Identifying  $\zeta, \eta, \theta$  with bundle morphisms  $TM \rightarrow TM$  as in (3.2.i), and using the multiplicative notation for their composites, we have

$$(5.4) \quad \theta\eta = -\eta\theta = \theta, \quad \zeta\eta = -\eta\zeta = -\zeta.$$

In fact, by (v), both sides in each equality agree separately on  $\mathcal{V} = \operatorname{Ker} \zeta$  and on  $\mathcal{H} = \operatorname{Ker} \theta$ .

Defining commutators of 2-forms, as usual, in terms of their composites, we see that  $[R(v, w), \eta]$  introduced above becomes such a commutator if we identify  $R(v, w)$  with the 2-form  $R(v, w, \cdot, \cdot)$ , cf. (3.2.i). In fact, this is immediate from the Ricci identity (3.7) applied to  $C$  which corresponds to  $b = \eta$  as in (3.2.i). By (ii),  $[R(v, w), \eta] = -\eta(v, w)\zeta + K[\xi \wedge \xi', \eta]$ , with  $\xi, \xi'$  as in (ii), since  $[\eta, \eta] = 0$ , while (5.4) gives  $[\zeta, \eta] = -2\zeta$ . Consequently,

$$(5.5) \quad \langle \theta, [R(v, w), \eta] \rangle = 2\eta(w, v) + 2K\theta(w, v), \quad \langle \zeta, [R(v, w), \eta] \rangle = -2K\zeta(w, v).$$

To verify (5.5), note that  $\langle \zeta, \theta \rangle = 2$  and  $\langle \zeta, \zeta \rangle = 0$  by (5.1.ii), while  $-2\langle \theta, [\xi \wedge \xi', \eta] \rangle = \operatorname{tr} \theta[\xi \wedge \xi', \eta] = \operatorname{tr} (\xi \wedge \xi')\eta\theta - \operatorname{tr} \theta\eta(\xi \wedge \xi') = -2\operatorname{tr} (\xi \wedge \xi')\theta = 4\langle \theta, \xi \wedge \xi' \rangle = 4\theta(v, w)$  by (3.9.b), (3.10) and (5.4), so that  $\langle \theta, [\xi \wedge \xi', \eta] \rangle = -2\theta(v, w)$  and, similarly,  $\langle \zeta, [\xi \wedge \xi', \eta] \rangle = 2\zeta(v, w)$ .

Combining (5.3) with (5.5), we obtain (iv).

Finally, both sides in (vi) are, by (v), sections of the null distribution  $\mathcal{H}$ . Their inner products with any section of  $\mathcal{H}$  (or, respectively, any section  $v$  of  $\mathcal{V}$ ) thus are both zero, (or, respectively, are equal in view of the last equality in (iii)). This yields (vi).  $\square$

Given  $(M, g)$  as in Lemma 5.2, with the corresponding objects  $\zeta, \eta, \theta, \mathcal{V}$  and  $\mathcal{H}$ , we may choose, locally, sections  $w, w'$  of  $\mathcal{H}$  such that  $\zeta(w, w') = 1$ . Setting  $v = -\zeta w'$  and  $v' = \zeta w$ , we obtain a local trivialization  $w, w', v, v'$  of the tangent bundle  $TM$ , in which the only nonzero components of  $g, \zeta, \eta$  and  $\theta$  are  $g(v, w) = g(v', w') = 1$ ,  $\zeta(w, w') = 1$ ,  $\eta(v, w) = \eta(v', w') = 1$ ,  $\theta(v, v') = 2$ , and those arising from them due to symmetry of  $g$  and skew-symmetry of  $\zeta, \eta, \theta$ .

In fact, such  $w$  and  $w'$  exist since  $TM = \mathcal{H} \oplus \mathcal{V}$  and  $\mathcal{V} = \operatorname{Ker} \zeta$ , which also implies injectivity of  $\zeta_x : \mathcal{H}_x \rightarrow \mathcal{V}_x$ , at every point  $x$ . Hence  $w, w', v, v'$  form a local trivialization of  $TM$ . As  $\mathcal{H} = \operatorname{Ker} \theta$ , applying to  $w$  and  $w'$  the relation  $(\zeta\theta + \theta\zeta)/2 = -\operatorname{Id}$ , immediate from (5.1.ii) for  $\Psi = W^+$  and (3.11), we get  $\theta v' = -2w$  and  $\theta v = 2w'$ . As  $\mathcal{V}$  and  $\mathcal{H}$  are  $g$ -null, our claim is now obvious from (5.2).

**Remark 5.3.** All self-dual oriented Einstein four-manifolds of the neutral metric signature  $--++$ , which are of Petrov type III, are curvature homogeneous. This well-known fact is



an obvious consequence of the last paragraph: by Lemma 5.2(ii),  $w, w', v, v'$  chosen as above form, at any point, a basis of the tangent space providing a canonical expression for both the metric and the curvature tensor.

## 6. The Walker and strictly non-Walker cases

Suppose that  $\mathcal{T}$  is a pseudo-Euclidean 4-space of the neutral signature  $(--++)$ . By a *null plane* in  $\mathcal{T}$  we mean a null two-dimensional vector subspace of  $\mathcal{T}$ . We also use the inner product of  $\mathcal{T}$ , denoted by  $\langle \cdot, \cdot \rangle$ , to identify the space  $[\mathcal{T}^*]^{\wedge 2}$  of 2-forms with the space  $\mathcal{T}^{\wedge 2}$  of bivectors. Thus, if  $\mathcal{T}$  is oriented, we can treat the Hodge star  $*$  as an involution of  $\mathcal{T}^{\wedge 2}$ , and speak of self-dual or anti-self-dual bivectors in  $\mathcal{T}$ .

**Lemma 6.1.** *Let  $\mathcal{T}$  be a pseudo-Euclidean 4-space of the neutral metric signature.*

- (a) *Any null plane  $\mathcal{N}$  in  $\mathcal{T}$  naturally distinguishes an orientation of  $\mathcal{T}$ , namely, the one which, for some/any basis  $u, v$  of  $\mathcal{N}$ , makes the bivector  $u \wedge v$  self-dual.*
- (b) *If  $\mathcal{N}$  and  $\mathcal{N}'$  are null planes in  $\mathcal{T}$  and  $\dim(\mathcal{N} \cap \mathcal{N}') = 1$ , then  $\mathcal{N}$  and  $\mathcal{N}'$  distinguish, in the sense of (a), two opposite orientations of  $\mathcal{T}$ .*

**Proof.** A bivector in  $\mathcal{T}$  equals  $u \wedge v$  for some basis  $u, v$  of some null plane if and only if it is nonzero, null, and self-dual or anti-self-dual [8, p. 645, Lemma 37.8]. This yields (a), and at the same time shows that, under the assumptions of (b), if  $u, v, w \in \mathcal{T}$  and  $u, v$  span  $\mathcal{N}$ , while  $v, w$  span  $\mathcal{N}'$ , then  $u \wedge v$  and  $v \wedge w$  are linearly independent null bivectors, and each of them is self-dual or anti-self-dual. If both  $\mathcal{N}$  and  $\mathcal{N}'$  distinguished the same orientation of  $\mathcal{T}$ , this orientation would make both bivectors self-dual, and so they would have to be orthogonal, since, for any self-dual bivectors  $\zeta, \eta$ , the 4-form  $\zeta \wedge \eta$  is the product of  $\langle \zeta, *\eta \rangle = \langle \zeta, \eta \rangle$  and the volume form. Thus,  $u \wedge v$  and  $v \wedge w$  would span a null plane in the pseudo-Euclidean 3-space of self-dual bivectors, which cannot exist for dimensional reasons, as it would be transverse to a spacelike or timelike plane.  $\square$

For a null plane  $\mathcal{N}$  as in Lemma 6.1(a), we will say that  $\mathcal{N}$  is *compatible* with the orientation of  $\mathcal{T}$  distinguished by it. Thus, a two-dimensional null distribution on an oriented pseudo-Riemannian four-manifold  $(M, g)$  of the neutral metric signature is either *compatible with the orientation* (at every point), or not compatible with it at any point.

The orientation distinguished by  $\mathcal{N}$  also has a description that does not invoke self-duality. Consequently, it can be generalized to all even dimensions  $n \geq 4$  (see Remark 6.3).

Part (iii) of the next theorem is due to Díaz-Ramos, García-Río and Vázquez-Lorenzo [7].

**Theorem 6.2.** *Let  $\mathcal{V}$  and  $\beta$  be the two-dimensional null distribution and the 1-form, defined as in Lemma 5.2 for a given neutral-signature oriented self-dual Einstein four-manifold  $(M, g)$  of Petrov type III.*

- (i)  *$\mathcal{V}$  is compatible with the orientation.*
- (ii)  *$\mathcal{V}$  is parallel if and only if  $\beta = 0$  identically.*
- (iii) *If  $\mathcal{V}$  is parallel, the metric  $g$  must be Ricci-flat.*
- (iv) *If  $\beta \neq 0$  everywhere,  $(M, g)$  does not admit a two-dimensional null parallel distribution compatible with the orientation.*

**Proof.** Choosing  $w, w', v, v'$  as in the lines preceding Remark 5.3, we see that  $v \wedge v'$  is the bivector corresponding to  $\zeta$  via  $g$ . Since  $v$  and  $v'$  span  $\mathcal{V}$ , while  $\zeta$  is self-dual, (i) follows.

As  $\mathcal{V}$  has totally geodesic leaves (Lemma 5.1(b)),  $\nabla_v u$  is a section of  $\mathcal{V}$  if so are  $u$  and  $v$ . Thus, by Lemma 5.2(i), for  $\mathcal{V}$  to be parallel, it is necessary and sufficient that  $\nabla_w u$  be a section of  $\mathcal{V}$  whenever  $u$  is a section of  $\mathcal{V}$ , parallel in the direction of  $\mathcal{V}$ , and  $w$  is any  $\mathcal{V}$ -projectable local vector field. Since  $\mathcal{V} = \mathcal{V}^\perp$  and  $\theta \neq 0$  everywhere (cf. Lemma 5.2(v)), the last equality in Lemma 5.2(iii) shows that  $\mathcal{V}$  has the property just stated if and only if  $\beta$  is identically zero, which yields (ii). Next, if  $\beta = 0$ , the first equality in Lemma 5.2(iv) gives  $K = 0$  (since  $\zeta \neq 0$  by Lemma 5.2(v)), proving (iii).

Suppose now that  $\beta \neq 0$  everywhere and  $\mathcal{D}$  is any two-dimensional null parallel distribution on  $(M, g)$ . If  $v, w$  are sections of  $\mathcal{D}$ , setting  $\xi = g(v, \cdot)$  and  $\xi' = g(w, \cdot)$ , we have

$$(6.1) \quad \zeta(v, w)\eta + \eta(v, w)\zeta + 2K\xi \wedge \xi' = 0.$$

This is immediate from Lemma 5.2(ii), as  $R(v, w, u, w') = 0$  for arbitrary vector fields  $u, w'$ , due to the fact that, by (3.4),  $R(u, w')v$ , being a section of  $\mathcal{D}$ , must be orthogonal to  $w$ .

Therefore,  $\zeta(v, w) = 0$  whenever  $v, w$  are sections of  $\mathcal{D}$ . In fact, evaluating (6.1) on  $(v, w)$ , we get  $\zeta(v, w)\eta(v, w) = 0$ . Thus, at points where  $\zeta(v, w) \neq 0$  we would have  $\eta(v, w) = 0$  and, by (6.1),  $\eta$  would be a multiple of  $\xi \wedge \xi'$ , contrary to its nondegeneracy (cf. (5.2)).

As  $\beta \neq 0$  everywhere, there exists  $x \in M$  with  $\mathcal{D}_x \neq \mathcal{V}_x$  (or else  $\mathcal{V} = \mathcal{D}$  would be parallel, contradicting (ii)), and we may choose  $v \in \mathcal{D}_x$  such that  $v \notin \mathcal{V}_x = \text{Ker } \zeta_x$ . Thus,  $\zeta_x v \in \mathcal{V}_x \setminus \{0\}$  (by Lemma 5.2(v)), while, according to the last paragraph,  $\zeta_x v \in \mathcal{D}_x^\perp = \mathcal{D}_x$ . Hence  $\dim(\mathcal{V}_x \cap \mathcal{D}_x) = 1$ . Lemma 6.1(b) and (i) now show that  $\mathcal{D}$  is not compatible with the orientation at  $x$ , or, equivalently, at any point, and (iv) follows.  $\square$

Suppose that  $(M, g)$  is a type III self-dual oriented Einstein four-manifold of the neutral metric signature. We call a point  $x \in M$  *generic* if  $\beta_x \neq 0$  or  $\beta = 0$  at all points of some neighborhood of  $x$ . Generic points obviously form a dense open subset of  $M$ . Each connected component of this set represents either the *Walker case* ( $\beta = 0$  identically), or the *strictly non-Walker case* ( $\beta \neq 0$  everywhere). Our terminology is motivated by Theorem 6.2 and the fact that null parallel distributions on pseudo-Riemannian manifolds are described by Walker's classical theorem [13].

It should be pointed out that, in any neutral-signature oriented self-dual four-manifold  $(M, g)$  which is Ricci-flat but not flat, there exists a whole family, diffeomorphic to the circle, of two-dimensional null parallel distributions which are *not* compatible with the orientation. In fact, for such  $(M, g)$ , the Levi-Civita connection in the bundle  $\Lambda^-M$  is well-known to be flat. The distributions in question now arise from nonzero null parallel local sections of  $\Lambda^-M$ , treated as anti-self-dual bivector fields. See [8, p. 638, Proposition 37.1(i) and p. 645, Lemma 37.8].

On the other hand, not all manifolds with the stated properties admit two-dimensional null parallel distributions compatible with the orientation (cf. Section 22).

**Remark 6.3.** In a  $2m$ -dimensional pseudo-Euclidean space  $\mathcal{T}$  of the neutral metric signature ( $m$  pluses,  $m$  minuses), any  $m$ -dimensional null subspace  $\mathcal{N}$  naturally distinguishes an orientation of  $\mathcal{T}$ . Specifically, this is the orientation represented by the basis  $v_1, \dots, v_m, w_1, \dots, w_m$ , where  $v_1, \dots, v_m$  is any basis of  $\mathcal{N}$  and  $w_1, \dots, w_m \in \mathcal{T}$  are any vectors satisfying the inner-product relations  $\langle v_j, w_k \rangle = \delta_{jk}$ . Note that, given  $v_1, \dots, v_m$ , such  $w_1, \dots, w_m$  exist since the functionals  $\mathcal{N} \rightarrow \mathbb{R}$  forming the basis dual to  $v_1, \dots, v_m$  may be extended to  $\mathcal{T}$  and then represented as  $\langle \cdot, w_k \rangle$ ,  $k = 1, \dots, m$ . Also, the  $2m$ -tuple  $v_1, \dots, v_m, w_1, \dots, w_m$  is a basis

since the corresponding Gram matrix of inner products is nonsingular. Finally, the transition matrix between two such bases is easily seen to have a positive determinant.

When  $m = 2$ , the orientation distinguished by  $\mathcal{N}$  in the manner just described coincides with that of Lemma 6.1(a). See [8, p. 638, Proposition 37.1(i)].

### 7. Partial metrics and affine foliations

Let  $\mathcal{E}$  and  $\mathcal{F}$  be real vector bundles over a manifold  $\Sigma$ . By a *pairing* of  $\mathcal{E}$  and  $\mathcal{F}$  we mean any  $C^\infty$  section of  $(\mathcal{E} \otimes \mathcal{F})^*$ . In other words, such a pairing is a  $C^\infty$  assignment of a bilinear mapping  $\mathcal{E}_y \times \mathcal{F}_y \rightarrow \mathbb{R}$  to every  $y \in \Sigma$ , and may also be regarded as a vector-bundle morphism  $\mathcal{E} \rightarrow \mathcal{F}^*$ , or  $\mathcal{F} \rightarrow \mathcal{E}^*$ .

Given an  $m$ -dimensional distribution  $\mathcal{V}$  on a manifold  $M$  of dimension  $2m$ , we define a *partial metric* for  $(M, \mathcal{V})$  to be any pairing  $h$  of the vector bundles  $\mathcal{V}$  and  $TM$  over  $M$  which, treated as a morphism  $TM \rightarrow \mathcal{V}^*$ , has the kernel  $\mathcal{V}$ . (Cf. [6, Sec. IV].) Clearly,

$$(7.1) \quad \text{such } h \text{ amount to (arbitrary) vector-bundle isomorphisms } \mathcal{V} \rightarrow [(TM)/\mathcal{V}]^*.$$

An obvious example of a partial metric for  $(M, \mathcal{V})$  is the restriction  $h$  to  $\mathcal{V}$  and  $TM$  of the pairing of  $TM$  and  $TM$  provided by a pseudo-Riemannian metric  $g$  on  $M$  such that  $\mathcal{V}$  is  $g$ -null. In this case we refer to  $h$  (or,  $g$ ) as the *restriction* of  $g$  to  $\mathcal{V}$  and  $TM$  (or, respectively, a *total-metric extension* of  $h$ ).

By an *affine foliation* on a manifold  $M$  we mean a pair  $(\mathcal{V}, D)$  consisting of an integrable distribution  $\mathcal{V}$  on  $M$  along with a fixed choice of a flat torsionfree connection  $D$  on each leaf of  $\mathcal{V}$  such that, in an obvious sense, the connection depends  $C^\infty$ -differentiably on the leaf. Our notation ignores the dependence of  $D$  on the leaf, and, instead, treats  $D$  as a mapping that sends sections  $u, v$  of  $\mathcal{V}$  to a section  $D_u v$  of  $\mathcal{V}$ . Obviously, for  $(\mathcal{V}, D)$  as above,

$$(7.2) \quad \begin{array}{l} \text{the vector subbundle } \mathcal{V} \text{ of } TM \text{ is locally trivialized by sections of } \mathcal{V} \\ \text{that are } \mathcal{V}\text{-parallel in the sense of being } D\text{-parallel along each leaf of } \mathcal{V}. \end{array}$$

Let  $(\mathcal{V}, D)$  be an affine foliation of dimension  $m$  on a  $2m$ -dimensional manifold  $M$ , and let  $h$  be a partial metric for  $(\mathcal{V}, D)$  (that is, for  $(M, \mathcal{V})$ ). We will say that  $h$  is

- i) *affine* if, for any  $\mathcal{V}$ -parallel section  $v$  of  $\mathcal{V}$  and any  $\mathcal{V}$ -projectable vector field  $w$  (cf. Remark 3.1), both defined on any open subset of  $M$ , the function  $h(v, w)$  restricted to each leaf  $N$  of  $\mathcal{V}$  is locally affine or, equivalently, the 1-form on  $N$  obtained by restricting  $d[h(v, w)]$  to  $N$  is  $D$ -parallel,
- ii) *skew-affine* if  $d_v[h(v, w)] = 0$  for any  $v, w$  as in (i),
- iii) *trivial* if, for any  $v, w$  as in (i),  $h(v, w)$  is locally constant along every leaf of  $\mathcal{V}$ .

For  $(M, \mathcal{V})$  and a partial metric  $h$  as above, being trivial obviously implies being skew-affine, while being skew-affine implies being affine: the last claim is clear since, whenever  $u, v'$  are sections of  $\mathcal{V}$  parallel along  $\mathcal{V}$  and  $v, w$  are as in (i),  $d_u d_v[h(v', w)]$  must vanish due to its simultaneous skew-symmetry in  $v, v'$  and symmetry in  $u, v$ .

An affine foliation  $(\mathcal{V}, D)$  on  $M$  obviously arises when  $M$  is an open submanifold of the total space of an affine bundle over a manifold,  $\mathcal{V}$  is the restriction to  $M$  of the vertical distribution  $\text{Ker } d\pi$ , where  $\pi$  is the bundle projection, and  $D$  is the standard flat torsionfree connection of each fibre. Locally, there are no other examples: any affine foliation  $(\mathcal{V}, D)$  of dimension  $k$  on an  $n$ -dimensional manifold  $M$  is, locally, obtained in the manner just described. In fact, let us fix an  $(n - k)$ -dimensional submanifold  $\Sigma$  of  $M$ , transverse to  $\mathcal{V}$ , and treat it as the

zero section  $\Sigma \subset \mathcal{P}$  in the total space  $\mathcal{P}$  of the vector bundle over  $\Sigma$  which is the restriction of  $\mathcal{V}$  to  $\Sigma$ . Then, at any point  $y$  of the zero section  $\Sigma$ , the exponential mapping of  $\mathbb{D}$  sends a neighborhood of  $y$  in  $\mathcal{P}$  diffeomorphically onto an open set in  $M$ , in such a way that the vertical distribution in  $\mathcal{P}$  corresponds to  $\mathcal{V}$ .

The total space  $M = T^*\Sigma$  of the cotangent bundle of any manifold  $\Sigma$  carries a trivial partial metric  $h$ , for  $(\mathcal{V}, \mathbb{D})$  defined as in the last paragraph, obtained by setting  $h_x(\xi, w) = \xi(d\pi_x w)$  for any  $x \in T^*\Sigma = M$ , any vertical vector  $\xi \in \text{Ker } d\pi_x = T_y^*\Sigma$ , with  $y = \pi(x)$ , and any  $w \in T_x M$ , where  $\pi : M \rightarrow \Sigma$  is the bundle projection, cf. [12]. Again, these are, locally, the only examples: for any trivial partial metric  $h$  for  $(\mathcal{V}, \mathbb{D})$  on a manifold  $M$ , treating the leaves of  $\mathcal{V}$ , locally, as the fibres of a bundle projection  $\pi : M \rightarrow \Sigma$ , we obtain a natural bijective correspondence between  $\mathcal{V}$ -parallel sections  $v$  of  $\mathcal{V}$  and 1-forms  $\xi$  on  $\Sigma$ , given by  $\xi(d\pi w) = h(v, w)$ , where  $w$  is any  $\mathcal{V}$ -projectable local vector field in  $M$ , and  $d\pi w$  denotes its  $\pi$ -image in  $\Sigma$ . Thus,  $M$  can be identified, locally, with the total space of an affine bundle over  $\Sigma$ , the associated vector bundle of which is  $T^*\Sigma$ . The required local identification of  $M$  with  $T^*\Sigma$  may now be obtained by choosing an  $(n - k)$ -dimensional submanifold  $\Sigma$  of  $M$ , transverse to  $\mathcal{V}$ , as in the last paragraph.

The partial metrics that naturally appear in the geometric situation discussed in this paper are skew-affine, though not trivial.

## 8. Basic octuples

By a *basic octuple* we mean a system  $(M, \mathcal{V}, \mathbb{D}, h, \alpha, \beta, \theta, \zeta)$  formed by a skew-affine partial metric  $h$  for a two-dimensional affine foliation  $(\mathcal{V}, \mathbb{D})$  on a manifold  $M$  of dimension four, along with sections  $\alpha, \beta, \zeta, \theta$  of  $\mathcal{V}^*, T^*M, [T^*M]^{\wedge 2}$  and, respectively,  $[\mathcal{V}^*]^{\wedge 2}$ , such that  $\beta \neq 0$  everywhere,  $\text{rank } \zeta = 2$  everywhere, and

$$(8.1) \quad \begin{array}{ll} \text{a) } d_u[h(v, w)] = \beta(w)\theta(u, v), & \text{b) } d_u[\alpha(v)] = \alpha(u)\alpha(v), \\ \text{c) } d_u[\zeta(w, w')] = \alpha(u)\zeta(w, w'), & \text{d) } d_u[\beta(w)] = 2\alpha(u)\beta(w), \\ \text{e) } d_u[\theta(v, v')] = -2\alpha(u)\theta(v, v'), & \text{f) } \theta(v, \zeta w) = 2h(v, w), \\ \text{g) } \alpha(\zeta w) = 2\beta(w), & \text{h) } \mathcal{V} = \text{Ker } \zeta \end{array}$$

for any  $\mathcal{V}$ -parallel sections  $u, v, v'$  of  $\mathcal{V}$  (see Section 7) and  $\mathcal{V}$ -projectable local vector fields  $w, w'$  in  $M$  (cf. Remark 3.1). Unlike the 1-form  $\beta$  and the 2-form  $\zeta$  on  $M$ , the objects  $\alpha$  and  $\theta$  in a basic octuple are only “partial” differential forms:  $\alpha(v)$  and  $\theta(v, v')$  are not defined unless  $v$  and  $v'$  are sections of  $\mathcal{V}$ .

In (8.1.f),  $\zeta w$  denotes the unique section of  $\mathcal{V}$  with  $h(\zeta w, w') = \zeta(w, w')$  for all vector fields  $w'$ , the existence and uniqueness of  $\zeta w$  being clear from (7.1). By (8.1.h) and (8.1.g),

$$(8.2) \quad \text{i) } \mathcal{V} = \text{Im } \zeta, \quad \text{ii) } \mathcal{V} = \text{Ker } \zeta \subset \text{Ker } \beta,$$

where (i) expresses surjectivity of  $\zeta$  treated as a morphism  $TM \rightarrow \mathcal{V}$  acting by  $w \mapsto \zeta w$ . For sections  $u, v$  of  $\mathcal{V}$  and vector fields  $w, w'$ ,

$$(8.3) \quad \begin{array}{l} \text{a) } 2\beta(w)h(u, w') - 2\beta(w')h(u, w) = \alpha(u)\zeta(w, w'), \\ \text{b) } \theta(u, v)\zeta w = 2[h(u, w)v - h(v, w)u]. \end{array}$$

Namely, we obtain (8.3.a) (or, (8.3.b)) by first selecting a vector field  $w''$  with  $u = \zeta w''$  (or, vector fields  $w, w'$  with  $u = \zeta w, v = \zeta w'$ ), then using (8.1.g) (or, (8.1.f)) to replace  $\alpha(u) = \alpha(\zeta w'')$  by  $2\beta(w'')$  (or,  $\theta(u, v) = \theta(\zeta w, \zeta w')$  by  $2h(\zeta w, w') = 2\zeta(w, w')$ ), and, finally, applying Remark 3.4, at any  $x \in M$ , to  $\Pi = T_x M / \mathcal{V}_x$ , cf. (8.2.ii).

**Remark 8.1.** For a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , a section  $u$  of  $\mathcal{V}$ , a  $\mathcal{V}$ -projectable local vector field  $w$  in  $M$ , and  $\zeta w$  as above, we have  $D_u(\zeta w) = 2\alpha(u)\zeta w - 2\beta(w)u$ .

In fact, let  $v = D_u(\zeta w) - 2\alpha(u)\zeta w + 2\beta(w)u$ . That  $v = 0$  will clearly follow from (7.1) once we show that  $h(v, w') = 0$  for every  $\mathcal{V}$ -projectable local vector field  $w'$  in  $M$ . To this end, note that (8.1.c), (8.1.a) and the Leibniz rule give  $\alpha(u)\zeta(w, w') = d_u[h(\zeta w, w')] = \beta(w')\theta(u, \zeta w) + h(D_u(\zeta w), w')$ , and so, by (8.1.f),  $h(D_u(\zeta w), w') = \alpha(u)\zeta(w, w') - 2\beta(w')h(u, w)$ . Thus,  $h(v, w') = -\alpha(u)\zeta(w, w') - 2\beta(w')h(u, w) + 2\beta(w)h(u, w')$ , which vanishes in view of (8.3.a).

**Remark 8.2.** The conclusion in Remark 8.1 was obtained without using condition (8.1.b).

**Remark 8.3.** If  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  is a basic octuple, then  $\alpha = \theta(\bar{u}, \cdot)$  on  $\mathcal{V}$ , where  $\bar{u}$  is the unique section of  $\mathcal{V}$  with  $h(\bar{u}, \cdot) = \beta$ . (Its existence and uniqueness are immediate from (7.1), since (8.2.ii) allows us to treat  $\beta$  as a section of  $[(TM)/\mathcal{V}]^*$ .) Namely, writing an arbitrary section of  $\mathcal{V}$  as  $\zeta w$ , which is allowed in view of (8.2.i), we see that, by (8.1.g) and (8.1.f),  $\alpha(\zeta w) = 2\beta(w) = 2h(\bar{u}, w) = \theta(\bar{u}, \zeta w)$ .

Our interest in basic octuples is due to the fact that they naturally arise in the strictly non-Walker case of our geometric situation. Specifically, we have the following result.

**Theorem 8.4.** *Given a neutral-signature oriented self-dual Einstein four-manifold  $(M, g)$  of Petrov type III, let us define  $\alpha, \beta, \zeta, \theta, \mathcal{V}$  as in Lemma 5.2, denote by  $D$  the restriction of the Levi-Civita connection of  $g$  to the leaves of  $\mathcal{V}$ , and declare  $h$  to be the partial metric for  $(M, \mathcal{V})$  obtained by restricting  $g$  to  $\mathcal{V}$  and  $TM$ . If  $\beta \neq 0$  everywhere in  $M$ , then  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  is a basic octuple.*

**Proof.** According to Lemma 5.2(i),  $(\mathcal{V}, D)$  is an affine foliation, while the partial metric  $h$  for  $(\mathcal{V}, D)$  is skew-affine in view of skew-symmetry of  $\theta$  and the equality  $d_u[g(v, w)] = \beta(w)\theta(u, v)$  in Lemma 5.2(iii), which also yields (8.1.a). Next, (8.1.e) follows from the third equality in Lemma 5.1(c) with  $\Psi = W^+$ , as  $\nabla_u v = \nabla_u v' = 0$ . (Note that (5.2) gives  $\eta v = v$  and  $\eta(v, v') = g(v, v') = 0$ , since  $\mathcal{V}$  is null by Lemma 5.2(v).) Conditions (8.1.h) and (8.1.g) are in turn immediate from Lemma 5.2(v) and, respectively, the equality  $2\eta\beta + \zeta\alpha = 0$  in Lemma 5.1(f) with  $\Psi = W^+$ , as the vector field corresponding to  $\beta$  via  $g$  is a section of  $\mathcal{V}$  (see Lemmas 5.1(e) and 5.2(v)), so that  $\eta\beta = \beta$  by (5.2).

On the other hand, (5.1.ii) for  $\Psi = W^+$  and (3.11) imply that  $(\zeta\theta + \theta\zeta)/2 = -\text{Id}$ . Also, by Lemma 5.2(v),  $\zeta$  vanishes on  $\mathcal{V}$ , while  $\theta$  vanishes on  $\mathcal{H}$  and maps  $\mathcal{V}$  onto  $\mathcal{H}$ . The last equality now shows that the composite  $\theta\zeta$ , treated as a morphism  $TM \rightarrow TM$ , equals  $-2$  times  $\text{Id}$  on  $\mathcal{H}$ , and  $0$  on  $\mathcal{V}$ , which, combined with skew-symmetry of  $\theta$ , yields (8.1.f).

Furthermore, using the first equality in Lemma 5.1(c) with  $\Psi = W^+$  and the Leibniz rule, we see that the left-hand side in (8.1.c) equals  $\alpha(u)\zeta(w, w')$  plus  $\alpha(u)\zeta(w, w') + \zeta(\nabla_u w, w') + \zeta(w, \nabla_u w')$ , since  $\beta(u) = 0$  by Lemma 5.1(d). Replacing  $\nabla_u w$  with  $[\nabla_u w]^{\mathcal{H}} = [\nabla_w u]^{\mathcal{H}}$  (cf. (8.1.h) and Remark 3.1), and using Lemma 5.2(vi), and setting  $w'' = \theta u$ , we can rewrite this last expression as  $\alpha(u)\zeta(w, w') + \beta(w)\zeta(w'', w')\beta(w')\zeta(w, w'')$ , which vanishes in view of Remark 3.4. (As in the last paragraph, we see that the morphism  $\zeta\theta : TM \rightarrow TM$  equals  $-2$  times  $\text{Id}$  on  $\mathcal{V}$ , and so, by (8.1.g),  $\alpha(u) = -\alpha(\zeta w'')/2 = -\beta(w'')$ .)

Next, for  $u, w$  as in (8.1.d),  $\beta(u) = \beta([u, w]) = 0$  and  $\zeta u = 0$  by Lemma 5.1(d), Remark 3.1 and (8.1.h), so that, evaluating  $(d\beta + 2\beta \wedge \alpha)(u, w)$  from (3.1) and then using the first equality in Lemma 5.2(iv), we get (8.1.d).

Finally, by (7.2), equality (8.1.b) amounts to the relation  $[D_u \alpha](v) = \alpha(u)\alpha(v)$  for all sections  $u, v$  of  $\mathcal{V}$ . Thus, since  $\mathcal{V} = \text{Ker } \zeta = \text{Im } \zeta$  (see Lemma 5.2(v)), (8.1.b) will follow if we

prove the latter relation for  $v = \zeta w$ , where  $w$  is any  $\mathcal{V}$ -projectable local vector field. However, as  $[D_u \alpha](\zeta w) = d_u[\alpha(\zeta w)] - \alpha(D_u(\zeta w))$ , while  $D_u(\zeta w)$  may be replaced by  $2\alpha(u)\zeta w - 2\beta(w)u$  (which is allowed according to Remark 8.2), (8.1.b) is immediate from (8.1.d) and (8.1.g).  $\square$

Our discussion of basic octuples can be simplified as follows. Let us assume, for the remainder of this section, that  $h$  is a skew-affine partial metric for a two-dimensional affine foliation  $(\mathcal{V}, D)$  on a manifold  $M$  of dimension four.

Given  $\alpha, \beta, \zeta, \theta$  such that  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  is a basic octuple, we may choose, locally in  $M$ , a positive function  $\phi$  with  $\alpha(v) = -d_v \log \phi$  for all sections  $v$  of  $\mathcal{V}$ . (In fact, by (3.1) and (8.1.b), the restriction of  $\alpha$  to every leaf of  $\mathcal{V}$  is a closed 1-form.) Rephrased in terms of  $\phi$ , (8.1.b) states that  $d_u d_v \phi = 0$ , and hence

(i) the restriction of  $\phi$  to each leaf of  $\mathcal{V}$  is a nonconstant positive affine function.

Note that  $\phi$  is nonconstant since  $\beta \neq 0$  everywhere, and so  $\alpha \neq 0$  everywhere by (8.1.g).

If one now sets  $\hat{\alpha} = \phi\alpha$ ,  $\hat{\beta} = \phi^2\beta$ ,  $\hat{\zeta} = \phi\zeta$  and  $\hat{\theta} = \phi^{-2}\theta$ , then, according to (8.1),

(ii)  $\hat{\beta} \neq 0$  everywhere,  $\text{rank } \hat{\zeta} = 2$  everywhere, and  $\text{Ker } \hat{\zeta} = \mathcal{V}$ ,

(iii)  $\hat{\alpha}(u) = -d_u \phi$ ,  $\hat{\theta}(v, \hat{\zeta}w) = 2\phi^{-1}h(v, w)$ ,  $\hat{\alpha}(\hat{\zeta}w) = 2\hat{\beta}(w)$  and  $d_u[h(v, w)] = \hat{\beta}(w)\hat{\theta}(u, v)$ , while  $\hat{\beta}(w)$ ,  $\hat{\theta}(v, v')$  and  $\hat{\zeta}(w, w')$  are constant along  $\mathcal{V}$ , for any D-parallel sections  $u, v, v'$  of  $\mathcal{V}$  and  $\mathcal{V}$ -projectable local vector fields  $w, w'$  in  $M$ .

Conversely, if sections  $\hat{\alpha}, \hat{\beta}, \hat{\zeta}, \hat{\theta}$  of  $\mathcal{V}^*, T^*M, [T^*M]^{\wedge 2}, [\mathcal{V}^*]^{\wedge 2}$  and a function  $\phi : M \rightarrow \mathbb{R}$  satisfy (i) – (iii), a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  can clearly be defined by setting

$$(8.4) \quad \alpha = \phi^{-1}\hat{\alpha}, \quad \beta = \phi^{-2}\hat{\beta}, \quad \zeta = \phi^{-1}\hat{\zeta}, \quad \theta = \phi^2\hat{\theta}.$$

## 9. Two-plane systems

By a *two-plane system* we mean a sextuple  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  consisting of a real affine space  $\Sigma$  and a real vector space  $\Pi$  with  $\dim \Sigma = \dim \Pi = 2$ , two linearly independent constant 1-forms  $\xi, \tau$  on  $\Sigma$ , a nonzero constant vector field  $c$  on  $\Pi$ , and a nonzero constant 2-form  $\Omega$  on  $\Pi$ . (In other words,  $c \in \Pi \setminus \{0\}$  and  $\Omega \in [\Pi^*]^{\wedge 2} \setminus \{0\}$ .)

Any two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  gives rise to a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  defined as follows. Let  $\Pi_+ \subset \Pi$  be the open set on which  $\Omega(\cdot, c) > 0$ . Thus,  $\Pi_+$  is a connected component of  $\Pi \setminus \Lambda$ , for the line  $\Lambda = \mathbb{R}c$  spanned by  $c$  in  $\Pi$ . On the four-dimensional product manifold  $M = \Sigma \times \Pi_+$  one has the two-dimensional affine foliation  $(\mathcal{V}, D)$  formed by the distribution  $\mathcal{V}$  tangent to the  $\Pi_+$  factor and the standard flat torsionfree connection  $D$  on each leaf of  $\mathcal{V}$ , the leaf being identified with the open set  $\Pi_+$  in the plane  $\Pi$ . Next, we denote by  $X$  the *radial vector field* on  $\Pi$ , that is, the identity mapping  $\Pi \rightarrow \Pi$  treated as a vector field on  $\Pi$ . Vector fields on the factor manifolds  $\Sigma$  and  $\Pi_+$ , including constant fields  $v$  (such as  $v = c$ ) and the radial field  $X$  on  $\Pi$  (restricted to  $\Pi_+$ ), and all vector fields  $w$  on  $\Sigma$ , will also be treated as vector fields on  $M = \Sigma \times \Pi_+$ , tangent to the factor distributions. Similarly, we will use the same symbols for differential forms on  $\Sigma$  and  $\Pi_+$  as for their pullbacks to  $M$ . Thus,  $\xi$  and  $\tau$  are now 1-forms on  $M$ , and  $\Omega$  is a 2-form on  $M$ . Using these conventions, we declare  $h$  to be the partial metric for  $(M, \mathcal{V})$  such that, for all vector fields  $v, u$  on  $\Pi_+$  and  $w$  on  $\Sigma$ , treated as vector fields on  $M$ ,

$$(9.1) \quad \text{a) } h(v, u) = 0, \quad \text{b) } h(v, w) = \Omega(Y_w, v), \quad \text{where c) } Y_w = \xi(w)X + \tau(w)c,$$

$X$  being the radial field on  $\Pi$ . Thus, for  $u, v, w$  as above,  $d_u[h(v, w)] = \xi(w)\Omega(u, v)$ . Skew-symmetry of  $\Omega$  now implies that  $h$  is a skew-affine partial metric for  $(\mathcal{V}, D)$ .

Conditions (i) – (iii) at the end of Section 8 are in turn satisfied if one sets

$$(9.2) \quad \phi = \Omega(X, c), \quad \hat{\alpha} = -d\phi \text{ (on } \mathcal{V}), \quad \hat{\beta} = \xi, \quad \hat{\zeta} = 2\xi \wedge \tau, \quad \hat{\theta} = \Omega \text{ (on } \mathcal{V}).$$

In fact, (i) – (iii) follow since, for vector fields  $v$  on  $\Pi_+$  and  $w$  on  $\Sigma$ , one clearly has

$$(9.3) \quad h(c, w) = \xi(w)\phi, \quad h(X, w) = -\tau(w)\phi, \quad \hat{\zeta}w = -2\phi^{-1}Y_w, \quad \hat{\alpha}(v) = \Omega(c, v),$$

with  $\hat{\zeta}w$  defined as in the lines following (8.1).

Consequently, formula (8.4) defines a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , naturally associated with  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$ .

We will use the following well-known lemma to prove Theorem 9.2, stating that all basic octuples represent just one local diffeomorphic type.

**Lemma 9.1.** *Given a manifold  $\Sigma$  of dimension  $m$ , a point  $y \in \Sigma$ , a differential  $m$ -form  $\nu$  on  $\Sigma$ , and closed 1-forms  $\xi^1, \dots, \xi^{m-1}$  which are linearly independent at  $y$ , there exists a closed 1-form  $\tau$  on a neighborhood  $U$  of  $y$  such that  $\nu = \xi^1 \wedge \dots \wedge \xi^{m-1} \wedge \tau$  on  $U$ .*

Namely, choosing a closed 1-form  $\xi^m$  on a neighborhood of  $y$  so that  $\xi^1, \dots, \xi^m$  are linearly independent at  $y$ , we have  $\xi^j = dy^j$  for some local coordinates  $y^j$  at  $y$  and  $j = 1, \dots, m$ , so that we may set  $\tau = d\chi$ , where  $\chi$  is a function defined near  $y$  with the partial derivative  $\psi = \partial\chi/\partial y^m$  characterized by  $\nu = \psi\xi^1 \wedge \dots \wedge \xi^m$ .

**Theorem 9.2.** *All basic octuples, at all points in their underlying four-manifolds, represent one single type of local diffeomorphic equivalence.*

*In other words, if  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  is a fixed two-plane system, then every basic octuple is locally diffeomorphically equivalent to the basic octuple associated with  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$ .*

**Proof.** Given a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , let us choose  $\phi$  and the corresponding objects  $\hat{\alpha}, \hat{\beta}, \hat{\zeta}, \hat{\theta}$  as at the end of Section 8. A neighborhood of any given point of  $M$  may, clearly, be diffeomorphically identified with an open subset of the total space of a real affine-plane bundle  $\mathcal{A}$  over a surface  $\Sigma$  in such a way that  $\mathcal{V}$  and  $D$  become the vertical distribution and the standard flat torsionfree connection in each fibre of  $\mathcal{A}$  (treated as an open set in an affine plane). Since  $\phi$ , restricted to each fibre of  $\mathcal{A}$ , is a nonconstant affine function, its zero set is the total space of a real affine-line subbundle  $\mathcal{J}$  of  $\mathcal{A}$ . Conditions (i) – (iii) at the end of Section 8 will remain unaffected if we multiply  $\phi$  by any positive function  $\Sigma \rightarrow \mathbb{R}$  (pulled back to the total space  $\mathcal{A}$ ), at the same time multiplying  $\hat{\alpha}, \hat{\beta}, \hat{\zeta}, \hat{\theta}$  by its appropriate powers. A suitable choice of such a positive function allows us to assume that  $\hat{\beta}$  is closed. (Note that  $\hat{\beta}$  and  $\hat{\zeta}$  are the pullbacks to  $\mathcal{A}$  of a 1-form and a 2-form without zeros on the surface  $\Sigma$ .) Setting  $\xi = \hat{\beta}$ , we may now use Lemma 9.1 (with  $m = 2$ ,  $\nu = \hat{\zeta}/2$  and  $\xi^1 = \xi$ ) to select, locally, a closed 1-form  $\tau$  in  $\Sigma$  with  $\hat{\zeta} = 2\xi \wedge \tau$ .

Let us also set  $\Omega = \hat{\theta}$  and  $c = \phi^3\bar{u}$ , for the section  $\bar{u}$  of  $\mathcal{V}$  defined in Remark 8.3. Then, on  $\mathcal{V}$ , we have  $d\phi = \Omega(\cdot, c)$  (in other words,  $d_v\phi = \Omega(v, c)$  for every section  $v$  of  $\mathcal{V}$ ). This is clear since, on  $\mathcal{V}$ , (iii) in Section 8 and Remark 8.3 give  $d\phi = -\hat{\alpha} = -\phi\alpha = \phi^{-2}\theta(\cdot, c) = \hat{\theta}(\cdot, c) = \Omega(\cdot, c)$ . On the other hand, as  $h(\bar{u}, \cdot) = \beta$  (see Remark 8.3), it follows that  $h(c, \cdot) = \phi\xi$ , and, consequently,  $d_v[h(c, w)] = \xi(w)d_v\phi = \xi(w)\Omega(v, c)$  for every  $\mathcal{V}$ -projectable local vector field  $w$  in  $\mathcal{A}$  and every section  $v$  of  $\mathcal{V}$ . At the same time, (8.1.a) and the Leibniz rule give  $d_v[h(c, w)] - h(D_v c, w) = \beta(w)\theta(v, c) = \xi(w)\Omega(v, c)$ , so that  $D_v c = 0$ . Hence  $c$ , restricted to each fibre of  $\mathcal{A}$  (which is an open subset of an affine plane), is a nonzero constant vector field.

Any fixed local section  $z$  of  $\mathcal{J}$  gives rise to the section  $\tilde{X}$  of the vertical distribution  $\mathcal{V}$  on  $\mathcal{A}$ , with the value at  $x \in \mathcal{A}$  equal to  $x - z_{\pi(x)}$ , where  $\pi$  is the bundle projection. (Thus,  $\tilde{X}$  restricted to the fibre of  $\mathcal{A}$  containing  $x$  is the radial vector field relative to the origin  $z_{\pi(x)}$ .) Then  $\phi = \Omega(\tilde{X}, c)$ . In fact, as we saw above,  $d\phi = \Omega(\cdot, c)$  on  $\mathcal{V}$ , so that  $\phi$  and  $\Omega(\tilde{X}, c)$  have the same  $d_v$ -derivative for any section  $v$  of  $\mathcal{V}$ , and, consequently, differ in each fibre of  $\mathcal{A}$  by a constant, while, due to our choice of  $z$ , they both vanish at the origin  $z_{\pi(x)}$  in the fibre containing  $x$ . Defining a 1-form  $\tilde{\tau}$  on the total space  $\mathcal{A}$  by  $\tilde{\tau} = -\phi^{-1}h(\tilde{X}, \cdot)$ , we in turn obtain  $h(v, w) = \Omega(\tilde{Y}_w, v)$ , with  $\tilde{Y}_w = \xi(w)\tilde{X} + \tilde{\tau}(w)c$ , for all sections  $v$  of  $\mathcal{V}$  and all vector fields  $w$ . Namely, since  $c$  and  $\tilde{X}$  span  $\mathcal{V}$  away from  $\mathcal{J}$ , it suffices to consider the cases  $v = c$  and  $v = \tilde{X}$ , in which the required equality follows since  $h(c, w) = \xi(w)\phi = \xi(w)\Omega(\tilde{X}, c)$  (as we saw earlier) and  $h(\tilde{X}, w) = -\tilde{\tau}(w)\phi = \tilde{\tau}(w)\Omega(c, \tilde{X})$  (by the definition of  $\tilde{\tau}$ ). Suppose now that  $w$  is a  $\mathcal{V}$ -projectable local vector field in  $\mathcal{A}$  and  $v$  is a section of  $\mathcal{V}$ . By (8.1.a),  $d_v[h(u, w)] = \xi(w)\Omega(v, u)$  if  $u$  is a D-parallel section of  $\mathcal{V}$ , while, as shown above,  $d\phi = \Omega(\cdot, c)$  on  $\mathcal{V}$ . Thus, the Leibniz rule yields  $d_v[\tilde{\tau}(w)] = -d_v[\phi^{-1}h(\tilde{X}, w)] = -\phi^{-1}\xi(w)\Omega(v, \tilde{X}) - \phi^{-1}h(v, w) + \phi^{-2}\Omega(v, c)h(\tilde{X}, w)$ . Since  $h(v, w) = \Omega(\tilde{Y}_w, v)$ , we get  $d_v[\tilde{\tau}(w)] = 0$ , and so  $\tilde{\tau}$  is the pullback to  $\mathcal{A}$  of a 1-form in  $\Sigma$ . Furthermore,  $\hat{\zeta} = 2\xi \wedge \tilde{\tau}$  as a consequence of (8.3.a) with  $\xi = \hat{\beta} = \phi^2\beta$ , the definition of  $\tilde{\tau}$ , (3.1.ii), and the relation  $\alpha(\tilde{X}) = -1$  (immediate since, on  $\mathcal{V}$ , we have  $d\phi = -\hat{\alpha} = -\phi\alpha$  and  $d\phi = \Omega(\cdot, c)$ , while  $\phi = \Omega(\tilde{X}, c)$ ). As  $\hat{\zeta} = 2\xi \wedge \tau$  for the closed 1-form  $\tau$  selected earlier, there exists a function  $\psi$  in  $\Sigma$  with  $\tau = \tilde{\tau} + \psi\xi$ . Replacing  $z$  by  $z - \psi c$  causes  $\tilde{X}$  and  $\tilde{\tau}$  to be replaced by  $\tilde{X} + \psi c$  and, respectively, by  $\tau$ . With  $z - \psi c$  (the new choice of  $z$ ) declared the zero section, our affine-plane bundle  $\mathcal{A}$  may be treated as a vector bundle  $\mathcal{P}$ , in such a way that  $c$  and  $\Omega$  are sections, both without zeros, of  $\mathcal{P}$  and  $[\mathcal{P}^*]^{\wedge 2}$ . Choosing, locally, a section  $a$  of  $\mathcal{P}$  with  $\Omega(a, c) = 1$ , we obtain a local trivialization  $a, c$  of  $\mathcal{P}$ , which allows us to view  $\mathcal{P}$  as a product bundle of the form  $\Sigma \times \Pi$ . Sections of  $\mathcal{P}$  now become functions on  $\Sigma$  valued in the vector space  $\Pi$ , with  $c$  corresponding in this way to a constant function (an element of  $\Pi$ ). Finally,  $\Sigma$  may be identified, locally, with the space  $\mathbb{R}^2$  so as to make  $\xi$  and  $\tau$  correspond to  $dy^1$  and  $dy^2$ , for the standard coordinates  $y^j$  in  $\mathbb{R}^2$ . The resulting two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  clearly has  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  as its associated basic octuple.  $\square$

## 10. Horizontal distributions

By a *horizontal distribution* for a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , cf. Section 8, we mean a vector subbundle  $\mathcal{H}$  of  $TM$  with  $TM = \mathcal{H} \oplus \mathcal{V}$ .

Any such  $\mathcal{H}$  gives rise to a neutral-signature pseudo-Riemannian metric  $g$  on  $M$ . Namely,

$$(10.1) \quad g \text{ is the unique total-metric extension of } h \text{ such that } \mathcal{H} \text{ is } g\text{-null.}$$

We denote by  $\nabla$  the Levi-Civita connection of  $g$ , and by  $\gamma$  the 1-form on  $M$  with

$$(10.2) \quad g(\nabla_w w', w'') = -\gamma(w)\zeta(w', w'') \text{ for all vector fields } w \text{ and sections } w', w'' \text{ of } \mathcal{H},$$

$\gamma$  being well defined since  $\zeta$  trivializes  $\mathcal{H}^{\wedge 2}$ , while skew-symmetry of  $g(\nabla_w w', w'')$  in  $w', w''$  implies its *valuewise* dependence on  $w, w'$  and  $w''$ . If  $v$  is a section of  $\mathcal{V}$ , we let  $\theta v$  stand for the unique section of  $\mathcal{H}$  such that  $g(\theta v, u) = \theta(v, u)$  whenever  $u$  is a section of  $\mathcal{V}$ .



Next, we denote by  $R$  the curvature tensor of  $g$ , and by  $\eta$  the 2-form satisfying (5.2) with our  $\mathcal{V}, \mathcal{H}$  and  $g$ , so that, for sections  $v, v'$  of  $\mathcal{V}$  and  $w, w'$  of  $\mathcal{H}$ ,

$$(10.3) \quad \eta(v, v') = \eta(w, w') = 0, \quad \eta(v, w) = -\eta(w, v) = h(v, w).$$

The symbol  $\bar{w}$  will be used for the unique section of  $\mathcal{H}$  with

$$(10.4) \quad h(v, \bar{w}) = \alpha(v) \text{ for all sections } v \text{ of } \mathcal{V}.$$

That (10.4) defines  $\bar{w}$  uniquely is clear from (10.1), since  $TM = \mathcal{H} \oplus \mathcal{V}$ . For this  $\bar{w}$ ,

$$(10.5) \quad \text{i) } \zeta(w, \bar{w}) = \alpha(\zeta w) = 2\beta(w), \quad \text{ii) } \beta(\bar{w}) = 0,$$

where  $w$  in (i) is an arbitrary vector field. In fact, (10.4) and (8.1.g) yield (10.5.i), while (10.5.ii) follows from (10.5.i) along with skew-symmetry of  $\zeta$ .

Three further objects associated with a horizontal distribution  $\mathcal{H}$  for  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  are extensions of the ‘‘partial’’ differential forms  $\alpha, \theta$  (see Section 8) to differential forms on  $M$ , still written as  $\alpha, \theta$ , which are given by  $\alpha(w) = 2\gamma(\zeta w)$  and  $\theta(w, \cdot) = 0$  for sections  $w$  of  $\mathcal{H}$ , with  $\gamma$  and  $\zeta w$  as in (10.2) and (8.1.f), and a 1-form  $Z$  on  $M$  characterized by  $Z(w)\zeta = \nabla_w \zeta - 2\alpha(w)\zeta$  on  $\mathcal{H}$ , that is,  $Z(w)\zeta(w', w'') = 2[\nabla_w \zeta](w', w'') - 4\alpha(w)\zeta(w', w'')$  for any sections  $w', w''$  of  $\mathcal{H}$ , and any vector field  $w$ . Note that we thus have  $\mathcal{H} = \text{Ker } \theta$ , and  $Z$  is well defined, since  $\zeta$  trivializes  $\mathcal{H}^2$ .

For simplicity, our notation ignores the dependence of  $g, \nabla, \gamma, \theta v, R, \eta, \bar{w}, \alpha, \theta$  and  $Z$  on  $\mathcal{H}$ .

**Lemma 10.1.** *If  $\mathcal{H}$  is a horizontal distribution for a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , and  $\bar{w}$  is the section of  $\mathcal{H}$  defined by (10.4), while  $\phi$  is chosen as at the end of Section 8, then the vector field  $\phi\bar{w}$  is  $\mathcal{V}$ -projectable.*

**Proof.** Let  $w$  be a  $\mathcal{V}$ -projectable local section of  $\mathcal{H}$ , chosen so as to agree with  $\phi\bar{w}$  at all points of a given surface  $\Sigma'$  embedded in  $M$  and transverse to  $\mathcal{V}$ . Since  $\beta(w) = 0$  on  $\Sigma'$  by (10.5.ii), relation (8.1.d) combined with uniqueness of solutions for first-order linear ordinary differential equations gives  $\beta(w) = 0$  on the union  $U$  of all leaves of  $\mathcal{V}$  that intersect  $\Sigma'$ . However,  $\beta \neq 0$  everywhere, and so  $\mathcal{H} \cap \text{Ker } \beta$  is spanned by  $\bar{w}$ , cf. (10.5.ii) and (8.2.ii). Thus,  $w = \chi\phi\bar{w}$  on  $U$ , where  $\chi$  is some function without zeros. For any  $\mathcal{V}$ -projectable local section  $w'$  of  $\mathcal{H}$ , (iii) in Section 8 implies that  $\phi\zeta(w, w')$  and  $\phi^2\beta(w')$  are constant along  $\mathcal{V}$ , while, by (10.5.i),  $\phi\zeta(w, w') = \chi\phi^2\zeta(\bar{w}, w') = -2\chi\phi^2\beta(w')$ . Hence  $\chi$  is constant along  $\mathcal{V}$ , and  $\phi\bar{w} = \chi^{-1}w$  is  $\mathcal{V}$ -projectable.  $\square$

## 11. Properties of the associated metric

Let  $\mathcal{H}$  be a horizontal distribution for a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ . We have

$$(11.1) \quad \text{a) } \nabla_v u = D_v u, \quad \text{b) } \nabla_v w = \beta(w)\theta v - \gamma(v)\zeta w,$$

whenever  $w$  is a  $\mathcal{V}$ -projectable local section of  $\mathcal{H}$  and  $u, v$  are sections of  $\mathcal{V}$ . (As before,  $\zeta w$  denotes the section of  $\mathcal{V}$  appearing in (8.1.f), and  $g, \nabla, \theta v, \gamma$  are defined as in Section 10.) In fact, (11.1.a) follows since, by (3.3),  $\nabla_v u = 0$  for sections  $u, v$  of  $\mathcal{V}$  which are  $\mathcal{V}$ -parallel. Namely,  $g(\nabla_v u, w) = 0$  both when  $w$  is a section of  $\mathcal{V}$  (all six terms resulting from (3.3) then vanish as  $\mathcal{V}$  is integrable and  $g$ -null), and when  $w$  is a  $\mathcal{V}$ -projectable section of  $\mathcal{H}$  (the last four terms in (3.3) vanish, again, according to Remark 3.1, while the sum of the first two is zero in view of (8.1.a) and skew-symmetry of  $\theta$ ). Similarly, to obtain (11.1.b), we take the  $g$ -inner product of both sides with any section of  $\mathcal{H}$ , or, respectively, with any  $\mathcal{V}$ -parallel section  $u$

of  $\mathcal{V}$  (assuming  $v$  to be  $\mathcal{V}$ -parallel as well): in the former case the equality is obvious from (10.2); in the latter, as  $\mathcal{V}$  is  $g$ -null, the right-hand side yields  $\beta(w)\theta(v, u)$ , which, by (8.1.a), is the same as  $g(\nabla_v w, u) = d_v[g(w, u)] = d_v[h(u, w)]$ .

Also, for sections  $v$  of  $\mathcal{V}$  and  $w, w'$  of  $\mathcal{H}$ , with  $\theta v$  and  $Z$  as in Section 10,

$$(11.2) \quad \text{i) } \zeta\theta v = -2v, \quad \text{ii) } \theta\zeta w = -2w, \quad \text{iii) } Z(v) = 0, \quad \text{iv) } [\nabla_w w']^{\mathcal{V}} = -\gamma(w)\zeta w',$$

$[\ ]^{\mathcal{V}}$  denoting the  $\mathcal{V}$ -component relative to the decomposition  $TM = \mathcal{H} \oplus \mathcal{V}$ . Namely, for such  $v$  and  $w$ , (8.1.f) gives  $-g(\zeta\theta v, w) = g(\zeta w, \theta v) = \theta(v, \zeta w) = 2g(v, w)$  and, similarly,  $-g(\theta\zeta w, v) = \theta(v, \zeta w) = 2g(w, v)$ , so that (11.2.i) and (11.2.ii) follow as  $\mathcal{V}$  and  $\mathcal{H}$  are  $g$ -null. On the other hand, in view of (8.1.c) and the Leibniz rule, the definition of  $Z$  in Section 10 gives  $-Z(v)\zeta(w, w')/2 = \zeta(\nabla_v w, w') + \zeta(w, \nabla_v w') + \alpha(v)\zeta(w, w')$  for any  $\mathcal{V}$ -projectable local sections  $w, w'$  of  $\mathcal{H}$ . Since  $\zeta(\nabla_v w, w') = 2\beta(w)h(u, w')$  by (11.1.b), (8.2) and (11.2.i), the relation  $Z(v) = 0$  is now immediate from (8.3.a). Finally, (11.2.iv) is an obvious consequence of (10.2) and (10.1).

Furthermore, for any  $\mathcal{V}$ -parallel sections  $u, v$  of  $\mathcal{V}$ ,

$$(11.3) \quad \nabla_u(\theta v) = 2\gamma(u)v - 2\alpha(u)\theta v.$$

To verify (11.3), we will show that both sides have equal  $g$ -inner products with any  $\mathcal{V}$ -parallel section  $v'$  of  $\mathcal{V}$ , and with any  $\mathcal{V}$ -projectable section  $w$  of  $\mathcal{H}$ . For  $v'$ , this is clear as  $\theta(v, v')$  equals  $h(\theta v, v')$ , that is,  $g(\theta v, v')$ , and so applying  $d_u$  we get, from (8.1.e),  $g(\nabla_u(\theta v), v') = -2\alpha(u)\theta(v, v')$ , as required. (By (11.1.a),  $\nabla_u v' = 0$ , while  $g(v, v') = 0$  since  $\mathcal{V}$  is  $g$ -null.) For  $w$ , (10.1) allows us to differentiate by parts, obtaining  $g(\nabla_u(\theta v), w) = -g(\nabla_u w, \theta v)$ . In view of (10.2), the last expression equals  $\gamma(u)\zeta(w, \theta v) = -\gamma(u)g(\zeta\theta v, w)$ , which, by (11.2.i), coincides with  $2\gamma(u)g(v, w)$ .

**Lemma 11.1.** *Given a horizontal distribution  $\mathcal{H}$  for a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , let  $g, \nabla, \gamma, Z$  and  $R$  be as in Section 10. Then, with  $R(w, w', u, v)$  given by (3.6),*

- (i)  $R(v, v')v'' = 0$  for all sections  $v, v', v''$  of  $\mathcal{V}$ ,
- (ii)  $\gamma(w)\zeta(w', w'') = g(w, [w', w''])$  for all sections  $w, w', w''$  of  $\mathcal{H}$ ,
- (iii)  $R(w, u)v = \nabla_u \nabla_w v = D_u[w, v] + 2\beta(w)\gamma(u)v - D_u[\gamma(v)\zeta w]$  whenever  $u, v$  are  $\mathcal{V}$ -parallel sections of  $\mathcal{V}$  and  $w$  is a  $\mathcal{V}$ -projectable local vector field in  $M$ ,
- (iv)  $R(u, v, w, w') = [(d\gamma)(v, u)]\zeta(w, w') + \gamma(v)[\nabla_u \zeta](w, w') - \gamma(u)[\nabla_v \zeta](w, w')$  for all sections  $w, w'$  of  $\mathcal{H}$  and all vector fields  $u, v$ ,
- (v)  $R(w, w', \cdot, \cdot) = \zeta(w, w')[\Gamma - \gamma \wedge Z + \eta/2] + K\xi \wedge \xi'$  for any real constant  $K$  and sections  $w, w'$  of  $\mathcal{H}$ , with  $\xi = g(w, \cdot)$ ,  $\xi' = g(w', \cdot)$  and  $\Gamma = d\gamma + 2\alpha \wedge \gamma - (K\theta + \eta)/2$ .

**Proof.** Flatness of  $D$ , (3.4) and (11.1.a) yield (i). Next, (10.1) implies (ii): by (10.2) and (3.3),  $2\gamma(w)\zeta(w', w'') = -2g(\nabla_w w', w'') = -g(w', [w'', w]) - g(w'', [w, w']) + g(w, [w', w'']) = 2g(w, [w', w''])$ . The last equality follows here from Remark 3.4, since the dependence of  $g(w, [w', w''])$  on  $w', w''$  is both skew-symmetric and valewise (in view of (10.1), one may replace  $[w', w'']$  with its  $\mathcal{V}$ -component  $[w', w'']^{\mathcal{V}}$ ).

In (iii),  $R(w, u)v = \nabla_u \nabla_w v$  by (3.4) and Remark 3.1. The other equality is now immediate from (11.1.b), since  $\nabla_w v = [w, v] + \nabla_v w$ , and (11.1.a) combined with Remark 3.1 imply that  $\nabla_u[w, v] = D_u[w, v]$ , while  $\nabla_u[\beta(w)\theta v] = 2\beta(w)\gamma(u)v$  in view of (8.1.d) and (11.3).

For  $u, v, w, w'$  as in (iv), the Leibniz rule and (10.2) give  $g(\nabla_u \nabla_v w, w') = -d_u[\gamma(v)\zeta(w, w')] - g(\nabla_v w, \nabla_u w')$ . As  $\mathcal{V}$  and  $\mathcal{H}$  are  $g$ -null,  $g(\nabla_v w, \nabla_u w') = g([\nabla_v w]^{\mathcal{H}}, \nabla_u w') + g(\nabla_v w, [\nabla_u w']^{\mathcal{H}})$ , where  $[\ ]^{\mathcal{H}}$  denotes the  $\mathcal{H}$ -component. Using (10.2) and (8.1.h), we now obtain  $g(\nabla_v w, \nabla_u w') =$

$-\gamma(u)\zeta(w', \nabla_v w) - \gamma(v)\zeta(w, \nabla_u w')$ , and (iv) easily follows from the above equalities combined with (3.6), (3.4), (3.1.iii), (10.2) and the Leibniz rule.

Finally, (v) is immediate from (iv), the definitions of  $Z$  and  $\theta$  in Section 10, and (3.1.ii), since (8.3.b) gives  $\zeta(w, w')\theta(u, v) = 2[g(w, u)g(w', v) - g(w', u)g(w, v)]$  for any sections  $w, w'$  of  $\mathcal{H}$  and any vector fields  $u, v$ .  $\square$

**Lemma 11.2.** *Let  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  be the basic octuple obtained as in Section 9 from a given two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$ . Then the distribution  $\mathcal{H}$  on  $M = \Sigma \times \Pi_+$  tangent to the factor plane  $\Sigma$  is a horizontal distribution for  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , the metric  $g$  on  $M$  is flat,  $\mathcal{H}$  is  $g$ -parallel, and  $\gamma = 0$ , where  $g$  and the 1-form  $\gamma$  are associated with  $\mathcal{H}$  as in Section 10.*

**Proof.** We fix a function  $f : \Sigma \rightarrow \mathbb{R}$  with  $df = \xi$ . For  $u, v \in \Pi$ , let  $\chi^{u,v}$  be the 1-form on  $M$  equal to  $h(v, \cdot)$  on  $\mathcal{H}$  and to  $\Omega(u - fv, \cdot)$  on  $\mathcal{V}$ . As the 1-forms  $\xi, \tau$  on  $\Sigma$  are constant, and hence closed, using (9.1) and (3.1.iii) we easily verify that  $d\chi^{u,v} = 0$  for all  $u, v \in \Pi$ .

The assignment  $(u, v) \mapsto \chi^{u,v}$  is a linear operator, with the domain  $\Pi \times \Pi$ , and so its image  $\mathcal{X}$  is a vector space. The  $g$ -inner product  $g(\chi, \chi')$  of any  $\chi, \chi' \in \mathcal{X}$  is constant on  $M$ . In fact, we may assume that  $\chi = \chi' = \chi^{u,v}$ . Now, as  $v$  is the  $\mathcal{V}$ -component  $w^\mathcal{V}$  of the vector field  $w$  such that  $\chi = g(w, \cdot)$ , while  $\mathcal{V}$  and  $\mathcal{H}$  are  $g$ -null, we get  $g(\chi, \chi)/2 = g(w^\mathcal{H}, w^\mathcal{V}) = g(w, w^\mathcal{V}) = \chi(w^\mathcal{V}) = \chi(v) = \Omega(u, v)$  due to skew-symmetry of  $\Omega$ , as required.

Any fixed basis of  $\mathcal{X}$  thus consists of forms which, locally, are the differentials of functions forming a coordinate system in  $M$ . According to the last paragraph, the components of  $g$  in such coordinates are constant, so that  $g$  is flat, and all  $\chi^{u,v}$  are  $g$ -parallel. Hence  $\mathcal{H}$  is  $g$ -parallel, being the simultaneous kernel of all  $\chi^{u,v}$  with  $v = 0$ . Finally, as  $\mathcal{H}$  is  $g$ -parallel and  $g$ -null, (10.2) gives  $\gamma = 0$ .  $\square$

**Remark 11.3.** For  $(M, g)$  satisfying the assumptions of Theorem 8.4 and such that  $\beta \neq 0$  everywhere in  $M$ , let  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  be the corresponding basic octuple. Then  $\gamma$  defined by (10.2) is the same as in Lemma 5.1(c) with  $\Psi = W^+$ . In fact, letting  $\gamma$  stand for the latter, we have, by (5.2),  $-g(\nabla_w w', w'') = g(\nabla_w(\eta w'), w'') = g(\eta(\nabla_w w'), w'') - g([\nabla_w \eta]w', w'')$  whenever  $w', w''$  are sections of  $\mathcal{H}$  and  $w$  is any vector field. On the other hand, (5.2) gives  $g(\eta(\nabla_w w'), w'') = -g(\nabla_w w', \eta w'') = g(\nabla_w w', w'')$ , and, as  $\theta w' = 0$  (see Lemma 5.1(a)), using Lemma 5.1(c) we get  $g([\nabla_w \eta]w', w'') = 2\gamma(w)\zeta(w', w'')$ .

Furthermore, with  $\bar{u}$  denoting the section of  $\mathcal{V}$  defined in Remark 8.3, the function  $\gamma(\bar{u})$  is a local geometric invariant of  $\tilde{g}$ , since so are  $\mathcal{V}, \alpha, \beta, \gamma$  (due to the uniqueness assertions in Lemma 5.1), and, consequently,  $\bar{u}$ .

## 12. Curvature conditions

Our next goal is to determine which horizontal distributions  $\mathcal{H}$  for a given basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  lead to metrics  $g$  that are Einstein and, at the same time, self-dual of Petrov type III. Rather than approach this property of  $g$  directly, we begin by describing some conditions, namely, (a) – (d) in Theorem 12.2, which are equivalent to it, yet easier to verify. We refer to them as *curvature conditions*, since the curvature tensor  $R$  explicitly appears in (a), while (b) and (d) involve the curvature forms of the Levi-Civita connection in the bundle  $L^+M$ , expressed in terms of the connection forms  $\alpha, \beta, \gamma$ .

**Lemma 12.1.** *If  $\mathcal{H}$  is a horizontal distribution for a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , and  $K$  is a real constant, while  $g, R$  and  $\eta$  correspond to  $\mathcal{H}$  as in Section 10, then the following two conditions are equivalent:*

- (i)  *$M$  is orientable and, for a suitable orientation,  $(M, g)$  is a self-dual Einstein four-manifold of Petrov type III, its scalar curvature equals  $12K$ , while our  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  and  $\mathcal{H}$  coincide with those determined by  $g$  as in Theorem 8.4 and Lemma 5.2,*
- (ii)  *$2R = \zeta \otimes \eta + \eta \otimes \zeta + 2Kg \wedge g$ , where the notation of (3.8) is used.*

**Proof.** That (i) implies (ii) is obvious from Lemma 5.2(ii) and (3.1.i). Next, let us assume (ii). The Ricci tensor of  $g$  then equals  $3Kg$ . (The Ricci contraction applied to  $\zeta \otimes \eta + \eta \otimes \zeta$  yields 0, as  $\mathcal{V} = \text{Ker } \zeta = \text{Im } \zeta$  according to (8.2), so that  $\zeta$  sends  $\text{Ker}(\eta \pm \text{Id})$  into  $\text{Ker}(\eta \mp \text{Id})$ , and hence anticommutes with  $\eta$ .) Therefore, by (3.8), the Weyl tensor  $W$  of  $g$  is equal to  $(\zeta \otimes \eta + \eta \otimes \zeta)/2$ . The hypotheses of Lemma 4.3 are thus satisfied by our  $(M, g)$  and  $k = 2$ , since the 2-forms  $\zeta$  and  $\eta$ , spanning the image of  $W$ , are linearly independent at each point by (8.1.h) and (5.2), while the relations  $\langle \zeta, \eta \rangle = \langle \zeta, \zeta \rangle = 0$  and  $\langle \eta, \eta \rangle = -2$  (immediate from (3.9.b), as  $\zeta, \eta$  anticommute,  $\text{Ker } \zeta = \text{Im } \zeta$ , while  $\eta\eta = \text{Id}$ ) show that the image of  $W$  is  $\langle \cdot, \cdot \rangle$ -degenerate, but not  $\langle \cdot, \cdot \rangle$ -null. Now Lemma 4.3(iii) yields (i).  $\square$

**Theorem 12.2.** *Under the assumptions of Lemma 12.1, condition (i) in Lemma 12.1 holds if and only if, for all sections  $u, v$  of  $\mathcal{V}$ , and  $w$  of  $\mathcal{H}$ , with  $g, \nabla, R, \alpha, \theta, \gamma$  as in Section 10,*

- (a)  $R(w, u)v = Kh(v, w)u$ ,
- (b)  $d\beta + 2\beta \wedge \alpha = -K\zeta/2$ ,
- (c)  $[\nabla_w \theta](u, v) = -2\alpha(w)\theta(u, v)$ ,
- (d)  $(d\gamma + 2\alpha \wedge \gamma)(\cdot, w) = g(\cdot, w)/2$ .

**Proof.** Assuming condition (i) in Lemma 12.1, we obtain (c) (or, respectively, (b) and (d)) from Lemma 5.1(c) with  $\Psi = W^+$  or, respectively, from Lemma 5.2(iv), cf. Remark 11.3. Note that  $\eta$  satisfies (5.2), and hence (10.3), cf. Lemma 5.1(a), while  $\theta(\cdot, w) = 0$  by Lemma 5.2(v). Also,  $\alpha(w)$  defined in Section 10 is the same as in Lemma 5.1 with  $\Psi = W^+$ , as  $\theta w = 0$  and  $\eta w = -w$  by Lemma 5.2(v) and (5.2), and so the equality  $2\zeta\gamma + \eta\alpha + \theta\beta = 0$  in Lemma 5.1(f), evaluated on  $w$ , gives  $0 = -\beta(\theta w) - \alpha(\eta w) - 2\gamma(\zeta w) = \alpha(w) - 2\gamma(\zeta w)$ .

Next, if (i) in Lemma 12.1 holds, so does (ii). Since, for sections  $u, v$  of  $\mathcal{V}$ , Lemma 5.2(v) gives  $\zeta(\cdot, u) = \zeta(v, \cdot) = 0$  and  $g(u, v) = 0$ , while  $g(u, \cdot) = h(u, \cdot)$ , this yields (a).

Conversely, suppose that (a) – (d) are satisfied.

For sections  $w$  of  $\mathcal{H}$  and  $u$  of  $\mathcal{V}$ , using the notation of (3.2.i), we now have

$$(12.1) \quad [\nabla_w \theta]u = -2\alpha(w)\theta u + 2\gamma(w)u.$$

This is verified by taking the  $g$ -inner products of both sides in (12.1) with sections  $v$  of  $\mathcal{V}$  and  $w'$  of  $\mathcal{H}$ . In the former case, the agreement is obvious from (c), as  $\mathcal{V}$  is  $g$ -null. In the latter, the Leibniz rule implies that  $[\nabla_w \theta](u, w') = -\theta(u, \nabla_w w') = -\theta(u, [\nabla_w w']^{\mathcal{V}})$ , with  $[\ ]^{\mathcal{V}}$  denoting the  $\mathcal{V}$ -component (since  $\theta(\cdot, w') = 0$ , cf. Section 10). The required equality  $[\nabla_w \theta](u, w') = 2\gamma(w)h(u, w')$  now follows from (11.2.iv) and (8.1.f).

If  $w, w'$  are sections of  $\mathcal{H}$ , using the Leibniz rule we obtain  $\theta[(\nabla_w \zeta)w'] = \theta[\nabla_w(\zeta w')] - \theta\zeta\nabla_w w' = \nabla_w(\theta\zeta w') - [\nabla_w \theta]\zeta w' - \theta\zeta\nabla_w w'$ . By (11.2.ii) and (8.1.h),  $\theta\zeta w' = -2w'$  and  $\theta\zeta\nabla_w w' = [\nabla_w w']^{\mathcal{H}}$  (the  $\mathcal{H}$ -component of  $\nabla_w w'$ ), so that (12.1) applied to  $u = \zeta w'$  yields  $\theta[(\nabla_w \zeta)w'] = -2[\nabla_w w']^{\mathcal{V}} + 2\alpha(w)\theta\zeta w' - 2\gamma(w)\zeta w'$ . Thus, (11.2.iv) gives  $Z(w) = 0$  for all sections  $w$  of  $\mathcal{H}$ , where  $Z$  is the 1-form defined in Section 10. (We have just shown that  $Z(w)\zeta(w', w'') = 0$  for sections  $w, w', w''$  of  $\mathcal{H}$  with  $w'' = \theta u$  for some vector field  $u$ , while such  $w''$  range over all sections of  $\mathcal{H}$  due to (11.2.ii).)

Combined with (11.2.iii), the conclusion of the last paragraph yields  $Z = 0$ . In addition,  $\Gamma(w, \cdot) = 0$ , for the 2-form  $\Gamma$  appearing in Lemma 11.1(v), and all sections  $w$  of  $\mathcal{H}$ . Namely, (10.3) and the definition of  $\theta$  in Section 10 give  $\eta(\cdot, w) = h(\cdot, w) = g(\cdot, w)$ , cf. (10.1), and  $\theta(\cdot, w) = 0$ , so that  $\Gamma(\cdot, w) = 0$  by (d).

In view of Lemma 12.1, it now suffices to verify that both sides in Lemma 12.1(ii) yield the same value when applied to any quadruple of vector fields, each of which is a section of  $\mathcal{H}$  or  $\mathcal{V}$ . In the following discussion of the possible cases, we will evaluate the right-hand side in Lemma 12.1(ii) on the four vector fields using, without further explanation, relations (10.3) and (8.1.h) along with the fact that  $\mathcal{V}$  and  $\mathcal{H}$  are both  $g$ -null. Due to well-known symmetries of  $R$ , only four cases need to be considered.

When three or four of the vector fields are sections of  $\mathcal{V}$ , both sides vanish (Lemma 11.1(i)).

When the first vector field is a section of  $\mathcal{H}$ , while the second and third ones are sections of  $\mathcal{V}$ , both sides yield the same value in view of (a).

When the first two vector fields are sections  $v, u$  of  $\mathcal{V}$  and the third one is a section  $w$  of  $\mathcal{V}$ , the first Bianchi identity gives  $R(v, u)w = R(w, u)v - R(w, v)u$ , and our equality is an obvious consequence of (a).

Finally, when the first three vector fields are sections of  $\mathcal{H}$ , the required equality is immediate from Lemma 11.1(v), since, as we saw,  $\Gamma(w, \cdot) = 0$ .  $\square$

**Remark 12.3.** The reader may have noticed that relation (b) in Theorem 12.2 was not used in the second (sufficiency) part of the proof. In other words, (b) is a consequence of (a), (c) and (d). It is nevertheless convenient, due to the structure of our argument, to list (b) as a separate condition. See the proof of Lemma 17.2.

### 13. Deformations of horizontal distributions

Horizontal distributions for a fixed basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  may be thought of as arbitrary sections of a specific locally trivial bundle  $\mathcal{C}$  over  $M$ . Its fibre  $\mathcal{C}_x$  at  $x \in M$  consists of all vector subspaces  $\mathcal{H}_x \subset T_x M$  with  $T_x M = \mathcal{H}_x \oplus \mathcal{V}_x$ . One can turn  $\mathcal{C}$  into an affine bundle over  $M$ , having as its associated vector bundle the subbundle  $\mathcal{F}$  of  $\text{Hom}(TM, \mathcal{V})$  with the fibre  $\mathcal{F}_x$  at any  $x \in M$  formed by all operators  $T_x M \rightarrow \mathcal{V}_x$  sending  $\mathcal{V}_x$  to  $\{0\}$ . Thus,

(13.1) sections  $F$  of  $\mathcal{F}$  are morphisms  $TM \rightarrow TM$  valued in  $\mathcal{V}$  and vanishing on  $\mathcal{V}$ .

Specifically, given a horizontal distribution  $\mathcal{H}$  and a section  $F$  of  $\mathcal{F}$ , we declare the sum  $\tilde{\mathcal{H}} = \mathcal{H} + F$  to be a new horizontal distribution for  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , the sections of which have the form  $\tilde{w} = w + Fw$ , with  $w$  ranging over all sections of  $\mathcal{H}$ .

A section  $F$  of  $\mathcal{F}$  associates with any twice-covariant tensor field  $b$  on  $M$  two further tensor fields,  $F.b$  and  $F^*b$ , defined by  $(F.b)(v, v') = b(Fv, v') + b(v, Fv')$  and  $(F^*b)(v, v') = b(Fv, Fv')$  for arbitrary vector fields  $v, v'$ . Next, we denote by  $\llbracket F \rrbracket$  the function  $M \rightarrow \mathbb{R}$  equal to  $-1/4$  times the (pointwise) trace of the bundle morphism  $\mathcal{H} \rightarrow \mathcal{H}$  sending a section  $w$  of  $\mathcal{H}$  to the section  $\theta v$  of  $\mathcal{H}$  defined as in Section 10 with  $v = Fw$ . For any vector fields  $w, w'$  on  $M$ , and any morphism  $\Phi$  of  $TM$  into any vector bundle over  $M$ , vanishing on  $\mathcal{V}$ ,

$$(13.2) \quad \begin{aligned} \text{i)} \quad & h(Fw, w') - h(Fw', w) = 2\llbracket F \rrbracket \zeta(w, w'), \\ \text{ii)} \quad & 2\beta(w)\Phi w' - 2\beta(w')\Phi w = \zeta(w, w')\Phi \bar{w}, \end{aligned}$$

with  $\bar{w}$  as in (10.4). Namely, since  $\mathcal{V}$  is  $h$ -null, both sides in (13.2.i) and (13.2.ii) equal 0 due to (13.1) and (8.2.ii) when one of  $w, w'$  is a section of  $\mathcal{V}$ . We may therefore assume that

$w$  and  $w'$  are sections of  $\mathcal{H}$ . Remark 3.3 gives  $-4\llbracket F \rrbracket \zeta(w, w') = \zeta(\theta Fw, w') + \zeta(w, \theta Fw') = h(\zeta\theta Fw, w') - h(\zeta\theta Fw', w)$ , and so (11.2.i) now implies (13.2.i), while, by (10.5.i),  $2\beta(w) = \zeta(w, \bar{w})$  and  $-2\beta(w') = \zeta(\bar{w}, w')$ , and hence (13.2.ii) follows as  $\zeta(w, w')\Phi\bar{w}$  summed cyclically over  $w, w', \bar{w}$  yields 0 in view of Remark 3.4.

**Lemma 13.1.** *Let  $\mathcal{H}$  be a horizontal distribution for a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ . If  $g, \gamma, \bar{w}$  and the 1-form  $\alpha$  on  $M$  are associated with  $\mathcal{H}$  as in Section 10, and  $F$  is a section of  $\mathcal{F}$ , then  $g, \alpha, \gamma$  and their analogues  $\tilde{g}, \tilde{\alpha}, \tilde{\gamma}$  corresponding to the horizontal distribution  $\tilde{\mathcal{H}} = \mathcal{H} + F$  are related by*

$$(13.3) \quad \begin{aligned} \text{a)} & \quad \tilde{g} = g - F.g, \\ \text{b)} & \quad \tilde{\alpha} = \alpha - F^*\alpha - 2d_{\zeta(\cdot)}\llbracket F \rrbracket - 4\llbracket F \rrbracket\beta + 2h(F\bar{w}, \cdot), \\ \text{c)} & \quad \tilde{\gamma}(v) = \gamma(v) - d_v\llbracket F \rrbracket - \llbracket F \rrbracket\alpha(v) + \theta(F\bar{w}, v)/2, \\ \text{d)} & \quad [\tilde{\gamma}(\tilde{w}) - \gamma(w)]\zeta(w', w'') = h([w', Fw''], w) + h([Fw', w''], w) \\ & \quad \quad \quad + h([Fw', Fw''], w) - h(F[w', w''], w), \end{aligned}$$

for  $\llbracket F \rrbracket$  defined above, whenever  $v$  is a section of  $\mathcal{V}$  and  $w, w', w''$  are  $\mathcal{V}$ -projectable local sections of  $\mathcal{H}$ , the section  $\tilde{w}$  of  $\tilde{\mathcal{H}} = \mathcal{H} + F$  is given by  $\tilde{w} = w + Fw$ , while  $F^*\alpha$  and  $d_{\zeta(\cdot)}f$ , for any function  $f$ , denote the 1-forms such that  $[F^*\alpha](u) = \alpha(Fu)$  and  $[d_{\zeta(\cdot)}f](u) = d_{\zeta u}f$  for all vector fields  $u$ .

**Proof.** Let  $g'$  be the right-hand side of (13.3.a). We thus have  $g'(v, \cdot) = g(v, \cdot) = h(v, \cdot)$  for sections  $v$  of  $\mathcal{V}$  (in view of (13.1), since  $\mathcal{V}$  is  $g$ -null), and, for the same reason,  $g'(\tilde{w}, \tilde{w}) = 0$  if  $\tilde{w}$  is a section of  $\tilde{\mathcal{H}}$  (that is,  $\tilde{w} = w + Fw$  for some section  $w$  of  $\mathcal{H}$ ), which proves (13.3.a).

For  $\mathcal{V}$ -projectable sections  $w, w'$  of  $\mathcal{H}$  and a  $\mathcal{V}$ -parallel section  $v$  of  $\mathcal{V}$ , one has

$$(13.4) \quad \begin{aligned} -2\gamma(v)\zeta(w, w') &= d_w[h(v, w')] - d_{w'}[h(v, w)] \\ &+ h([w', v], w) + h([v, w], w') - h(v, [w, w']), \end{aligned}$$

where we write  $h$  rather than  $g$  since, in each inner product, one of the vector fields involved is a section of  $\mathcal{V}$ , cf. Remark 3.1. This is immediate from (10.2) combined with (3.3); the first of the six terms provided by (3.3) vanishes here in view of (10.1).

As (13.4) holds for any horizontal distribution, including  $\tilde{\mathcal{H}}$ , it remains valid if one replaces  $\gamma(v), w$  and  $w'$  with  $\tilde{\gamma}(v), \tilde{w} = w + Fw$  and  $\tilde{w}' = w' + Fw'$ . Since  $Fw$  and  $Fw'$  are sections of the  $h$ -null distribution  $\mathcal{V}$ , Remark 3.1 implies that the right-hand side of the analogue of (13.4) corresponding to the triple  $(\tilde{\mathcal{H}}, \tilde{w}, \tilde{w}')$  equals its original version for  $(\mathcal{H}, w, w')$  plus

$$(13.5) \quad d_{Fw}[h(v, w')] - d_{Fw'}[h(v, w)] + h([Fw', v], w) + h([v, Fw], w').$$

On the other hand, by (8.1.a) and (13.2.ii) with  $\Phi = F$ , for any  $\mathcal{V}$ -parallel section  $v$  of  $\mathcal{V}$ ,

$$\begin{aligned} \text{(i)} & \quad d_{Fw}[h(v, w')] - d_{Fw'}[h(v, w)] = \beta(w')\theta(Fw, v) - \beta(w)\theta(Fw', v), \\ \text{(ii)} & \quad \beta(w')\theta(Fw, v) - \beta(w)\theta(Fw', v) = \theta(v, F\bar{w})\zeta(w, w')/2, \\ \text{(iii)} & \quad h([Fw', v], w) + h([v, Fw], w') = h(D_v(Fw), w') - h(D_v(Fw'), w). \end{aligned}$$

Applying  $d_v$  to (13.2.i) and using (8.1.a) along with (8.1.c) and the Leibniz rule, we see that the difference of the right-hand sides in (iii) and (i) is  $2\zeta(w, w')[d_v\llbracket F \rrbracket + \llbracket F \rrbracket\alpha(v)]$ . Thus, by (i) – (iii), the expression (13.5) is equal to twice the right-hand side of (ii), plus the difference just mentioned, that is, to  $\zeta(w, w')[2d_v\llbracket F \rrbracket + 2\llbracket F \rrbracket\alpha(v) - \theta(F\bar{w}, v)]$ , and (13.3.c) follows. (Note that  $\zeta(w, w')$  does not change when the pair  $(w, w')$  is replaced with  $(w + Fw, w' + Fw')$ , since  $Fw$  and  $Fw'$  are sections of  $\mathcal{V} = \text{Ker } \zeta$ , cf. (13.1) and (8.1.h).)

Next, for sections  $v$  of  $\mathcal{V}$ , we have  $\tilde{\alpha}(v) = \alpha(v)$  (see the definition of the 1-form  $\alpha$  in Section 10). This is consistent with (13.3.b) in view of (13.1) and (8.2.ii), since  $\mathcal{V}$  is  $h$ -null.

Similarly,  $\tilde{\alpha}(\tilde{w}) = 2\tilde{\gamma}(\zeta w)$ , where  $\tilde{w} = w + Fw$  and  $w$  is any section of  $\mathcal{H}$ . (By (8.1.h),  $\zeta\tilde{w} = \zeta w$ .) Now (13.3.c) for  $v = \zeta w$ , (8.1.g) and (8.1.f) give (13.3.b).

Finally, (13.3.d) is immediate from Lemma 11.1(ii) applied to both  $\mathcal{H}$  and  $\tilde{\mathcal{H}} = \mathcal{H} + F$  (where, in the latter case,  $w, w', w''$  are replaced by  $\tilde{w} = w + Fw$ ,  $\tilde{w}' = w' + Fw'$  and  $\tilde{w}'' = w' + Fw''$ ), along with (a), (13.1), (10.1) and Remark 3.1. As before,  $\zeta(\tilde{w}', \tilde{w}'') = \zeta(w', w'')$ .  $\square$

In a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , the 2-form  $\zeta$  treated as a morphism  $TM \rightarrow TM$  (cf. (8.2)) is a section of  $\mathcal{F}$  according to (13.1) and (8.2). By (13.2.i),  $[\zeta] = 1$ . Any function  $f : M \rightarrow \mathbb{R}$  thus gives rise to the section  $F = f\zeta$  of  $\mathcal{F}$ , with  $[f\zeta] = f$ .

Consequently, every section of  $\mathcal{F}$  can be uniquely written as  $F + f\zeta$ , where  $f : M \rightarrow \mathbb{R}$  and  $F$  is a section of  $\mathcal{F}$  with  $[F] = 0$ . Given two horizontal distributions  $\mathcal{H}$  and  $\tilde{\mathcal{H}} = \mathcal{H} + (F + f\zeta)$ , where  $[F] = 0$ , relations (13.3.b) and (13.3.c) now yield

$$(13.6) \quad \begin{aligned} \text{i)} \quad & \tilde{\alpha} = \alpha - 2d_{\zeta(\cdot)}f - 10f\beta + h(F\bar{w}, \cdot), \\ \text{ii)} \quad & \tilde{\gamma}(v) = \gamma(v) - d_v f - 2f\alpha(v) + \theta(F\bar{w}, v)/2 \end{aligned}$$

for any section  $v$  of  $\mathcal{V}$ . In fact, (10.4) and (13.2.i) give  $[F^*\alpha](w) = \alpha(Fw) = h(Fw, \bar{w}) = h(F\bar{w}, w)$  for any vector field  $w$ , if  $[F] = 0$ . Thus,

$$(13.7) \quad F^*\alpha = h(F\bar{w}, \cdot) \text{ for sections } F \text{ of } \mathcal{F} \text{ with } [F] = 0.$$

On the other hand, by (10.5.i),  $\zeta^*\alpha = 2\beta$  and  $h(\zeta\bar{w}, \cdot) = -2\beta$ , while (8.1.f) and (10.4) imply that  $\theta(\zeta\bar{w}, v)/2 = -h(v, \bar{w}) = -\alpha(v)$ .

#### 14. The first three conditions in Theorem 12.2

Let us fix a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  and a real constant  $K$ .

Given a horizontal distribution  $\mathcal{H}$  for  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , we denote by  $g, \nabla, R, \alpha, \theta$  and  $\bar{w}$  the corresponding objects described in Section 10. Setting  $\Xi(u, v, w) = R(w, u)v - Kh(v, w)u$  for sections  $u, v$  of  $\mathcal{V}$  and a vector field  $w$ , we obtain a section  $\Xi$  of the vector bundle  $\text{Hom}(\mathcal{V} \otimes \mathcal{V} \otimes TM, TM)$  over  $M$ . We also define sections  $B$  of  $[T^*M]^{\wedge 2}$  and  $\Theta$  of  $\mathcal{H}^*$ , by  $B = d\beta + 2\beta \wedge \alpha + K\zeta/2$  and  $\Theta(w)\theta(u, v) = [\nabla_w \theta](u, v) + 2\alpha(w)\theta(u, v)$ , for sections  $w$  of  $\mathcal{H}$  and  $u, v$  of  $\mathcal{V}$ . Obviously,  $\Xi, B$  and  $\Theta$  depend on  $\mathcal{H}$ , and  $\Theta$  is well defined, since  $\mathcal{V}^{\wedge 2}$  is trivialized by  $\theta$ .

Conditions (a), (b) and (c) in Theorem 12.2 amount, respectively, to  $\Xi = 0$ ,  $B = 0$  and  $\Theta = 0$ . (See Remark 14.1 below.) The simultaneous vanishing of  $\Xi, B$  and  $\Theta$  is a special property of  $\mathcal{H}$ . To determine which choices of  $\mathcal{H}$  have this property, we first describe the transformations that  $\Xi, B$  and  $\Theta$  undergo when  $\mathcal{H}$  is replaced by another horizontal distribution  $\tilde{\mathcal{H}}$ . As pointed out in Section 13,  $\tilde{\mathcal{H}}$  is always the result of adding to  $\mathcal{H}$  a section of  $\mathcal{F}$ . Writing an arbitrary section of  $\mathcal{F}$  uniquely as  $F + f\zeta$  with  $[F] = 0$  and  $f : M \rightarrow \mathbb{R}$  (cf. the end of Section 13), and denoting by  $\tilde{\Xi}, \tilde{B}$  and  $\tilde{\Theta}$  the analogues of  $\Xi, B$  and  $\Theta$  for the new horizontal distribution  $\tilde{\mathcal{H}} = \mathcal{H} + (F + f\zeta)$ , we have, as shown in the next section,

$$(14.1) \quad \begin{aligned} \text{a)} \quad & \tilde{\Xi} = \Xi - \mathbb{I}PF, \\ \text{b)} \quad & \tilde{B} = B - 2\beta \wedge [2d_{\zeta(\cdot)}f - h(F\bar{w}, \cdot)], \\ \text{c)} \quad & \tilde{\Theta}(\tilde{w}) = \Theta(w) + \text{div}^{\mathcal{V}}(Fw) - 4d_{\zeta w}f - 24f\beta(w) + 3h(F\bar{w}, w). \end{aligned}$$

Here  $d_{\zeta(\cdot)}f$  is defined as in Lemma 13.1,  $\mathbb{I}PF$  is given by

$$(14.2) \quad (\mathbb{I}PF)(u, v, w) = D_u D_v(Fw) - \beta(w)\theta(F\bar{w}, u)v + D_u[\theta(F\bar{w}, v)\zeta w]/2$$

for any  $\mathcal{V}$ -projectable vector field  $w$  and  $\mathcal{V}$ -parallel sections  $u, v$  of  $\mathcal{V}$ , while, in (14.1.c),  $w$  stands for an arbitrary section of  $\mathcal{H}$  and  $\tilde{w} = w + (F + f\zeta)w$ . (Thus,  $\tilde{w}$  is a section of  $\tilde{\mathcal{H}}$ .) Finally, the  $\mathcal{V}$ -divergence  $\operatorname{div}^{\mathcal{V}}u : M \rightarrow \mathbb{R}$  of any section  $u$  of  $\mathcal{V}$  is the (pointwise) trace of the bundle morphism  $Du : \mathcal{V} \rightarrow \mathcal{V}$  sending each section  $v$  of  $\mathcal{V}$  to  $D_vu$ , cf. Section 7.

**Remark 14.1.** For  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ ,  $\mathcal{H}$  and  $K$  as above, the condition  $\Xi = 0$  is equivalent to vanishing of  $\Xi(u, v, w)$  whenever  $u, v$  are sections of  $\mathcal{V}$ , while  $w$ , rather than being an arbitrary vector field on  $M$ , is assumed to be a section of  $\mathcal{H}$ . In fact, as  $\mathcal{V}$  is  $h$ -null, Lemma 11.1(i) gives  $\Xi(u, v, w) = 0$  if  $u, v, w$  are sections of  $\mathcal{V}$ .

**Example 14.2.** For the basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  associated, as in Section 9, with a fixed two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$ , let  $\mathcal{H}$  be the horizontal distribution appearing in Lemma 11.2, and let  $K$  be a real constant. The objects  $\Xi, B$  and  $\Theta$  then are given by

$$(14.3) \quad \Xi(u, v, w) = -K\Omega(Y_w, v)u, \quad B = K\phi^{-1}\xi \wedge \tau, \quad \Theta = 0,$$

where  $Y_w = \xi(w)X + \tau(w)c$  and  $X$  is the radial vector field on  $\Pi$ , while (14.2) takes the form

$$(14.4) \quad \begin{array}{l} \text{i) } (IPF)(u, v, w) = D_u(H_vw) - \xi(w)\Omega(F\bar{w}, u)v, \quad \text{with} \\ \text{ii) } H_vw = D_v(Fw) - \Omega(F\bar{w}, v)Y_w. \end{array}$$

To justify (14.3) and (14.4), first note that  $\Xi(u, v, w) = -Kh(v, w)u = -K\Omega(Y_w, v)u$  by (9.1). Next, as an immediate consequence of (9.2), if  $u$  is a section of  $\mathcal{V}$ ,

$$(14.5) \quad \text{i) } \phi = \Omega(X, c), \quad \text{ii) } d_u\phi = \Omega(u, c), \quad \text{iii) } d_c\phi = 0.$$

On the other hand, (9.1), (14.5.i) and (3.1.ii) give

$$(14.6) \quad h(c, w) = \xi(w)\phi, \quad h(X, w) = -\tau(w)\phi, \quad h(Y_w, w') = -\phi(\xi \wedge \tau)(w, w')$$

for sections  $w$  of  $\mathcal{H}$ . Furthermore, for such  $w$ ,

$$(14.7) \quad \text{a) } \beta = \phi^{-2}\xi, \quad \text{b) } \theta = \phi^2\Omega, \quad \text{c) } \zeta = 2\phi^{-1}\xi \wedge \tau, \quad \text{d) } \zeta w = -2\phi^{-2}Y_w,$$

where the first three equalities are obvious from (9.2), (8.4), and the definition of  $\theta$  in Section 10, while the last one is easily verified by taking the  $g$ -inner products of both sides with any section  $w'$  of  $\mathcal{H}$ , and using (14.7.c) along with the last formula in (14.6).

Also,  $\alpha = -d \log \phi$ , as both sides agree on  $\mathcal{V}$  (by (9.2) and (8.4)), and vanish on  $\mathcal{H}$  (due to the definition of  $\alpha$  in Section 10, where  $\gamma = 0$ , cf. Lemma 11.2). Since  $d\xi = 0$ , (14.7.a) thus yields  $d\beta + 2\beta \wedge \alpha = 0$ , and so  $B = K\zeta/2 = K\phi^{-1}\xi \wedge \tau$  (see (14.7.c)), as required in (14.3). The relation  $\gamma = 0$  in Lemma 11.2, combined with (12.1), implies in turn the last equality in (14.3). Now (14.2) and (14.7) give (14.4).

In addition, for  $\bar{w}$  and  $\bar{u}$  as in (10.4) and Remark 8.3 we have, in this case,

$$(14.8) \quad \text{a) } \xi(\bar{w}) = 0, \quad \text{b) } \tau(\bar{w}) = \phi^{-1}, \quad \text{c) } \zeta\bar{w} = -2\phi^{-3}c, \quad \text{d) } \bar{u} = \phi^{-3}c.$$

In fact,  $\xi(\bar{w}) = 0$  by (14.7.a) and (10.5.ii), while (13.2.ii) with  $\Phi = \tau$ , (14.7.a), (14.7.c) and (3.1.ii) yield  $\tau(\bar{w}) = \phi^{-1}$ , and, as  $Y_w = \xi(w)X + \tau(w)c$ , the third equality is immediate from the first two and (14.7.d). Finally,  $h(\phi^{-3}c, \cdot) = \beta$  by (14.7.a), the first formula in (14.6), and (9.1.a), so that (14.8.d) follows.



### 15. Proof of (14.1)

Equality (14.1.b) is obvious from (13.6.i), since  $\tilde{B} - B = 2\beta \wedge (\tilde{\alpha} - \alpha)$  due to the fact that  $\beta$  and  $\zeta$  do not depend on  $\mathcal{H}$ .

We now establish (14.1.a), assuming that  $u, v$  are  $\mathcal{V}$ -parallel sections of  $\mathcal{V}$  (cf. (7.2)), while  $w$ , in addition to being  $\mathcal{V}$ -projectable, is a section of  $\mathcal{V}$  or a section of  $\mathcal{H}$ . In the former case, both sides equal 0. Namely, Remark 14.1 then shows that  $\Xi(u, v, w) = 0$  for any choice of a horizontal distribution, including  $\tilde{\Xi}(u, v, w) = 0$  for  $\tilde{\mathcal{H}}$ , while  $(\mathbb{P}F)(u, v, w) = 0$  by (14.2) since, for sections  $w$  of  $\mathcal{V}$ , (13.1) and (8.2.ii) yield  $Fw = \zeta w = 0$  and  $\beta(w) = 0$ .

In the latter case, where  $w$  is a section of  $\mathcal{H}$ , the relation  $\tilde{\mathcal{H}} = \mathcal{H} + (F + f\zeta)$  implies that  $\tilde{w} = w + (F + f\zeta)w$  is a  $\mathcal{V}$ -projectable section of  $\tilde{\mathcal{H}}$ , and, according to the preceding paragraph,  $\tilde{\Xi}(u, v, w) = \tilde{\Xi}(u, v, \tilde{w})$ . Let us now evaluate  $\Xi(u, v, w)$  (or,  $\tilde{\Xi}(u, v, \tilde{w})$ ) with the aid of the equality  $R(w, u)v = D_u[w, v] + 2\beta(w)\gamma(u)v - D_u[\gamma(v)\zeta w]$  in Lemma 11.1(iii) (or, respectively, its analogue for  $\tilde{\mathcal{H}}$ ). Since  $\beta(\tilde{w}) = \beta(w)$  and  $\zeta\tilde{w} = \zeta w$ , cf. (8.2.ii), we may thus express  $\Xi(u, v, w) - \tilde{\Xi}(u, v, w) = R(w, u)v - \tilde{R}(\tilde{w}, u)v$  as a sum of some terms containing  $F$  and some terms involving  $f$ . By (13.6.ii), the former terms add up to  $(\mathbb{P}F)(u, v, w)$ . (Since  $v$  is  $\mathcal{V}$ -parallel,  $[w, v] - [\tilde{w}, v] = D_v[(F + f\zeta)w]$ .) On the other hand, the sum  $S$  of the latter terms is zero. Namely, (13.6.ii) gives  $S = D_u D_v(f\zeta w) + 2\beta(w)[d_u f + 2f\alpha(u)]v - (d_u d_v f)\zeta w - 2(d_u f)\alpha(v)\zeta w - 2f\{d_u[\alpha(v)]\}\zeta w - [d_v f + 2f\alpha(v)]D_u(\zeta w)$ . (The last four terms arise when  $D_u$  is applied to  $-[d_v f + 2f\alpha(v)]\zeta w$ .) However, by the Leibniz rule,  $D_u D_v(f\zeta w) = (d_u d_v f)\zeta w + (d_u f)D_v(\zeta w) + (d_v f)D_u(\zeta w) + fD_u D_v(\zeta w)$ , while, according to Remark 8.1 and (8.1.d),  $D_u D_v(\zeta w) = 2D_u[\alpha(v)\zeta w - \beta(w)v] = 2\{d_u[\alpha(v)]\}\zeta w + 2\alpha(v)D_u(\zeta w) - 4\alpha(u)\beta(w)v$ . The resulting cancellations show that  $S$  equals  $[D_v(\zeta w) - 2\alpha(v)\zeta w + 2\beta(w)v]d_u f$ , and so  $S = 0$  in view of Remark 8.1. This yields (14.1.a).

To prove (14.1.c), let us fix  $\mathcal{V}$ -parallel sections  $u, v$  of  $\mathcal{V}$  and a  $\mathcal{V}$ -projectable section  $w$  of  $\mathcal{H}$ . Since  $\nabla$  is torsionfree, the Leibniz rule gives  $[\Theta(w) - 2\alpha(w)]\theta(u, v) = [\nabla_w \theta](u, v) = d_w[\theta(u, v)] - \theta(\nabla_u w, v) - \theta(u, \nabla_v w) - \theta([w, u], v) - \theta(u, [w, v])$ . However,  $\mathcal{H} = \text{Ker } \theta$ , cf. Section 10, so that  $\nabla_u w$  and  $\nabla_v w$  can be replaced here with their  $\mathcal{V}$  components, equal, by (11.1.b), to  $-\gamma(u)\zeta w$  and  $-\gamma(v)\zeta w$ . Consequently,  $-\theta(\nabla_u w, v) - \theta(u, \nabla_v w) = \gamma(u)\theta(\zeta w, v) + \gamma(v)\theta(u, \zeta w)$  which, by Remark 3.4, equals  $\gamma(\zeta w)\theta(u, v) = \alpha(w)\theta(u, v)/2$ , as  $\alpha(w) = 2\gamma(\zeta w)$  (see Section 10). Thus,  $\Theta(w)\theta(u, v) = d_w[\theta(u, v)] + \theta([u, w], v) + \theta(u, [v, w]) + 5\alpha(w)\theta(u, v)/2$ .

Suppose now that  $\tilde{\mathcal{H}} = \mathcal{H} + F$ , where  $F$  is an arbitrary section of  $\mathcal{F}$ , not necessarily one with  $\llbracket F \rrbracket = 0$ . For  $\tilde{w} = w + Fw$ , the preceding equality, applied to both  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$ , yields  $[\tilde{\Theta}(\tilde{w}) - \Theta(w)]\theta(u, v) = d_{Fw}[\theta(u, v)] + \theta([u, Fw], v) + \theta(u, [v, Fw]) + 5[\tilde{\alpha}(\tilde{w}) - \alpha(w)]\theta(u, v)/2$ . As  $u$  and  $v$  are  $\mathcal{V}$ -parallel,  $[u, Fw] = D_u(Fw)$  and  $[v, Fw] = D_v(Fw)$ , so that Remark 3.3 gives  $\theta([u, Fw], v) + \theta(u, [v, Fw]) = [\text{div}^{\mathcal{V}}(Fw)]\theta(u, v)$ . Hence, by (8.1.e),  $\tilde{\Theta}(\tilde{w}) - \Theta(w) = -2\alpha(Fw) + \text{div}^{\mathcal{V}}(Fw) + 5[\tilde{\alpha}(\tilde{w}) - \alpha(w)]/2$ . Substituting for  $F$ , in this last equality, the sum  $F + f\zeta$  with  $\llbracket F \rrbracket = 0$ , we obtain (14.1.c), as  $\text{div}^{\mathcal{V}}(\zeta w) = 0$  by Remark 8.1 and (8.1.g), and so  $\text{div}^{\mathcal{V}}[(F + f\zeta)w] = \text{div}^{\mathcal{V}}(Fw) + d_{\zeta w}f$ , while (13.7) and (8.1.g) give  $\alpha((F + f\zeta)w) = h(F\bar{w}, w) + 2f\beta(w)$ , and (13.6.i) with  $\tilde{w} = w + (F + f\zeta)w$  yields  $\tilde{\alpha}(\tilde{w}) - \alpha(w) = -2d_{\zeta w}f - 10f\beta(w) + h(F\bar{w}, w) + \alpha((F + f\zeta)w)$ .

### 16. Dimension of a solution space

Let  $\mathcal{H}$  be a horizontal distribution for a basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ .

Formula (14.2) makes sense also when  $F$ , rather than being a section of  $\mathcal{F}$ , is just a section of the restriction of  $\mathcal{F}$  to a surface  $\Pi'$  embedded in  $M$  and contained in a leaf of  $\mathcal{V}$ . (In fact, (14.2) involves only covariant derivatives in directions tangent to  $\mathcal{V}$ .) The kernel of the operator  $IP$ , acting on those sections  $F$  of the restriction of  $\mathcal{F}$  to  $\Pi'$  which satisfy the additional condition  $\llbracket F \rrbracket = 0$ , is at most eight-dimensional.

To show this, we first observe that, if  $F$  is a section of  $\mathcal{F}$  with  $IPF = 0$  and  $\llbracket F \rrbracket = 0$ , then

$$(16.1) \quad \begin{array}{ll} \text{i)} & h(D_v(Fw), w') - h(D_v(Fw'), w) = \theta(v, F\bar{w})\zeta(w, w')/2, \\ \text{ii)} & h(Fw, w') = h(Fw', w), \quad \text{iii)} \quad \theta(D_u(F\bar{w}), v) = \theta(D_v(F\bar{w}), u) \end{array}$$

for any sections  $u, v$  of  $\mathcal{V}$  and any  $\mathcal{V}$ -projectable local sections  $w, w'$  of  $\mathcal{H}$ .

Namely, (16.1.ii) is obvious from (13.2.i), while (16.1.i) easily follows if we apply  $d_v$  to (16.1.ii), use (8.1.a) along with the Leibniz rule, and then set  $\Phi w = \theta(v, Fw)$  in (13.2.ii). Finally, (16.1.iii) is immediate if one skew-symmetrizes the right-hand side of (14.2) in  $u, v$ , assuming  $u, v$  to be  $\mathcal{V}$ -parallel (so that  $D_u D_v = D_v D_u$  as  $D$  is flat), and then uses (8.1.e) along with the Leibniz rule and Remark 8.1.

Note that the assumption  $\llbracket F \rrbracket = 0$  alone yields (16.1.ii) and (16.1.i), while Remark 3.3 allows us to rewrite (16.1.iii) as  $\text{div}^{\mathcal{V}}(F\bar{w}) = 0$ , and (by (13.1) and (8.2.ii))  $w, w'$  in (16.1) may equivalently be just any  $\mathcal{V}$ -projectable local vector fields.

Condition  $IPF = 0$  implies that  $F$ , restricted to any D-geodesic contained in  $\Pi'$ , satisfies a system of second-order linear ordinary differential equations solved for the second derivatives. To make sense of  $F\bar{w}$  in this context, here and below we choose  $\phi$  as at the end of Section 8, thus getting  $F\bar{w} = \phi^{-1}F\phi\bar{w}$ , where  $\phi\bar{w}$  is  $\mathcal{V}$ -projectable by Lemma 10.1. Let us fix a point  $x \in \Pi'$ . Due to uniqueness of solutions,  $F$  is completely determined, on  $\Pi'$ , by the pair  $(F, b)$  consisting of its value at  $x$ , still denoted by  $F$ , and its  $\mathcal{V}$ -differential  $b$  at  $x$ . More precisely,  $F$  is a linear operator  $\mathcal{H}_x \rightarrow \mathcal{V}_x$ , and  $b$  may be treated as a bilinear mapping  $\mathcal{V}_x \times \mathcal{H}_x \rightarrow \mathcal{V}_x$  with  $b(v, w) = D_v(Fw)$  (at  $x$ , for  $v, w$  as in (16.1)). Thus, solutions  $F$  to the equations  $IPF = 0$  and  $\llbracket F \rrbracket = 0$  on  $\Pi'$  lead, at  $x$ , to pairs  $(F, b)$  which are vectors in a 12-dimensional space  $\mathcal{W}$ . Rather than being arbitrary vectors in  $\mathcal{W}$ , such  $(F, b)$  are subject to the additional constraints stemming from (16.1), which state that, for all  $v \in \mathcal{V}_x$  and  $w, w' \in \mathcal{H}_x$ ,

$$(16.2) \quad \begin{array}{ll} \text{i)} & h(b(v, w), w') - h(b(v, w'), w) = \phi^{-1}\theta(v, F\phi\bar{w})\zeta(w, w')/2, \\ \text{ii)} & h(Fw, w') = h(Fw', w), \\ \text{iii)} & \theta(b(u, \phi\bar{w}), v) - \theta(b(v, \phi\bar{w}), u) = -h(F\phi\bar{w}, \bar{w})\theta(u, v). \end{array}$$

We obtain (16.2.iii) from (16.1.iii), the relation  $\alpha = -d \log \phi$  on  $\mathcal{V}$  (cf. Section 8), combined with Remark 3.4 for the expression  $\alpha(u)\theta(F\phi\bar{w}, v)$  (trilinear in  $u, F\phi\bar{w}, v$ ), and (13.7). Since  $\phi\bar{w}$  is  $\mathcal{V}$ -projectable (see above), it is useful here to rewrite  $F\bar{w}$  as  $\phi^{-1}F\phi\bar{w}$ .

By assigning to a linear operator  $\Phi : \mathcal{H}_x \rightarrow \mathcal{V}_x$  the bilinear form on  $\mathcal{H}_x$  that sends  $w, w'$  to  $(\Phi w, w')$ , we obtain an isomorphism between the space of operators  $\mathcal{H}_x \rightarrow \mathcal{V}_x$  and the space of bilinear forms  $\mathcal{H}_x \times \mathcal{H}_x \rightarrow \mathbb{R}$ . (See (7.1).) We will say that a linear operator  $\Phi : \mathcal{H}_x \rightarrow \mathcal{V}_x$  is *h-self-adjoint* if it corresponds under this isomorphism to a form which is symmetric, that is, if  $(\Phi w, w') = h(\Phi w', w)$  for all  $w, w' \in \mathcal{H}_x$ . The space of *h-self-adjoint* operators  $\mathcal{H}_x \rightarrow \mathcal{V}_x$  is, obviously, three-dimensional.

Denoting by  $\mathcal{W}'$  the subspace of  $\mathcal{W}$  formed by all  $(F, b) \in \mathcal{W}$  satisfying (16.2.ii) and (16.2.i), we have  $\dim \mathcal{W}' = 9$ . In fact, the assignment  $(F, b) \mapsto (F, \hat{b})$ , given by  $\hat{b}(v, w) = b(v, w) - \phi^{-1}\theta(v, F\phi\bar{w})\zeta w/4$ , is an isomorphism  $\mathcal{W} \rightarrow \mathcal{W}$  sending  $\mathcal{W}'$  onto the space of all

$(F, \hat{b}) \in \mathcal{W}$  such that  $F$  and  $\hat{b}(v, \cdot)$  are  $h$ -self-adjoint for every  $v \in \mathcal{V}_x$ , while the latter space is nine-dimensional (cf. the last paragraph).

Finally, the three conditions (16.2) together define an eight-dimensional subspace of  $\mathcal{W}$ . In fact, the subspace in question is the kernel of a linear functional on the space  $\mathcal{W}'$  with  $\dim \mathcal{W}' = 9$ . (Note that (16.2.iii) amounts to a single scalar equation, due to its skew-symmetry in  $u, v$ .) The functional in question is nonzero, since condition (16.2.iii) for  $(F, b) \in \mathcal{W}$  is *not* a consequence of (16.2.i) and (16.2.ii). An example  $(F, b)$  proving the last claim may be defined as follows. We choose  $F$  which is both  $h$ -self-adjoint and such that  $h(F\phi\bar{w}, \bar{w}) \neq 0$ . In other words,  $\phi\bar{w}$  is not null for the symmetric bilinear form corresponding to  $F$ . Thus, (16.2.ii) holds. Then we set  $b(v, w) = \phi^{-1}\theta(v, F\phi\bar{w})\zeta w/4$ , which clearly gives (16.2.i). However, the left-hand side of (16.2.iii) equals here  $1/4$  times  $\theta(u, F\phi\bar{w})\theta(\zeta\bar{w}, v) - \theta(v, F\phi\bar{w})\theta(\zeta\bar{w}, u)$ , which coincides with  $\theta(\zeta\bar{w}, F\phi\bar{w})\theta(u, v)/4$  (from Remark 3.4 applied to the expression  $\theta(u, F\phi\bar{w})\theta(\zeta\bar{w}, v)$ , trilinear in  $u, \zeta\bar{w}, v$ ). Now, by (11.2.ii), the left-hand side of (16.2.iii) is equal to  $-h(F\phi\bar{w}, \bar{w})\theta(u, v)/2$ , and hence different from the right-hand side, due to our choice of  $F$ .

## 17. Explicit solutions for a two-plane system

Let  $K$  be a fixed real constant. We will now describe the set of all horizontal distributions  $\mathcal{H}$ , for any given basic octuple  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$ , which have the properties (a), (b) and (c) in Theorem 12.2.

Our discussion is local. Theorem 9.2 thus allows us to fix a two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  and assume, without loss of generality, that  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  is its associated basic octuple. We denote by  $\mathcal{H}$  the horizontal distribution appearing in Lemma 11.2. Horizontal distributions  $\tilde{\mathcal{H}}$  satisfying (a) – (c) in Theorem 12.2, written as  $\tilde{\mathcal{H}} = \mathcal{H} + (F + f\zeta)$ , where  $F$  is a section of  $\mathcal{F}$  with  $\llbracket F \rrbracket = 0$  and  $f : M \rightarrow \mathbb{R}$  (see the end of Section 13), are characterized, according to Section 14, by simultaneous vanishing of  $\tilde{\Xi}, \tilde{B}$  and  $\tilde{\Theta}$  in (14.1), which is a system of three (usually nonhomogeneous) linear partial differential equations with the unknowns  $F$  and  $f$ . Specifically,  $\tilde{\Xi} = 0$  is a second-order equation involving  $F$  only,  $\tilde{B} = 0$  is of first order in  $f$  and of order zero in  $F$ , while  $\tilde{\Theta} = 0$  is of first order in both  $f$  and  $F$ . By (14.1.a),  $\tilde{\Xi} = 0$  if and only if  $IPF = \Xi$  (under the assumption that  $\llbracket F \rrbracket = 0$ ).

In all three equations, only derivatives in  $\Pi$  directions occur, so that we may fix  $y \in \Sigma$  and restrict the unknowns  $f$  and  $F$  to the subset  $\{y\} \times \Pi_+ \approx \Pi_+$ , thus treating them as a function  $f : \Pi_+ \rightarrow \mathbb{R}$  and, respectively,

$$(17.1) \quad \text{a linear operator } w \mapsto Fw \text{ from } \dot{\Sigma} \text{ into the space of vector fields on } \Pi_+,$$

vector fields being identified with mappings  $\Pi_+ \rightarrow \Pi$ . Here  $\dot{\Sigma}$  is the translation vector space of  $\Sigma$ , canonically isomorphic both to  $T_y\Sigma$  and to the fibre of  $\mathcal{H}$  at any point of  $\{y\} \times \Pi_+$ . The three equations, phrased in terms of such identifications, with fixed  $y$ , are solved below. As before,  $X$  denotes the restriction to  $\Pi_+$  of the radial vector field,  $Y_w = \xi(w)X + \tau(w)c$  and  $\Xi(u, v, w) = -K\Omega(Y_w, v)u$ , cf. (14.3), while  $\phi : \Pi_+ \rightarrow (0, \infty)$  is given by  $\phi = \Omega(X, c)$ . The symbol  $F\bar{w}$  stands for  $\phi^{-1}F(\phi\bar{w})$ . (By (14.8.a) – (14.8.b), or Lemma 10.1,  $\phi\bar{w}$  is a  $\mathcal{V}$ -projectable section of  $\mathcal{H}$ , and may be identified with a constant vector field on  $\Sigma$ , so that this convention about the meaning of  $F\bar{w}$  agrees with our previous usage, such as in (14.4).)

**Lemma 17.1.** *Among all operators (17.1), those with  $IPF = \Xi$  and  $\llbracket F \rrbracket = 0$  form an eight-dimensional affine space  $\mathcal{S}$  containing  $F^K$  given by  $F^K w = K\phi\tau(w)X/2$ . The translation*

vector space  $\dot{\mathcal{S}}$  of  $\mathcal{S}$  is the direct sum of three subspaces, of dimensions 2, 3 and 3, consisting, respectively, of the operators  $F^q, F^\lambda, F^\mu$  sending  $w$  to

- (a)  $2\xi(w)q - (d_q\phi)\phi^{-1}Y_w$ , for any constant vector field  $q$  on  $\Pi$ ,
- (b)  $\phi^{-1}[\xi(w)\lambda(X, X)c + \tau(w)\lambda(c, c)X]$ , for any constant symmetric 2-tensor field  $\lambda$  on  $\Pi$ ,
- (c)  $\phi^{-1}[\mu(X, X)Y_w - 2\mu(Y_w, X)X]$ , for any constant symmetric 2-tensor field  $\mu$  on  $\Pi$ .

If  $F = F^K + F^q + F^\lambda + F^\mu$ , then  $\phi^2 F\bar{w} = EX + Lc$ , where  $E = \lambda(c, c) - 2\mu(c, X) + K\phi^2/2$  and  $L = \mu(X, X) - \Omega(q, c)$ . Finally, for any  $w, w' \in \dot{\Sigma}$ ,

$$(17.2) \quad \begin{aligned} h(F^K w, w') &= -K\phi^2\tau(w)\tau(w')/2, \\ h(F^q w, w') &= 2\Omega(X, q)\xi(w)\xi(w') - \Omega(q, c)[\xi(w)\tau(w') + \xi(w')\tau(w)], \\ h(F^\lambda w, w') &= \lambda(X, X)\xi(w)\xi(w') - \lambda(c, c)\tau(w)\tau(w'), \\ h(F^\mu w, w') &= \mu(X, X)[\xi(w)\tau(w') + \xi(w')\tau(w)] + 2\mu(c, X)\tau(w)\tau(w'). \end{aligned}$$

**Proof.** First, (14.6), (3.1.ii), (9.1) and (14.5.ii) yield (17.2). Hence  $\llbracket F^K \rrbracket = \llbracket F^q \rrbracket = \llbracket F^\lambda \rrbracket = \llbracket F^\mu \rrbracket = 0$  due to (13.2.i) and symmetry in  $w, w'$  of the right-hand sides in (17.2).

Since  $\xi(\bar{w}) = 0$ ,  $\tau(\bar{w}) = \phi^{-1}$  and  $Y_{\bar{w}} = \phi^{-1}c$  (see (14.8) and (9.1.c)), we have, by (14.5.ii),

$$(17.3) \quad \begin{aligned} F^K \bar{w} &= KX/2, \quad F^q \bar{w} = \Omega(c, q)\phi^{-2}c, \quad F^\lambda \bar{w} = \lambda(c, c)\phi^{-2}X, \\ F^\mu \bar{w} &= \phi^{-2}[\mu(X, X)c - 2\mu(c, X)X]. \end{aligned}$$

We denote by  $H_v^K w, H_v^q w, H_v^\lambda w$  or  $H_v^\mu w$  the expression (14.4.ii) with  $F$  replaced by  $F^K, F^q, F^\lambda$  or  $F^\mu$ . Let  $\mathcal{S}$  be the affine space of all operators (17.1) with  $IPF = \Xi$  and  $\llbracket F \rrbracket = 0$ .

Using (14.5) and noting that  $\Omega(X, c)v - \Omega(X, v)c = \Omega(v, c)X$  (see Remark 3.4), we obtain  $H_v^K w = K[\Omega(v, c)\tau(w) - \Omega(X, v)\xi(w)/2]X$ . Now (14.4.i) and (17.3) combined with the equality  $\Omega(u, v)X + \Omega(X, u)v = \Omega(X, v)u$  (immediate from Remark 3.4) yield  $IPF^K = \Xi$ , with  $\Xi(u, v, w) = -K\Omega(Y_w, v)u$ . As  $\llbracket F^K \rrbracket = 0$ , it follows that  $F^K \in \mathcal{S}$ .

Furthermore, (14.5) gives  $H_v^q w = \Omega(c, q)\phi^{-1}\xi(w)v$ , and so  $IPF^q = 0$  by (14.4.i) and (17.3).

Next,  $H_v^\lambda w = 2\phi^{-1}\xi(w)\lambda(v, X)c - \phi^{-2}\xi(w)[\lambda(c, c)\Omega(X, v)X + \Omega(v, c)\lambda(X, X)c]$  by (14.5), as  $\Omega(X, c)v - \Omega(X, v)c = \Omega(v, c)X$  (see above). The relation  $IPF^\lambda = 0$  is now easily verified using (14.4.i) and (17.3) along with the equalities  $\Omega(u, v)X + \Omega(X, u)v = \Omega(X, v)u$ ,  $(d_u\phi)X - \phi u = \Omega(u, c)X + \Omega(c, X)u = \Omega(u, X)c$  (and hence  $\Omega(u, c)\lambda(X, X) - \phi\lambda(u, X) = \Omega(u, X)\lambda(c, X)$ ),  $\Omega(u, c)\lambda(X, v) - \phi\lambda(u, v) = \Omega(u, c)\lambda(X, v) + \Omega(c, X)\lambda(u, v) = \lambda(c, v)\Omega(u, X)$  and  $\lambda(c, c)\Omega(X, v) + \lambda(c, v)\Omega(c, X) + \lambda(c, c)\Omega(X, v) = 0$ , due to Remark 3.4 and (14.5.i).

Similarly,  $H_v^\mu w = 2\phi^{-1}\tau(w)[\mu(v, X)c - \mu(v, c)X] - \phi^{-1}\mu(X, X)\xi(w)v$ , since  $\Omega(X, v)c + \Omega(v, c)X = \Omega(X, c)v$ , which also gives  $\Omega(X, v)\mu(c, X) + \Omega(v, c)\mu(X, X) = \Omega(X, c)\mu(v, X) = \phi\mu(v, X)$ . Therefore,  $IPF^\mu = 0$ , in view of the relation  $\Omega(u, c)X - \phi u = \Omega(u, X)c$  (see above) and two further equalities, which are its immediate consequences:  $\Omega(X, u)\mu(c, X) + \Omega(u, c)\mu(X, X) = \phi\mu(c, X)$  and  $\Omega(u, c)\mu(v, X) = \phi\mu(u, v) + \mu(c, v)\Omega(u, X)$ .

As established above,  $F^q, F^\lambda, F^\mu \in \dot{\mathcal{S}}$ , where  $\dot{\mathcal{S}}$  is the translation vector space of  $\mathcal{S}$ . The operators  $F^q, F^\lambda, F^\mu$  together span a vector space of dimension 8. In fact, assuming that  $[F^q + F^\lambda + F^\mu]\phi$  vanishes identically, we will show that  $q = 0$  and  $\lambda = \mu = 0$ . Namely,  $[F^q + F^\lambda + F^\mu]\phi$ , as a function on  $\Pi_+$  valued in the space of operators  $\dot{\Sigma} \rightarrow \Pi$ , is a polynomial of degree at most 3, with some homogeneous components  $H^{\text{cst}}, H^{\text{lin}}, H^{\text{qdr}}, H^{\text{cub}}$  of degrees 0, 1, 2, 3. Clearly,  $H^{\text{cst}}w$  equals  $-d_q\phi$  times the constant term in  $Y_w = \xi(w)X + \tau(w)c$ , that is,  $0 = H^{\text{cst}}w = (d_q\phi)\tau(w)c$ . (Our assumption is that  $H^{\text{cst}} = H^{\text{lin}} = H^{\text{qdr}} = H^{\text{cub}} = 0$ .) Hence  $d_q\phi = 0$ . Next,  $0 = H^{\text{lin}}w = 2\xi(w)\phi q + \tau(w)\lambda(c, c)X$ , and so  $q = 0$  due to linear independence of  $\xi$  and  $\tau$ . Similarly,  $H^{\text{cub}}w$  is the cubic term in  $\phi F^\mu w$ , and, therefore,

$0 = H^{\text{cub}}w = -\xi(w)\mu(X, X)X$ , so that  $\mu = 0$ . Thus,  $F^\lambda = 0$  and hence  $\lambda = 0$ , as  $\xi$  and  $\tau$  are linearly independent.

Consequently,  $\dim \dot{S} \geq 8$ , while  $\dim \dot{S} \leq 8$  according to Section 16, which shows that  $\dot{S}$  is both eight-dimensional and spanned by all  $F^q, F^\lambda, F^\mu$ , completing the proof.  $\square$

**Lemma 17.2.** *For  $F$  with  $IPF = \Xi$  and  $\llbracket F \rrbracket = 0$ , written uniquely as  $F = F^K + F^q + F^\lambda + F^\mu$ , cf. Lemma 17.1, and a function  $f : \Pi_+ \rightarrow \mathbb{R}$ , the section  $F + f\zeta$  of  $\mathcal{F}$  satisfies the conditions  $\tilde{B} = 0$  and  $\tilde{\Theta} = 0$  if and only if  $f = r\phi^3 + [\lambda(c, X) - \mu(X, X) - d_q\phi]\phi/4$  for some  $r \in \mathbb{R}$ .*

**Proof.** By (14.1.b) with  $B = K\phi^{-1}\xi \wedge \tau$  and  $\beta = \phi^{-2}\xi$  (cf. (14.3) and (14.7.a)), the condition  $\tilde{B} = 0$  is equivalent to  $K\phi^2\xi \wedge \tau = 2\xi \wedge [2\phi d_{\zeta(\cdot)}f - h(\phi F\bar{w}, \cdot)]$ . The formula for  $\phi^2 F\bar{w}$  in Lemma 17.1 gives  $h(\phi F\bar{w}, \cdot) = -[\lambda(c, c) - 2\mu(c, X) + K\phi^2/2]\tau + [\mu(X, X) - d_q\phi]\xi$ , since  $h(X, \cdot) = -\phi\tau$  and  $h(c, \cdot) = \phi\xi$  (see (14.6)). Next,  $\xi \wedge d_{\zeta(\cdot)}f = -2\phi^{-2}(d_c f)\xi \wedge \tau$ , as one sees evaluating both sides on  $(\bar{w}, w)$ , for  $w \in \dot{S}$ , and using (14.8). Thus,  $\tilde{B} = 0$  if and only if

$$(17.4) \quad 4d_c f = [\lambda(c, c) - 2\mu(c, X)]\phi.$$

Let us now assume (17.4). As  $F = F^K + F^q + F^\lambda + F^\mu$ , the formulae in Lemma 17.1 easily yield  $\phi \operatorname{div}^{\mathcal{V}}(Fw) = [\lambda(c, c) - 2\mu(c, X) + 3K\phi^2/2]\tau(w) + [2\lambda(c, X) - 3\mu(X, X) - d_q\phi]\xi(w)$  and  $3\phi h(F\bar{w}, w) = -3[\lambda(c, c) - 2\mu(c, X) + K\phi^2/2]\tau(w) + 3[\mu(X, X) - d_q\phi]\xi(w)$ , while (14.7.d) and (14.7.a) give  $-4\phi d_{\zeta w}f = 8\phi^{-1}(d_c f)\tau(w) + 8\phi^{-1}(d_X f)\xi(w)$  and  $-24\phi f\beta(w) = -24f\phi^{-1}\xi(w)$ . Adding the last four equalities side-by-side, and using the relation  $\Theta = 0$  (cf. (14.3)), we see that, by (14.1.c),  $\tilde{\Theta} = 0$  if and only if  $4(d_X f - 3f)\phi^{-1} = 2d_q\phi - \lambda(c, X)$ , or, equivalently,  $4d_X(f\phi^{-3}) = 2\phi^{-2}d_q\phi - \phi^{-2}\lambda(c, X)$ . (The terms involving  $\tau(w)$  add up to zero as a consequence of (17.4).) The system formed by this last equation and  $4d_c(f\phi^{-3}) = \phi^{-2}\lambda(c, c) - 2\phi^{-2}\mu(c, X)$  (which is immediate from (17.4) and (14.5.iii)) determines the solution  $f\phi^{-3} : \Pi_+ \rightarrow \mathbb{R}$  uniquely up to an additive constant. Our assertion now follows, since a solution may be defined by  $4f\phi^{-3} = [\lambda(c, X) - \mu(X, X) - d_q\phi]\phi^{-2}$ .  $\square$

## 18. The remaining condition in Theorem 12.2

Suppose that  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  and  $\mathcal{H}$  are chosen as in Lemma 11.2, for a fixed two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$ , while  $K$  is a given real constant,  $F$  is a section of  $\mathcal{F}$ , and the new horizontal distribution  $\tilde{\mathcal{H}} = \mathcal{H} + F$  satisfies conditions (a) – (c) in Theorem 12.2. Thus,  $F$  can be uniquely written as  $F = F^K + F^q + F^\lambda + F^\mu + f\zeta$ , with the summands defined as in Lemmas 17.1 and 17.2, and hence depending, for any fixed  $y \in \Sigma$ , on the quadruple  $(q, \lambda, \mu, r) \in V \times \mathbb{R}$ , where  $V = \Pi \times [\Pi^*]^{\odot 2} \times [\Pi^*]^{\odot 2}$  (thus,  $\dim V = 8$ ). As  $(q, \lambda, \mu, r)$  varies with  $y$ , rather than being a single element of  $V \times \mathbb{R}$ , it constitutes  $(V \times \mathbb{R})$ -valued function on some connected open set  $U$  in the affine plane  $\Sigma$ .

**Theorem 18.1.** *Under the above assumptions, given  $a \in \Pi$  with  $\Omega(a, c) = 1$ , condition (d) in Theorem 12.2 is satisfied by  $\tilde{\mathcal{H}} = \mathcal{H} + F$  if and only if, for some function  $s : U \rightarrow \mathbb{R}$ ,*

$$(18.1) \quad \begin{aligned} [2\mu_1(X, X) - \lambda_2(X, X)]\phi &= 4\lambda(c, X)\mu(X, X) - 4\mu(c, X)\lambda(X, X) - 2K\phi^2\Omega(X, q), \\ \lambda_1(c, X) + 2\Omega(X, q_2) &= 4\mu(q, X) + \Omega(X, a) - \phi \det_{\Omega}\lambda - s\phi, \end{aligned}$$

where  $X$  is the radial vector field on  $\Pi$  and  $\phi = \Omega(X, c)$ , as well as

$$(18.2) \quad K\Omega(q_1, c) + s_2 = 2K\lambda(c, q) + 8r.$$

The subscripts denote here the partial derivatives with respect to the affine coordinates  $y^j$  in  $\Sigma$  such that  $dy^1 = \xi$  and  $dy^2 = \tau$ .

The symbol  $\det_{\Omega}\lambda$  in (18.1) represents the function  $U \rightarrow \mathbb{R}$  assigning to  $y \in U$  the ratio  $(\det \Omega)^{-1}\det \lambda$ , in which  $\lambda$  stands for the value of  $\lambda$  at  $y$ , and the determinants of the bilinear forms  $\lambda, \Omega \in [\Pi^*]^{\otimes 2}$  are evaluated in any basis of  $\Pi$ . Thus, by (14.5.i),

$$(18.3) \quad [\lambda(c, X)]^2 - \lambda(c, c)\lambda(X, X) = -\phi^2 \det_{\Omega}\lambda,$$

since one may use the basis  $c, X$  (even though it depends on a point of  $\Pi_+$ ).

A proof of Theorem 18.1 will be given in Sections 20 and 21.

**Theorem 18.2.** *Under the same hypotheses as above, the assignment  $(q, \lambda, \mu, r) \mapsto (q, \lambda, \mu)$  defines a bijective correspondence between functions  $(q, \lambda, \mu, r) : U \rightarrow V \times \mathbb{R}$  satisfying conditions (18.1) – (18.2) for some  $s : U \rightarrow \mathbb{R}$ , and functions  $(q, \lambda, \mu) : U \rightarrow V$  with*

$$(18.4) \quad \begin{aligned} [2\mu_1(X, X) - \lambda_2(X, X)]\phi &= 4\lambda(c, X)\mu(X, X) - 4\mu(c, X)\lambda(X, X) - 2K\phi^2\Omega(X, q), \\ \lambda_1(c, c) + 2\Omega(c, q_2) &= 4\mu(q, c) - 1. \end{aligned}$$

**Proof.** As  $d_c\phi = 0$  by (14.5.iii), applying  $d_c$  to the second equality in (18.1) we see that (18.1) implies (18.4). Conversely, suppose that  $q, \lambda, \mu$  satisfy (18.4) and  $a \in \Pi$  is a fixed vector with  $\Omega(a, c) = 1$ . For any  $y \in \Sigma$ , if  $\psi$  denotes the restriction of the function  $\lambda_1(c, X) + 2\Omega(X, q_2) - 4\mu(q, X) - \Omega(X, a) + \phi \det_{\Omega}\lambda$  to the set  $\{y\} \times \Pi_+ \subset M$  (which we may identify with  $\Pi_+$ ), the second equality in (18.4) and (14.5.iii) give  $d_c\psi = 0$ . Since  $\psi$  is a linear functional, (14.5.iii) implies that  $\psi = -s\phi$  for some scalar  $s$ , depending on  $y$ , which yields the second relation in (18.1). Our assertion now follows, as  $q, \lambda, \mu$  uniquely determine  $s$  (and  $r$ ) via the second equality in (18.1) and then, respectively, via (18.2).  $\square$

**Example 18.3.** Let us fix  $K \in \mathbb{R}$  and  $a \in \Pi$  with  $\Omega(a, c) = 1$ . A solution  $(q, \lambda, \mu) : \Sigma \rightarrow V$  to (18.4) can obviously be defined by setting  $q = 0$ ,  $\mu = 0$  and choosing the real-valued component functions  $\lambda(c, c), \lambda(a, a), \lambda(a, c)$  of  $\lambda$  relative to the basis  $c, a$  of  $\Pi$  in such a way that  $\lambda(c, c)$  equals a constant minus the coordinate function  $y^1$ , while  $\lambda(a, a)$  and  $\lambda(a, c)$  are arbitrary functions of  $y^1$ .

**Remark 18.4.** For  $\lambda, \mu : U \rightarrow [\Pi^*]^{\otimes 2}$  as above, let  $\delta$  be the function on  $U$  valued in endomorphisms of  $\Pi$  which corresponds to  $\lambda$  via  $\Omega$ , so that  $\lambda(u, v) = \Omega(\delta u, v)$  for all  $u, v \in \Pi$ . Thus,  $\lambda(c, X)\mu(X, X) - \mu(c, X)\lambda(X, X) = \mu(c, X)\Omega(X, \delta X) + \mu(X, X)\Omega(\delta X, c)$ . Hence

$$(18.5) \quad \lambda(c, X)\mu(X, X) - \mu(c, X)\lambda(X, X) = \phi\mu(\delta X, X)$$

by (19.5) with  $v = \delta X$ . This allows us to rewrite the first equality in (18.4) or (18.1) as

$$(18.6) \quad 2\mu_1(X, X) - \lambda_2(X, X) = 4\mu(\delta X, X) - 2K\phi\Omega(X, q).$$

We also have  $\delta = \Omega^{-1}\lambda$ , meaning that  $\delta$  is, at each  $y \in U$ , the composite of  $\lambda$  and the inverse of  $\Omega$ , where the values of  $\lambda$  and  $\Omega$  at  $y$  are treated as linear operators  $\Pi \rightarrow \Pi^*$  sending  $u$  to  $\lambda(u, \cdot)$  and  $\Omega(u, \cdot)$ . Consequently, by (14.5.i), equations (18.4) now take the form

$$(18.7) \quad \begin{aligned} 2\mu_1(X, X) - \lambda_2(X, X) &= 4\mu((\Omega^{-1}\lambda)X, X) - 2K\Omega(X, c)\Omega(X, q), \\ \lambda_1(c, c) + 2\Omega(c, q_2) &= 4\mu(q, c) - 1. \end{aligned}$$

Fixing  $a \in \Pi$  such that  $\Omega(a, c) = 1$  and using it to represent  $q, \lambda$  and  $\mu$  by their components  $\mu(c, c), \mu(a, a), \mu(a, c), \lambda(c, c), \lambda(a, a), \lambda(a, c), \Omega(c, q), \Omega(a, q)$  relative to the basis  $c, a$  of  $\Pi$ , which form an octuple of functions  $U \rightarrow \mathbb{R}$ , we see that (18.7) (and, therefore, (18.4)) is

equivalent to the following system of four first-order quasi-linear partial differential equations with eight unknown real-valued functions of two real variables:

$$(18.8) \quad \begin{aligned} 2\mu_1(c, c) - \lambda_2(c, c) &= 4\lambda(c, c)\mu(a, c) - 4\mu(c, c)\lambda(a, c), \\ 2\mu_1(a, a) - \lambda_2(a, a) &= 4\lambda(a, c)\mu(a, a) - 4\mu(a, c)\lambda(a, a) - 2K\Omega(a, q), \\ 2\mu_1(a, c) - \lambda_2(a, c) &= 4\lambda(a, c)\mu(a, c) - 4\mu(c, c)\lambda(a, a) - 2K\Omega(c, q), \\ \lambda_1(c, c) + 2\Omega(c, q_2) &= 4\Omega(a, q)\mu(c, c) - 4\Omega(c, q)\mu(a, c) - 1. \end{aligned}$$

The subscripts in  $\mu_1(c, c)$ ,  $\Omega(c, q_2)$ , etc., stand for partial derivatives of  $\mu(c, c)$  and  $\Omega(c, q)$ .

In fact, one obtains (18.8) by applying  $d_c d_c$ ,  $d_a d_a$  and  $d_c d_a$  to the first equality in (18.7) and noting that  $\mu(q, c) = \Omega(a, c)\mu(q, c) = \Omega(a, q)\mu(c, c) - \Omega(c, q)\mu(a, c)$  (cf. Remark 3.4).

### 19. Some lemmas

Throughout this section we make the same assumptions as in Section 18, and  $w, w', w''$  always stand for constant vector fields on the affine plane  $\Sigma$ , treated also as  $\mathcal{V}$ -projectable sections of  $\mathcal{H}$ . As  $\xi$  and  $\tau$  are constant 1-forms on  $\Sigma$ , it follows that  $\xi(w), \tau(w), \tau(w')$ , etc., are constant functions on  $M = \Sigma \times \Pi_+$ . We also set  $\tilde{w} = w + Fw$ .

For  $U \subset \Sigma$  as in Section 18, let the functions  $Q, E, L, L^\pm : U \times \Pi_+ \rightarrow \mathbb{R}$  be given by

$$(19.1) \quad \begin{aligned} \text{i)} \quad Q &= \lambda(X, X) + 2\Omega(X, q), & E &= \lambda(c, c) - 2\mu(c, X) + K\phi^2/2, \\ \text{ii)} \quad L &= \mu(X, X) - \Omega(q, c), & L^\pm &= L \pm 2f\phi^{-1}. \end{aligned}$$

Since  $F = F^K + F^q + F^\lambda + F^\mu + f\zeta$ , Lemmas 17.1 – 17.2, (14.5) and (9.1.c) yield

$$(19.2) \quad \begin{aligned} \text{i)} \quad 2L^+ &= \mu(X, X) + \lambda(c, X) - 3\Omega(q, c) + 4r\phi^2, \\ \text{ii)} \quad 2L^- &= 3\mu(X, X) - \lambda(c, X) - \Omega(q, c) - 4r\phi^2, \\ \text{iii)} \quad \phi Fw &= [E\tau(w) - L^+\xi(w)]X + [L^-\tau(w) + Q\xi(w)]c, \\ \text{iv)} \quad d_{Fw}\phi &= E\tau(w) - L^+\xi(w). \end{aligned}$$

(By (14.5.i) and Remark 3.4,  $\phi = \Omega(X, c)$  and  $\Omega(q, c)X + \Omega(X, q)c + \Omega(c, X)q = 0$ , so that  $q = \phi^{-1}\Omega(q, c)X + \phi^{-1}\Omega(X, q)c$ , while (14.7.d) gives  $\zeta w = -2\phi^{-2}Y_w$ .) Now, from (14.6),

$$(19.3) \quad h(Fw, w') = Q\xi(w)\xi(w') + L^+\xi(w)\tau(w') + L^-\tau(w)\xi(w') - E\tau(w)\tau(w').$$

**Remark 19.1.** Any degree  $k$  homogeneous polynomial function on  $\Pi$ , valued in an arbitrary finite-dimensional vector space, is

- (a) an eigenvector of  $d_X$  for the eigenvalue  $k$ ,
- (b) an eigenvector of  $d_X - \text{Id}$  for the eigenvalue  $k - 1$ .

(In fact, (b) is obvious from (a).) Examples of such functions include  $X$  (valued in  $\Pi$ , with  $k = 1$ ), as well as the real-valued functions such as  $\phi$  or  $\lambda(c, X)$  (with  $k = 1$ ) and  $\mu(X, X)$  (with  $k = 2$ ). Thus, for instance,  $d_X\phi = \phi$ .

**Lemma 19.2.** *The Lie bracket  $[Fw, Fw']$  equals  $\phi^{-2}(\xi \wedge \tau)(w, w')$  times*

$$[(2Q - d_X Q)E + (2L^- - d_X L^-)L^+ + Qd_c L^- - L^- d_c Q]c + (Ed_X L^+ - L^+ d_X E + L^- d_c L^+ + Qd_c E)X.$$

**Proof.** We have  $\phi^2[Fw, Fw'] = [\phi Fw, \phi Fw'] - \phi(d_{Fw}\phi)Fw' + \phi(d_{Fw'}\phi)Fw$ . By (19.2.iv), (19.2.iii) and (3.1.ii), the last two terms add up to  $(L^+L^- + EQ)[(\xi \wedge \tau)(w, w')]c$ . As  $[c, X] = d_c X = c$ , our assertion follows if we evaluate  $[\phi Fw, \phi Fw']$  using (19.2.iii) and (3.1.ii).  $\square$

**Lemma 19.3.** *The  $h$ -inner product  $h([Fw', Fw''], w)$  equals  $\zeta(w', w'')/2$  times*

$$\begin{aligned} [(2Q - d_X Q)E + (2L^- - d_X L^-)L^+ + Qd_c L^- - L^- d_c Q]\xi(w) \\ + (L^+ d_X E - Ed_X L^+ - L^- d_c L^+ - Qd_c E)\tau(w). \end{aligned}$$

**Proof.** This is obvious from Lemma 19.2, (14.6) and (14.7.c).  $\square$

As in the statement of Theorem 18.1, here and in the next section the subscripts  $( )_j$ ,  $j = 1, 2$ , denote the directional derivatives in the directions of the constant vector fields  $\partial_j$  on  $\Sigma$  forming the basis of  $\dot{\Sigma}$  dual to the basis  $\xi, \tau$  of  $\dot{\Sigma}^*$ . In other words,  $( )_j = \partial/\partial y^j$  for the affine coordinates  $y^j$  with  $dy^1 = \xi$  and  $dy^2 = \tau$ .

**Remark 19.4.** For any function  $\chi$  on  $\Sigma$  one has  $d_w[\chi\xi(w')] - d_{w'}[\chi\xi(w)] = -\chi_2(\xi \wedge \tau)(w, w')$  and  $d_w[\chi\tau(w')] - d_{w'}[\chi\tau(w)] = \chi_1(\xi \wedge \tau)(w, w')$ , since  $d\chi = \chi_1\xi + \chi_2\tau$ . (Cf. (3.1).)

**Lemma 19.5.** *The expression  $h([w', Fw''], w) + h([Fw', w''], w)$  equals  $(\xi \wedge \tau)(w, w')$  times  $(L_1^- - Q_2)\xi(w) - (E_1 + L_2^+)\tau(w)$ , that is,  $\zeta(w', w'')/2$  times  $(L_1^- - Q_2)\phi\xi(w) - (E_1 + L_2^+)\phi\tau(w)$ .*

**Proof.** Treating  $Fw'$  and  $[w, Fw']$  as functions  $U \rightarrow \Pi$ , we have  $[w, Fw'] = d_w(Fw')$ , so that, by (19.2.iv) and Remark 19.4,  $\phi[w, Fw'] + \phi[Fw', w'] = d_w(\phi Fw') - d_{w'}(\phi Fw)$  equals  $(\xi \wedge \tau)(w, w')$  times  $(L_1^- - Q_2)c + (E_1 + L_2^+)X$ . Now (14.5) and (14.7.c) yield our claim.  $\square$

**Lemma 19.6.** *If  $\tilde{w} = w + Fw$ , then*

$$\begin{aligned} 8\tilde{\gamma}(\tilde{w}) &= 2[3\mu_1(X, X) - 2\lambda_2(X, X) - 4\Omega(X, q_2) - \Omega(q_1, c) - \lambda_1(c, X) - 4\phi^2 r_1]\phi\xi(w) \\ &\quad + 2[4\mu_1(c, X) - \mu_2(X, X) + 3\Omega(q_2, c) - \lambda_2(c, X) - 2\lambda_1(c, c) - 4\phi^2 r_2]\phi\tau(w) \\ &\quad + [12\mu(c, X)\lambda(X, X) - 13\lambda(c, X)\mu(X, X) + 4K\phi^2\Omega(X, q) + 12r\phi^2\lambda(c, X)]\xi(w) \\ &\quad + [10\Omega(q, c)\mu(X, X) - 2\lambda(c, c)\lambda(X, X) + 3(\lambda(c, X))^2]\xi(w) \\ &\quad + [8\mu(c, X)\Omega(X, q) - 24r\Omega(q, c)\phi^2]\xi(w) \\ &\quad + [4\lambda(c, c)\Omega(X, q) + \Omega(q, c)\lambda(c, X)]\xi(w) + 2[\Omega(q, c)]^2\xi(w) \\ &\quad + [24r\phi^2\mu(c, X) - 2\mu(c, X)\mu(X, X) + K\phi^2\lambda(c, X)]\tau(w) \\ &\quad + [8\mu(c, c)\lambda(X, X) - 7\lambda(c, c)\mu(X, X)]\tau(w) \\ &\quad + [2\lambda(c, X)\mu(c, X) - 12r\lambda(c, c)\phi^2 - 6K\Omega(q, c)\phi^2]\tau(w) \\ &\quad + [16\mu(c, c)\Omega(X, q) - \lambda(c, c)\lambda(c, X) + 14\Omega(q, c)\mu(c, X)]\tau(w) \\ &\quad + \lambda(c, c)\Omega(q, c)\tau(w). \end{aligned}$$

**Proof.** As  $\gamma = 0$  (see Lemma 11.2) and  $[w', w''] = 0$ , (13.3.d) implies that  $\tilde{\gamma}(\tilde{w})\zeta(w', w'')$  equals the sum of the inner-product expressions appearing in Lemmas 19.3 and 19.5. Our assertion now follows from these two lemmas, (19.1.i), (19.2.i), (19.2.ii) and Remark 19.1. (By Remark 19.1(b),  $2Q - d_X Q = 2\Omega(X, q)$  and  $2L^- - d_X L^- = -\Omega(q, c) - \lambda(c, X)/2$ .)  $\square$

For  $f$  as in Lemma 17.2, any fixed vector  $a \in \Pi$  with  $\Omega(a, c) = 1$ , and any section  $v$  of  $\mathcal{V}$ ,

$$\begin{aligned} \text{a)} \quad &4\tilde{\gamma}(v) = (E + K\phi^2/2)\Omega(X, v) - (L + 4r\phi^2)\Omega(v, c), \\ \text{b)} \quad &\phi\tilde{\alpha}(\tilde{w}) = (L + 4r\phi^2)\xi(w) - (E + K\phi^2/2)\tau(w), \\ \text{c)} \quad &\tilde{\alpha}(\tilde{w}) + \phi^{-1}d_{Fw}\phi = -2A\xi(w) - K\phi\tau(w)/2, \quad \text{for } A = f\phi^{-2} - 2r\phi, \\ \text{d)} \quad &\phi^{-2}h(v, w) = d_v[\phi^{-1}\tau(w) - \phi^{-1}\Omega(X, a)\xi(w)], \\ \text{e)} \quad &\phi^{-2}\tilde{\gamma}(v) = d_v A + K\Omega(X, v)/4, \quad \text{where } A = f\phi^{-2} - 2r\phi. \end{aligned} \tag{19.4}$$

In fact, (13.3.c) with  $[F] = f$  and  $\gamma = 0$  (cf. Section 17 and Lemma 11.2) yields  $\tilde{\gamma}(v) = -\phi d_v(f\phi^{-1}) + \Omega(EX + L^-c, v)/2$ , since  $F\bar{w} = \phi^{-2}(EX + L^-c)$  (from (19.2.iii) and (14.8)), while  $\phi^{-2}\theta = \Omega$  (by (14.7.b)), and  $\alpha(v) = -\phi^{-1}d_v\phi$  (see (8.4) and (9.2)). Thus, (14.5) and the formula for  $f$  in Lemma 17.2 yield  $4\tilde{\gamma}(v) = 2E\Omega(X, v) - (2L^- + 8r\phi^2)\Omega(v, c) + [2\mu(v, X) - \lambda(c, v)]\phi$ . Relation (19.4.a), that is, vanishing of  $4\tilde{\gamma}(v) - (E + K\phi^2/2)\Omega(X, v) + (L + 4r\phi^2)\Omega(v, c)$ , is now immediate from (19.1.i), as (19.2.ii) gives  $2L^- + 8r\phi^2 = 3\mu(X, X) - \lambda(c, X) - \Omega(q, c) + 4r\phi^2$ , while

$$\begin{aligned} \text{(19.5)} \quad &\phi\lambda(c, v) = \Omega(v, c)\lambda(c, X) + \lambda(c, c)\Omega(X, v), \\ &\phi\mu(v, X) = \mu(c, X)\Omega(X, v) + \mu(X, X)\Omega(v, c), \end{aligned}$$



as a consequence of (14.5.i) and Remark 3.4.

Since  $\tilde{\alpha}(\tilde{w}) = 2\tilde{\gamma}(\zeta\tilde{w}) = 2\tilde{\gamma}(\zeta w)$  (see Section 10, (13.1) and (8.1.h)), (19.4.b) follows from (19.4.a), (14.7.d), (9.1.c) and (14.5.i).

Next, (19.4.b) and (19.2.iv) give  $[\tilde{\alpha}(\tilde{w}) + \phi^{-1}d_{Fw}\phi]\phi = (L - L^+ + 4r\phi^2)\xi(w) - K\phi^2\tau(w)/2$ . As  $L - L^+ = -2f\phi^{-1}$  (see (19.1.ii)), this equals  $-[2A\xi(w) + K\phi\tau(w)/2]\phi$ , which yields (19.4.c).

Subtracting the left-hand side of (19.4.d) from its right-hand side, and evaluating the difference with the aid of (14.5.ii) and (9.1), we obtain zero, since (14.5.i) and Remark 3.4 give  $\Omega(v, c)\Omega(X, a) - \Omega(v, a)\phi = \Omega(v, X)\Omega(c, a) = \Omega(X, v)$ . We thus obtain (19.4.d).

Finally, using differentiation by parts and (14.5.ii), we obtain  $\phi^2d_vA = \phi d_v(\phi A) - \Omega(v, c)\phi A$ , for  $A$  as in (19.4.e). Since, by Lemma 17.2 and (14.5.ii),

$$(19.6) \quad 4\phi A = \lambda(c, X) - \mu(X, X) - \Omega(q, c) - 4r\phi^2,$$

the preceding equality, (19.4.a), (14.5.ii) and the formulae for  $E$  and  $L$  in (19.1) yield an expression for  $4\tilde{\gamma}(v) - 4\phi^2d_vA$  involving  $K, q, \lambda, \mu, r$  and  $\Omega, c, X, \phi, v$  (but not  $f, Q, E, L$  or  $L^\pm$ ), which equals  $K\phi^2\Omega(X, v)$  in view of (19.5). This proves (19.4.e).

## 20. Proof of Theorem 18.1, first part

We use the same assumptions and notations as at the beginning of Section 19.

Condition (d) in Theorem 12.2 may be naturally split into two parts, which read

$$(20.1) \quad \text{i) } (d\tilde{\gamma} + 2\tilde{\alpha} \wedge \tilde{\gamma})(v, \tilde{w}) = h(v, w)/2, \quad \text{ii) } (d\tilde{\gamma} + 2\tilde{\alpha} \wedge \tilde{\gamma})(\tilde{w}, \tilde{w}') = 0$$

for all sections  $w, w'$  of  $\mathcal{H}$  and  $v$  of  $\mathcal{V}$ , where  $\tilde{w} = w + Fw$  and  $\tilde{w}' = w' + Fw'$ . By (3.1),  $(d\tilde{\gamma} + 2\tilde{\alpha} \wedge \tilde{\gamma})(v, \tilde{w}) - h(v, w)/2 = d_v[\tilde{\gamma}(\tilde{w})] - 2\phi^{-1}(d_v\phi)\tilde{\gamma}(\tilde{w}) - d_{\tilde{w}}[\tilde{\gamma}(v)] - 2\tilde{\alpha}(\tilde{w})\tilde{\gamma}(v) - \tilde{\gamma}([v, \tilde{w}]) - h(v, w)/2$ , since, according to Section 10, (8.4) and (9.2),  $\tilde{\alpha}(v) = \alpha(v) = -\phi^{-1}d_v\phi$ . Multiplied by  $\phi^{-2}$ , this becomes

$$(20.2) \quad d_v[\phi^{-2}\tilde{\gamma}(\tilde{w})] - d_{\tilde{w}}[\phi^{-2}\tilde{\gamma}(v)] - \phi^{-2}\tilde{\gamma}([v, \tilde{w}]) - 2[\tilde{\alpha}(\tilde{w}) + \phi^{-1}d_{Fw}\phi]\phi^{-2}\tilde{\gamma}(v) - \phi^{-2}h(v, w)/2,$$

where we have first rewritten  $\phi^{-2}d_{\tilde{w}}[\tilde{\gamma}(v)]$  as  $d_{\tilde{w}}[\phi^{-2}\tilde{\gamma}(v)] + 2\phi^{-3}(d_{\tilde{w}}\phi)\tilde{\gamma}(v)$ , using differentiation by parts, and then noted that  $d_{\tilde{w}}\phi = d_{w+Fw}\phi = d_{Fw}\phi$ .

**Lemma 20.1.** *For  $A$  given by (19.6) and a fixed vector  $a \in \Pi$  with  $\Omega(a, c) = 1$ , the expression (20.2), that is,  $[(d\tilde{\gamma} + 2\tilde{\alpha} \wedge \tilde{\gamma})(v, \tilde{w}) - h(v, w)/2]\phi^{-2}$ , is the result of applying  $d_v$  to*

$$(20.3) \quad \begin{aligned} & \phi^{-2}\tilde{\gamma}(\tilde{w}) + (2\phi)^{-1}\Omega(X, a)\xi(w) - (2\phi)^{-1}\tau(w) - d_{\tilde{w}}A + 2A^2\xi(w) + K\phi A\tau(w) \\ & + K[\Omega(X, q) + \lambda(X, X)/4]\xi(w) - K\Omega(X, Fw/4) + K[\mu(X, X) - \lambda(c, X)]\tau(w)/2. \end{aligned}$$

**Proof.** The first and last terms in (20.2) are the  $d_v$ -images of the first three terms in (20.3), cf. (19.4.d). In the sum of the remaining three terms in (20.2), let us replace  $\tilde{\alpha}(\tilde{w}) + \phi^{-1}d_{Fw}\phi$  by the right-hand side of (19.4.c),  $\phi^{-2}\tilde{\gamma}(v)$  by the right-hand side of (19.4.e), and  $\phi^{-2}\tilde{\gamma}([v, \tilde{w}])$  by an analogous expression involving, instead of  $v$ , the section  $[v, \tilde{w}]$  of  $\mathcal{V}$  (see Remark 3.1). After rearranging terms and noting that  $K\phi(d_vA)\tau(w) = d_v[K\phi A\tau(w)] - KA\Omega(v, c)\tau(w)$  (cf. (14.5.ii)), the result is

$$(20.4) \quad -d_{\tilde{w}}d_vA - d_{[v, \tilde{w}]}A + 4A(d_vA)\xi(w) + d_v[K\phi A\tau(w)] - d_v[K\Omega(X, Fw/4)]$$

plus, as explained next,  $K/4$  times

$$(20.5) \quad 4A[\Omega(X, v)\xi(w) - \Omega(v, c)\tau(w)] + K\phi\Omega(X, v)\tau(w) + 2\Omega(v, Fw).$$

Namely, the terms  $-K[\Omega(d_{\tilde{w}}X, v) + \Omega(X, [v, \tilde{w}])]/4$ , originally present in the resulting expression, can be rewritten as  $-d_v[K\Omega(X, Fw/4)]$  plus  $K/4$  times  $2\Omega(v, Fw)$ , since  $[v, \tilde{w}] = [v, w + Fw] = [v, Fw] = d_v(Fw)$ , while  $d_vX = v$  and  $d_{\tilde{w}}X = d_{w+FW}X = d_{FW}X = Fw$ .

As  $-d_{\tilde{w}}d_v - d_{[v, \tilde{w}]} = -d_vd_{\tilde{w}}$ , (20.4) is the  $d_v$ -image of the sum of the fourth, fifth and sixth terms in (20.3) along with the middle term in the second line of (20.3). Next, cancelling the  $K$  factor in the second line in (20.3), minus the middle term, and then applying  $4d_v$ , we get

$$(20.6) \quad [4\Omega(v, q) + 2\lambda(v, X)]\xi(w) + [4\mu(v, X) - 2\lambda(c, v)]\tau(w),$$

Therefore, our claim will follow if we show that (20.5) equals (20.6). To this end, let us rewrite (20.5), noting that  $4A = [\lambda(c, X) - \mu(X, X) - \Omega(q, c)]\phi^{-1} - 4r\phi$  (see (19.6)), and  $Fw$ , in  $\Omega(v, Fw)$ , may be replaced by  $\phi^{-1}$  times the right-hand side of (19.2.iii). Substituting for the resulting occurrences of  $Q, E$  and  $L^\pm$  the expressions in (19.1.i), (19.2.i) and (19.2.ii), and using (19.5) along with the equalities  $\lambda(c, X)\Omega(X, v) + \lambda(X, X)\Omega(v, c) = \lambda(v, X)\phi$  and  $\Omega(v, c)\Omega(X, q) + \Omega(q, c)\Omega(v, X) = \Omega(v, q)\phi$ , both of which immediate from (14.5.i) and Remark 3.4, we see that (20.5) in fact coincides with (20.6).  $\square$

In the whole discussion following (20.1), which includes Lemma 20.1 and its proof, we never made use of the fact that  $\tilde{\gamma}(\tilde{w})$  is a specific function  $U \times \Pi_+ \rightarrow \mathbb{R}$ , given by the formula in Lemma 19.6. This now allows us to solve (20.1.i) as a system of differential equations imposed on  $\tilde{\gamma}(\tilde{w})$  treated  $\tilde{\gamma}(\tilde{w})$  as an arbitrary function. The only assumption made about  $\tilde{\gamma}(\tilde{w})$  is that its dependence on  $w$  (via the relation  $\tilde{w} = w + Fw$ ) should be valuewise and linear, or, equivalently,  $\tilde{\gamma}(\tilde{w})$  is a combination of  $\xi(w)$  and  $\tau(w)$  with some coefficients which are functions on a connected open set in  $M$ .

**Lemma 20.2.** *Solving the system (20.1.i) for the unknown function  $\tilde{\gamma}(\tilde{w})$ , the dependence of which on  $w$  is valuewise and linear, we obtain*

$$\begin{aligned} 8\tilde{\gamma}(\tilde{w}) &= 2[\lambda_1(c, X) - \mu_1(X, X) - \Omega(q_1, c) - 4\phi^2r_1]\phi\xi(w) \\ &+ 2[\lambda_2(c, X) - \mu_2(X, X) - \Omega(q_2, c) - 4\phi^2r_2]\phi\tau(w) \\ &+ [3\lambda(c, X)\mu(X, X) - 4\mu(c, X)\lambda(X, X) - 4K\phi^2\Omega(X, q) + 12r\phi^2\lambda(c, X)]\xi(w) \\ &+ [2\lambda(c, c)\lambda(X, X) - (\lambda(c, X))^2 - 6\Omega(q, c)\mu(X, X)]\xi(w) \\ &- [8\mu(c, X)\Omega(X, q) + 24r\Omega(q, c)\phi^2 + 4\phi\Omega(X, a) - 4s\phi^2]\xi(w) \\ &+ [4\lambda(c, c)\Omega(X, q) + \Omega(q, c)\lambda(c, X)]\xi(w) + 2[\Omega(q, c)]^2\xi(w) \\ &+ [24r\phi^2\mu(c, X) - 2\mu(c, X)\mu(X, X) + K\phi^2\lambda(c, X)]\tau(w) \\ &+ [\lambda(c, c)\mu(X, X) + 2\lambda(c, X)\mu(c, X) - 12r\lambda(c, c)\phi^2 + 2K\Omega(q, c)\phi^2 + 4t\phi^2]\tau(w) \\ &+ [4\phi - \lambda(c, c)\lambda(c, X) - 2\Omega(q, c)\mu(c, X)]\tau(w) + \lambda(c, c)\Omega(q, c)\tau(w), \end{aligned}$$

where  $a \in \Pi$  is fixed, with  $\Omega(a, c) = 1$ , and  $s, t$  are arbitrary functions defined on an open subset of  $\Sigma$ , so that, as functions in  $M = \Sigma \times \Pi_+$ , they are constant in the  $\Pi_+$  direction.

**Proof.** The dependence of  $\tilde{\gamma}(\tilde{w})$  on  $w$  is assumed to be valuewise and linear. Thus, by Lemma 20.1, a function  $\tilde{\gamma}(\tilde{w})$  is a solution to (20.1.i) if and only if (20.3) is constant in the  $\Pi_+$  direction, that is, equal to  $[s\xi(w) + t\tau(w)]/2$  for some functions  $s, t$  defined on an open set in  $\Sigma$ . On the other hand,  $8\phi^2$  times (20.3) equals  $4[s\xi(w) + t\tau(w)]\phi^2$  if and only if  $8\tilde{\gamma}(\tilde{w})$  is given by the formula displayed in the lemma, as one easily verifies using the last identity in

Remark 19.4, for  $\chi = 4\phi A$ , and (19.6), along with four equalities, justified below:

$$(20.7) \quad \begin{aligned} & \text{a) } 4\phi^2 d_{\tilde{w}} A = \phi d_w(4\phi A) + [\lambda(c, c) - 2\mu(c, X)][L^-\tau(w) + Q\xi(w)] \\ & \quad + [\Omega(q, c) - \mu(X, X) - 4r\phi^2][E\tau(w) - L^+\xi(w)], \\ & \text{b) } 16\phi^2 A^2 = [\lambda(c, X) - \mu(X, X) - \Omega(q, c) - 4r\phi^2]^2, \\ & \text{c) } 4K\phi A = K[\lambda(c, X) - \mu(X, X) - \Omega(q, c) - 4r\phi^2], \\ & \text{d) } K\phi\Omega(X, Fw) = K[L^-\tau(w) + Q\xi(w)]\phi, \end{aligned}$$

and then replacing  $Q, E, L^\pm$  by right-hand sides of (19.1.i), (19.2.i) and (19.2.ii).

Equalities in (20.7.b,c,d) are obvious: (20.7.b,c) from (19.6), and (20.7.d) from (19.2.iii) along with (14.5.i). As for (20.7.a), it follows since  $\tilde{w} = w + Fw$  and  $4\phi^2 d_{Fw} A = d_{\phi Fw}(4\phi A) - (4\phi A)d_{Fw}\phi$  (differentiation by parts), which, by (19.2), equals  $[E\tau(w) - L^+\xi(w)](d_X - \text{Id})(4\phi A)$  plus  $[L^-\tau(w) + Q\xi(w)]d_c(4\phi A)$ , while (19.6) gives  $(d_X - \text{Id})(4\phi A) = \Omega(q, c) - \mu(X, X) - 4r\phi^2$  (see Remark 19.1(b)) and  $d_c(4\phi A) = \lambda(c, c) - 2\mu(c, X)$  (cf. (14.5.iii)).  $\square$

**Remark 20.3.** Conditions necessary and sufficient for equality of the right-hand sides of the two formulae for  $8\tilde{\gamma}(\tilde{w})$ , provided by Lemmas 19.6 and 20.2, can be described as follows. The coefficients of both  $\xi(w)$  and  $\tau(w)$  in the two right-hand sides, when restricted to  $\{y\} \times \Pi_+ \approx \Pi_+$  for any given  $y \in \Sigma$ , are cubic polynomial functions in  $\Pi$ , and so one may proceed by equating their cubic, quadratic, linear and constant homogeneous components.

(a) Equality between the cubic or, respectively, quadratic homogeneous components of the coefficient of  $\xi(w)$  is equivalent to the first or, respectively, second relation in (18.1), as one easily sees using (18.3) and noting that  $\Omega(q, c)\mu(X, X) + \mu(c, X)\Omega(X, q) = \mu(q, X)\phi$ , in view of Remark 3.4 and (14.5.i).

(b) The quadratic or, respectively, linear homogeneous components of the coefficient of  $\tau(w)$  in the two right-hand sides are equal if and only if

$$(20.8) \quad [2\mu_1(c, X) - \lambda_2(c, X)]\phi = 2\lambda(c, c)\mu(X, X) - 2\mu(c, c)\lambda(X, X) + 2K\Omega(q, c)\phi^2 + t\phi^2,$$

or, respectively,  $[\lambda_1(c, c) + 2\Omega(c, q_2)]\phi = 4\mu(c, c)\Omega(X, q) + 4\Omega(q, c)\mu(c, X) - \phi$ . The last equation may also be rewritten as  $\lambda_1(c, c) + 2\Omega(c, q_2) = 4\mu(c, q) - 1$ , since, according to Remark 3.4 and (14.5.i),

$$(20.9) \quad \mu(c, c)\Omega(X, q) + \Omega(q, c)\mu(c, X) = \mu(c, q)\Omega(X, c) = \mu(c, q)\phi.$$

(c) The corresponding equalities involving other homogeneous components of either coefficient function are always satisfied.

(d) Subtracting from (20.8) one-half of the equation obtained by applying  $d_c$  to the first equality in (18.1), and using (14.5.iii), we get  $t = K\Omega(c, q)$ . Similarly,  $d_c$  applied to the second equality in (18.1) yields  $\lambda_1(c, c) + 2\Omega(c, q_2) = 4\mu(c, q) - 1$ .

**Lemma 20.4.** *Under the hypotheses of Theorem 18.1, condition (20.1.i) is satisfied by all sections  $w$  of  $\mathcal{H}$  and  $v$  of  $\mathcal{V}$ , with  $\tilde{w} = w + Fw$ , if and only if (18.1) holds for some function  $s : U \rightarrow \mathbb{R}$ .*

*Furthermore, the relations (18.1), for any specific function  $s$ , imply that  $8\tilde{\gamma}(\tilde{w})$  is given by the formula in Lemma 20.2 with this  $s$  and  $t = K\Omega(c, q)$ . In addition, then*

$$(20.10) \quad \begin{aligned} & [2\mu_1(c, X) - \lambda_2(c, X)]\phi = 2\lambda(c, c)\mu(X, X) - 2\mu(c, c)\lambda(X, X) + K\Omega(q, c)\phi^2, \\ & \lambda_1(c, c) + 2\Omega(c, q_2) = 4\mu(c, q) - 1. \end{aligned}$$

**Proof.** Condition (20.1.i) is clearly equivalent to equality of the right-hand sides in Lemmas 19.6 and 20.2, for suitably chosen functions  $s, t : U \rightarrow \mathbb{R}$ . Thus, according to Remark 20.3,



Also,  $d_w[\tilde{\gamma}(\tilde{w}')] - d_{w'}[\tilde{\gamma}(\tilde{w})] = (N_1 - G_2)[(\xi \wedge \tau)(w, w')]$ . In fact, setting  $\chi = \tilde{\gamma}(\tilde{w}')$ , we get  $d_w \chi = \chi_1 \xi(w') + \chi_2 \tau(w')$  from Remark 19.4, while, by (21.4.i),  $\chi_j = (T_{1j} + G_j)\xi(w) + (T_{2j} + N_j)\tau(w)$  for  $j = 1, 2$ , and our claim follows from (3.1.ii), as  $T_{12} = T_{21}$ .

Finally, (19.4.b), (21.4.i) and (3.1.ii) imply that  $[(\tilde{\alpha} \wedge \tilde{\gamma})(\tilde{w}, \tilde{w}')] \phi$  equals  $(\xi \wedge \tau)(w, w')$  times  $(L + 4r\phi^2)(T_2 + N) + (E + K\phi^2/2)(T_1 + G)$ .

Our assertion is now immediate from (3.1.iii) and the above equalities, including (21.4.i).  $\square$

The conclusion of Theorem 18.1 is a trivial consequence of the next result. In fact, of the two conditions (20.1.i) and (20.1.ii), together constituting (d) in Theorem 12.2, the first holds, according to Lemma 20.4, if and only if (18.1) does. Under the assumption (20.1.i) (or, (18.1)), made throughout this section, (20.1.ii) amounts, by Lemma 21.1, to vanishing of  $\psi$  in (21.3), while (21.5) shows that this is equivalent to (18.2).

**Theorem 21.2.** *The function  $\psi$  in Lemma 21.1 can also be expressed as*

$$(21.5) \quad \psi = [2K\lambda(c, q) + 8r - K\Omega(q_1, c) - s_2]\phi^3.$$

**Proof.** We need to evaluate the right-hand side of (21.3), replacing  $T, G, N$  (or,  $Q, E, L$ ) with the expressions provided by (21.2) (or, (19.1)), and  $L^\pm$  with the right-hand sides in (19.2.i) – (19.2.ii). To make this task manageable, we note that the ingredients of (21.3), when restricted to  $\{y\} \times \Pi_+ \approx \Pi_+$  for any given  $y \in \Sigma$ , are polynomial functions in  $\Pi$ , of degrees which are, at most: one (for  $\phi$ ), two (for  $Q, E, L, L^\pm$  as well as  $d_c(T_2 + N)$  and  $d_c(T_1 + G)$ ), three (for  $N_1 - G_2$  along with  $T_2 + N, T_1 + G$  and their  $d_X$ -images), and, consequently, five for  $\psi$ . Rather than describing  $\psi$  directly, we will derive formulae for its homogeneous components  $\psi^{\text{qnt}}, \psi^{\text{qrt}}, \psi^{\text{cub}}, \psi^{\text{qdr}}, \psi^{\text{lin}}, \psi^{\text{cst}}$  of degrees 5, 4, 3, 2, 1 and 0. The number or the resulting terms will be reduced, since some of them can be consolidated or eliminated due to the fact that

$$(21.6) \quad Q^{\text{cst}} = (L + 4r\phi^2)^{\text{lin}} = [d_X(\dots)]^{\text{cst}} = [d_c(\dots)]^{\text{cub}} = 0.$$

where ... stands for  $T_2 + N$  or  $T_1 + G$ , and each homogeneous component of  $d_X(T_2 + N)$  or  $d_X(T_1 + G)$  equals the degree times  $T_2 + N$  or  $T_1 + G$  (cf. Remark 19.1(b)).

Proceeding as stated above, we will show that  $\psi^{\text{qnt}} = \psi^{\text{cst}} = \psi^{\text{lin}} = \psi^{\text{qrt}} = \psi^{\text{qdr}} = 0$ , while  $\psi^{\text{cub}}$  equals the right-hand side of (21.5). Specifically,

$$\begin{aligned} \psi^{\text{qnt}} &= 3(L + 4r\phi^2 - 2L^+)^{\text{qdr}}(T_2 + N)^{\text{cub}} - 3(E - K\phi^2/2)^{\text{qdr}}(T_1 + G)^{\text{cub}}, \\ \psi^{\text{cst}} &= 3(L + 4r\phi^2)^{\text{cst}}(T_2 + N)^{\text{cst}} + 3(E + K\phi^2/2)^{\text{cst}}(T_1 + G)^{\text{cst}} - (2L^-)^{\text{cst}}d_c[(T_1 + G)^{\text{lin}}], \\ \psi^{\text{lin}} &= 2(N_1 - G_2)^{\text{cst}}\phi + (3L + 12r\phi^2 - 2L^+)^{\text{cst}}(T_2 + N)^{\text{lin}} + (E + 3K\phi^2/2)^{\text{cst}}(T_1 + G)^{\text{lin}} \\ &\quad + 3(E + K\phi^2/2)^{\text{lin}}(T_1 + G)^{\text{cst}} + 2Q^{\text{lin}}d_c[(T_2 + N)^{\text{lin}}] \\ &\quad - (2L^-)^{\text{lin}}d_c[(T_1 + G)^{\text{lin}}] - (2L^-)^{\text{cst}}d_c[(T_1 + G)^{\text{qdr}}], \\ \psi^{\text{qrt}} &= 2(N_1 - G_2)^{\text{cub}}\phi + (3L + 12r\phi^2 - 6L^+)^{\text{lin}}(T_2 + N)^{\text{cub}} \\ &\quad + (3L + 12r\phi^2 - 4L^+)^{\text{qdr}}(T_2 + N)^{\text{qdr}} \\ &\quad - 3(E - K\phi^2/2)^{\text{lin}}(T_1 + G)^{\text{cub}} - (E - 3K\phi^2/2)^{\text{qdr}}(T_1 + G)^{\text{qdr}} \\ &\quad + 2Q^{\text{qdr}}d_c[(T_2 + N)^{\text{cub}}] - (2L^-)^{\text{qdr}}d_c[(T_1 + G)^{\text{cub}}]. \end{aligned}$$

As  $(L + 4r\phi^2 - 2L^+)^{\text{qdr}} = (E - K\phi^2/2)^{\text{qdr}} = 0$ , we thus have  $\psi^{\text{qnt}} = 0$ . To conclude that  $\psi^{\text{cst}}, \psi^{\text{lin}}$  and  $\psi^{\text{qrt}}$  vanish as well, we use the relations

$$\begin{aligned} 8(T_2 + N)^{\text{cst}} &= \lambda(c, c)\Omega(q, c), \quad 4(T_1 + G)^{\text{cst}} = [\Omega(q, c)]^2, \\ 8(T_2 + N)^{\text{lin}} &= 2\Omega(c, q_2)\phi + 4\phi - 2\Omega(q, c)\mu(c, X) - \lambda(c, c)\lambda(c, X), \\ 8(T_1 + G)^{\text{lin}} &= 2\Omega(c, q_1)\phi + \Omega(q, c)\lambda(c, X) + 4\lambda(c, c)\Omega(X, q), \\ 8(T_2 + N)^{\text{qdr}} &= 2\lambda_2(c, X)\phi + \lambda(c, c)\mu(X, X) + 2\lambda(c, X)\mu(c, X) - 12r\lambda(c, c)\phi^2 - 2K\Omega(q, c)\phi^2, \\ 8(T_1 + G)^{\text{qdr}} &= 2\lambda_1(c, X)\phi + 2\lambda(c, c)\lambda(X, X) - [\lambda(c, X)]^2 - 6\Omega(q, c)\mu(X, X) \\ &\quad - 8\mu(c, X)\Omega(X, q) - 24r\Omega(q, c)\phi^2 - 4\phi\Omega(X, a) + 4s\phi^2, \\ 8(T_2 + N)^{\text{cub}} &= -2\phi\mu_2(X, X) - 8r_2\phi^3 + 24r\phi^2\mu(c, X) - 2\mu(c, X)\mu(X, X) + K\phi^2\lambda(c, X), \\ 8(T_1 + G)^{\text{cub}} &= -2\phi\mu_1(X, X) - 8r_1\phi^3 + 3\lambda(c, X)\mu(X, X) - 4\mu(c, X)\lambda(X, X) \\ &\quad - 4K\phi^2\Omega(X, q) + 12r\phi^2\lambda(c, X). \end{aligned}$$

Since  $\psi^{\text{cst}} = -3\Omega(q, c)(T_2 + N)^{\text{cst}} + 3\lambda(c, c)(T_1 + G)^{\text{cst}} + \Omega(q, c)d_c[(T_1 + G)^{\text{lin}}]$ , and, by (14.5.iii),  $8d_c[(T_1 + G)^{\text{lin}}] = -3\Omega(q, c)\lambda(c, c)$ , we get  $\psi^{\text{cst}} = 0$ . Also, one easily verifies that

$$\begin{aligned} 8(N_1 - G_2)^{\text{cst}} &= \Omega(q, c)\lambda_1(c, c) + \lambda(c, c)\Omega(q_1, c) - 4\Omega(q, c)\Omega(q_2, c), \\ 8(N_1 - G_2)^{\text{lin}} &= 2\mu(c, X)\Omega(c, q_1) - \lambda(c, c)\lambda_1(c, X) - \lambda(c, X)\lambda_1(c, c) \\ &\quad - 2\Omega(q, c)\mu_1(c, X) - 4\lambda(c, c)\Omega(X, q_2) - 4\Omega(X, q)\lambda_2(c, c) \\ &\quad + \lambda(c, X)\Omega(c, q_2) - \Omega(q, c)\lambda_2(c, X), \\ 8(N_1 - G_2)^{\text{qdr}} &= \lambda(c, c)\mu_1(X, X) + \mu(X, X)\lambda_1(c, c) + 2\lambda(c, X)\mu_1(c, X) + 2\mu(c, X)\lambda_1(c, X) \\ &\quad - 12\lambda(c, c)\phi^2r_1 - 12r\phi^2\lambda_1(c, c) - 2K\phi^2\Omega(q_1, c) - 2\lambda(c, c)\lambda_2(X, X) - 4\phi^2s_2 \\ &\quad - 2\lambda(X, X)\lambda_2(c, c) + 2\lambda(c, X)\lambda_2(c, X) + 6\Omega(q, c)\mu_2(X, X) + 6\mu(X, X)\Omega(q_2, c) \\ &\quad + 8\mu(c, X)\Omega(X, q_2) + 8\Omega(X, q)\mu_2(c, X) + 24\Omega(q, c)\phi^2r_2 + 24r\phi^2\Omega(q_2, c), \\ 8(N_1 - G_2)^{\text{cub}} &= 24r\phi^2\mu_1(c, X) + 24\phi^2\mu(c, X)r_1 - 2\mu(c, X)\mu_1(X, X) \\ &\quad - 2\mu(X, X)\mu_1(c, X) + K\phi^2\lambda_1(c, X) - 3\lambda(c, X)\mu_2(X, X) \\ &\quad - 3\mu(X, X)\lambda_2(c, X) + 4\mu(c, X)\lambda_2(X, X) + 4\lambda(X, X)\mu_2(c, X) \\ &\quad + 4K\phi^2\Omega(X, q_2) - 12r\phi^2\lambda_2(c, X) - 12\phi^2\lambda(c, X)r_2. \end{aligned}$$

As  $(3L + 12r\phi^2 - 2L^+)^{\text{cst}} = 0$ , the above formula for  $\psi^{\text{lin}}$  gives

$$\begin{aligned} \psi^{\text{lin}} &= 2(N_1 - G_2)^{\text{cst}}\phi + \lambda(c, c)(T_1 + G)^{\text{lin}} - 6\mu(c, X)(T_1 + G)^{\text{cst}} \\ &\quad + 4\Omega(X, q)d_c[(T_2 + N)^{\text{lin}}] + \lambda(c, X)d_c[(T_1 + G)^{\text{lin}}] + \Omega(q, c)d_c[(T_1 + G)^{\text{qdr}}], \end{aligned}$$

so that  $4\psi^{\text{lin}} = \phi\Omega(q, c)[\lambda_1(c, c) + 2\Omega(c, q_2) + 1] - 4\Omega(q, c)[\mu(c, c)\Omega(X, q) + \Omega(q, c)\mu(c, X)]$ . Thus,  $4\psi^{\text{lin}} = \phi\Omega(q, c)[\lambda_1(c, c) + 2\Omega(c, q_2) - 4\mu(c, q) + 1]$ , since  $\mu(c, c)\Omega(X, q) + \Omega(q, c)\mu(c, X) = \mu(c, q)$  (see (20.9)). The second equation in (20.10) now yields  $\psi^{\text{lin}} = 0$ . Similarly,

$$\begin{aligned} \psi^{\text{qrt}} &= 2(N_1 - G_2)^{\text{cub}}\phi - 3\lambda(c, X)(T_2 + N)^{\text{cub}} + [\mu(X, X) + 4r\phi^2](T_2 + N)^{\text{qdr}} \\ &\quad + 6\mu(c, X)(T_1 + G)^{\text{cub}} + K\phi^2(T_1 + G)^{\text{qdr}} \\ &\quad + 2\lambda(X, X)d_c[(T_2 + N)^{\text{cub}}] + [4r\phi^2 - 3\mu(X, X)]d_c[(T_1 + G)^{\text{cub}}], \end{aligned}$$

and so, by (18.3) and (14.5.iii),  $4\psi^{\text{qrt}}$  equals

$$\begin{aligned} &[2\mu(X, X) + 8r\phi^2][2\phi\mu_1(c, X) - \phi\lambda_2(c, X) - 2\lambda(c, c)\mu(X, X) + 2\mu(c, c)\lambda(X, X) - K\Omega(q, c)\phi^2] \\ &\quad - 4\mu(c, X)[2\phi\mu_1(X, X) - \phi\lambda_2(X, X) - 4\lambda(c, X)\mu(X, X) + 4\mu(c, X)\lambda(X, X) + 2K\phi^2\Omega(X, q)] \\ &\quad + 2K\phi^3[\lambda_1(c, X) + 2\Omega(X, q_2) - \Omega(X, a) + \phi\det_{\Omega}\lambda + s\phi] \\ &\quad - 8K\phi^2\Omega(X, q)\mu(c, X) - 8K\phi^2\Omega(q, c)\mu(X, X). \end{aligned}$$

According to Lemma 20.4, we have (20.10) and (18.1). Thus, in the above expression, the first two lines vanish, while the third is equal to  $8K\phi^3\mu(q, X)$ . As Remark 3.4 and (14.5.i) give

$$(21.7) \quad \phi\mu(q, X) = \Omega(X, q)\mu(c, X) + \Omega(q, c)\mu(X, X),$$

we see that  $\psi^{\text{qrt}} = 0$ . Next, as before, by (21.6),  $\psi^{\text{qdr}}$  equals  $2(N_1 - G_2)^{\text{lin}}\phi$  plus

$$\begin{aligned} & (3L + 12r\phi^2 - 4L^+)^{\text{cst}}(T_2 + N)^{\text{qdr}} - (2L^+)^{\text{lin}}(T_2 + N)^{\text{lin}} + 3(L + 4r\phi^2)^{\text{qdr}}(T_2 + N)^{\text{cst}} \\ & - (E - 3K\phi^2/2)^{\text{cst}}(T_1 + G)^{\text{qdr}} + (E + 3K\phi^2/2)^{\text{lin}}(T_1 + G)^{\text{lin}} \\ & + 3(E + K\phi^2/2)^{\text{qdr}}(T_1 + G)^{\text{cst}} + 2Q^{\text{lin}}d_c[(T_2 + N)^{\text{qdr}}] + 2Q^{\text{qdr}}d_c[(T_2 + N)^{\text{lin}}] \\ & - (2L^-)^{\text{cst}}d_c[(T_1 + G)^{\text{cub}}] - (2L^-)^{\text{lin}}d_c[(T_1 + G)^{\text{qdr}}] - (2L^-)^{\text{qdr}}d_c[(T_1 + G)^{\text{lin}}], \end{aligned}$$

which can easily be rewritten as

$$\begin{aligned} \psi^{\text{qdr}} &= 2(N_1 - G_2)^{\text{lin}}\phi + 3\Omega(q, c)(T_2 + N)^{\text{qdr}} - \lambda(c, X)(T_2 + N)^{\text{lin}} \\ &+ [3\mu(X, X) + 12r\phi^2](T_2 + N)^{\text{cst}} - \lambda(c, c)(T_1 + G)^{\text{qdr}} - 2\mu(c, X)(T_1 + G)^{\text{lin}} \\ &+ 3K\phi^2(T_1 + G)^{\text{cst}} + 4\Omega(X, q)d_c[(T_2 + N)^{\text{qdr}}] + 2\lambda(X, X)d_c[(T_2 + N)^{\text{lin}}] \\ &+ \Omega(q, c)d_c[(T_1 + G)^{\text{cub}}] + \lambda(c, X)d_c[(T_1 + G)^{\text{qdr}}] + [4r\phi^2 - 3\mu(X, X)]d_c[(T_1 + G)^{\text{lin}}]. \end{aligned}$$

Since  $\Omega(a, c) = 1$ , we thus obtain

$$\begin{aligned} 4\psi^{\text{qdr}} &= \Omega(c, q)[2\phi\mu_1(c, X) - \phi\lambda_2(c, X) - 2\lambda(c, c)\mu(X, X) + 2\mu(c, c)\lambda(X, X) - K\Omega(q, c)\phi^2] \\ &- 2\phi\lambda(c, c)[\lambda_1(c, X) + 2\Omega(X, q_2) - \Omega(X, a) + \phi\det_\Omega\lambda + s\phi] \\ &+ 8\lambda(c, c)[\Omega(X, q)\mu(c, X) + \Omega(q, c)\mu(X, X)], \end{aligned}$$

where we have used (18.3) and formulae displayed earlier in the proof. Again, as Lemma 20.4 yields (20.10) and (18.1), in the above equality the right-hand side of the first line vanishes, and the second line equals  $-8\phi\lambda(c, c)\mu(q, X)$ . Consequently, by (21.7),  $\psi^{\text{qdr}} = 0$ .

Finally, repeating the same steps for  $\psi^{\text{cub}}$ , we verify that  $\psi^{\text{cub}}$  equals

$$\begin{aligned} & 2(N_1 - G_2)^{\text{qdr}}\phi + 3(L + 4r\phi^2 - 2L^+)^{\text{cst}}(T_2 + N)^{\text{cub}} + (3L + 12r\phi^2 - 4L^+)^{\text{lin}}(T_2 + N)^{\text{qdr}} \\ & + (3L + 12r\phi^2 - 2L^+)^{\text{qdr}}(T_2 + N)^{\text{lin}} - 3(E - K\phi^2/2)^{\text{cst}}(T_1 + G)^{\text{cub}} \\ & - (E - 3K\phi^2/2)^{\text{lin}}(T_1 + G)^{\text{qdr}} + (E + 3K\phi^2/2)^{\text{qdr}}(T_1 + G)^{\text{lin}} + 2Q^{\text{lin}}d_c[(T_2 + N)^{\text{cub}}] \\ & + 2Q^{\text{qdr}}d_c[(T_2 + N)^{\text{qdr}}] - (2L^-)^{\text{lin}}d_c[(T_1 + G)^{\text{cub}}] - (2L^-)^{\text{qdr}}d_c[(T_1 + G)^{\text{qdr}}], \end{aligned}$$

which we can again rewrite as

$$\begin{aligned} \psi^{\text{cub}} &= 2(N_1 - G_2)^{\text{qdr}}\phi + 6\Omega(q, c)(T_2 + N)^{\text{cub}} - 2\lambda(c, X)(T_2 + N)^{\text{qdr}} \\ &+ [2\mu(X, X) + 8r\phi^2](T_2 + N)^{\text{lin}} - 3\lambda(c, c)(T_1 + G)^{\text{cub}} + 2\mu(c, X)(T_1 + G)^{\text{qdr}} \\ &+ 2K\phi^2(T_1 + G)^{\text{lin}} + 4\Omega(X, q)d_c[(T_2 + N)^{\text{cub}}] + 2\lambda(X, X)d_c[(T_2 + N)^{\text{qdr}}] \\ &+ \lambda(c, X)d_c[(T_1 + G)^{\text{cub}}] + [4r\phi^2 - 3\mu(X, X)]d_c[(T_1 + G)^{\text{qdr}}]. \end{aligned}$$

Consequently,  $4\psi^{\text{cub}}$  is equal to

$$\begin{aligned} & 2\lambda(c, c)[2\phi\mu_1(X, X) - \phi\lambda_2(X, X) - 4\lambda(c, X)\mu(X, X) + 4\mu(c, X)\lambda(X, X) + 2K\phi^2\Omega(X, q)] \\ & + 4[2\mu(X, X) + 8r\phi^2][\mu(c, c)\Omega(X, q) + \Omega(q, c)\mu(c, X)] \\ & - [2\phi\mu(X, X) + 8r\phi^3][\lambda_1(c, c) + 2\Omega(c, q_2) + 1] \\ & + 4\phi\mu(c, X)[\lambda_1(c, X) + 2\Omega(X, q_2) - \Omega(X, a) + \phi\det_\Omega\lambda + s\phi] \\ & - 16\mu(c, X)[\mu(c, X)\Omega(X, q) + \mu(X, X)\Omega(q, c)] \\ & + 4[8r - K\Omega(q_1, c) - s_2]\phi^3 + 8K\phi^2[\Omega(q, c)\lambda(c, X) + \lambda(c, c)\Omega(X, q)]. \end{aligned}$$

Each of the six lines forming the above expression can now be evaluated as follows. The first line vanishes due to (18.1). The second and third lines add up to zero as a consequence of (20.9) and (18.4). Similarly, the fourth and fifth lines cancel each other in view of (18.1) and (19.5) for  $v = q$ . Next, again by (19.5) for  $v = q$ , the last line equals  $4[2K\lambda(c, q) + 8r - K\Omega(q_1, c) - s_2]\phi^3$ , which completes the proof.  $\square$

## 22. The local-structure theorem

Given a two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  (see Section 9) and a real constant  $K$ , let us choose  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  and  $\mathcal{H}$  as in Lemma 11.2. If  $(q, \lambda, \mu) : U \rightarrow V$  is a solution to (18.4), defined on a nonempty connected open set  $U \subset \Sigma$ , and  $(q, \lambda, \mu, r)$  corresponds to  $(q, \lambda, \mu)$  in the sense of Theorem 18.2, then, setting  $\tilde{\mathcal{H}} = \mathcal{H} + F$  for the section  $F = F^K + F^q + F^\lambda + F^\mu + f\zeta$  of  $\mathcal{F}$ , with  $F^K, F^q, F^\lambda, F^\mu$  and  $f$  given by the formulae in Lemmas 17.1 – 17.2, we obtain a new horizontal distribution  $\tilde{\mathcal{H}}$  on  $U \times \Pi_+$ , which gives rise to the neutral-signature metric  $\tilde{g}$  on the four-manifold  $U \times \Pi_+$  characterized by (10.1) (for  $\tilde{\mathcal{H}}$  rather than  $\mathcal{H}$ ).

**Theorem 22.1.** *Suppose that  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  is a fixed two-plane system and  $K \in \mathbb{R}$ .*

- (a) *For any nonempty connected open set  $U \subset \Sigma$  and any  $(q, \lambda, \mu) : U \rightarrow V$  with (18.4), a suitable orientation of  $U \times \Pi_+$  makes  $\tilde{g}$  constructed above a strictly non-Walker self-dual neutral Einstein metric of Petrov type III with the scalar curvature  $12K$ .*
- (b) *Conversely, in any strictly non-Walker self-dual neutral Einstein four-manifold  $(M, g)$  of Petrov type III with the scalar curvature  $12K$ , every point has a neighborhood isometric to an open submanifold of  $(U \times \Pi_+, \tilde{g})$ , where  $\tilde{g}$  is obtained as above using our fixed  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  and  $K$  along with some solution  $(q, \lambda, \mu) : U \rightarrow V$  to (18.4).*

**Proof.** As  $(q, \lambda, \mu, r)$  satisfies (18.1) – (18.2) (see Theorem 18.2), with some  $s : U \rightarrow \mathbb{R}$  and  $a \in \Pi$  such that  $\Omega(a, c) = 1$ , Theorem 18.1 implies condition (d) in Theorem 12.2 for  $\tilde{\mathcal{H}} = \mathcal{H} + F$ , while conditions (a) – (c) in Theorem 12.2, for  $\tilde{\mathcal{H}} = \mathcal{H} + F$ , are guaranteed by our use of the formulae in Lemmas 17.1 – 17.2 to define  $F$  (cf. Section 14). Assertion (a) now follows from Theorem 12.2. (That  $\tilde{g}$  represents the strictly non-Walker case is clear as  $\beta \neq 0$  everywhere, according to (14.7.a).)

Conversely, for  $(M, g)$  as in (b),  $(M, \mathcal{V}, D, h, \alpha, \beta, \theta, \zeta)$  used in the above construction may be assumed, by Theorem 9.2, to coincide with the basic octuple that  $(M, g)$  gives rise to in Theorem 8.4. Under this identification  $g$  corresponds to a metric  $\tilde{g}$  on a connected open subset of  $\Sigma \times \Pi_+$  defined as in (10.1), that is, forming the unique total-metric extension of  $h$  such that  $\tilde{\mathcal{H}}$  is  $\tilde{g}$ -null, for  $\tilde{\mathcal{H}}$  associated with  $\tilde{g}$  as in Lemma 5.2. Denoting by  $\mathcal{H}$  the horizontal distribution of Lemma 11.2, we have  $\tilde{\mathcal{H}} = \mathcal{H} + F$  for some section  $F$  of  $\mathcal{F}$ . (See Section 13.) Conditions (a) – (d) in Theorem 12.2 thus are satisfied by  $\tilde{\mathcal{H}}$  and  $\tilde{g}$ . The discussion at the beginning of Section 17, combined with Lemmas 17.1 and 17.2, now shows that  $F = F^K + F^q + F^\lambda + F^\mu + f\zeta$ , with  $(q, \lambda, \mu, r)$  (depending on  $y \in \Sigma$ ) and  $f$  as in Lemmas 17.1 and 17.2. Theorems 18.1 and 18.2 now yield (18.4), completing the proof.  $\square$

Since the manifolds described in Theorem 22.1 are all curvature homogeneous (Remark 5.3), it is natural to ask if they must also be locally homogeneous. As the next result explicitly shows, this is not the case, and not all such manifolds are Ricci-flat. The latter conclusion answers a question raised by Díaz-Ramos, García-Río and Vázquez-Lorenzo [7, Remark 3.5].

**Theorem 22.2.** *For any given  $K \in \mathbb{R}$ , applying Theorem 22.1(a) to  $(q, \lambda, \mu) : \Sigma \rightarrow V$  chosen as in Example 18.3 we obtain a strictly non-Walker self-dual neutral Einstein four-manifold of Petrov type III with the scalar curvature  $12K$ , which is not locally homogeneous.*

**Proof.** In view of Theorem 22.1(a), we only need to verify that the resulting metric  $\tilde{g}$  on  $M = \Sigma \times \Pi_+$  is not locally homogeneous. This will clearly follow once we show that the function  $\tilde{\gamma}(\bar{u}) : M \rightarrow \mathbb{R}$ , constituting a local invariant of  $\tilde{g}$  (see the end of Remark 11.3), is nonconstant on  $\{y\} \times \Pi_+$  for some  $y \in \Sigma$ .



By (19.4.a) for  $v = \bar{u}$ , (14.8.d) and (14.5.i),  $\tilde{\gamma}(\bar{u}) = K + [\lambda(c, c) - 2\mu(c, X)]\phi^{-2}$ . As  $\lambda(c, c), \mu(c, X)$  and  $\phi^2$  restricted to  $\{y\} \times \Pi_+ \approx \Pi_+$  are homogeneous polynomial functions of degrees 0, 1 and 2, the restriction of  $\tilde{\gamma}(\bar{u})$  to  $\{y\} \times \Pi_+$  is nonconstant on  $\{y\} \times \Pi_+$  for every  $y \in \Sigma$  at which  $\lambda(c, c) \neq 0$ . However, our choice of  $(q, \lambda, \mu)$  gives  $\lambda(c, c) \neq 0$  at all  $y$  lying outside a specific line in the plane  $\Sigma$ , which completes the proof.  $\square$

In Theorem 22.2 we used some solutions of (18.4) to obtain examples of metrics with interesting geometric properties. A description of all solutions to (18.4) will be given in Section 25.

### 23. The method of characteristics

Given an open set  $U \subset \mathbb{R}^2$  with the Cartesian coordinates  $y^1, y^2$  and an open interval  $I \subset \mathbb{R}$ , let  $z : U \rightarrow I$  be the unknown function in a first-order quasi-linear equation

$$(23.1) \quad \rho z_1 + \sigma z_2 = \chi, \quad \text{where } z_j = \partial z / \partial y^j \text{ and } \rho, \sigma, \chi \text{ are functions of } (y^1, y^2, z) \in U \times I.$$

One solves (23.1) using the method of characteristics, based on the observation that a function  $z$  of the variables  $(y^1, y^2)$ , defined on  $U$  and valued in  $I$ , satisfies (23.1) if and only if its graph surface in  $U \times I$  is a union of integral curves of the vector field  $(\rho, \sigma, \chi)$ .

Thus, if  $y \in \Upsilon$  for a curve  $\Upsilon$  embedded in  $U$ , the operation of restricting functions to  $\Upsilon$  is a bijective correspondence between germs at  $y$  of solutions  $z : U \rightarrow \mathbb{R}$  to (23.1) such that

$$(23.2) \quad \text{the vector } (\rho(y^1, y^2, z), \sigma(y^1, y^2, z)) \text{ at } y, \text{ with } z = z(y), \text{ is not tangent to } \Upsilon,$$

and germs at  $y$  of functions  $z : \Upsilon \rightarrow I$  having the property (23.2).

### 24. Connections in plane bundles over surfaces

In this and the next sections, by a *plane bundle* (or, *line bundle*) we always mean a real vector bundle of fibre dimension 2 (or, respectively, 1).

For a vector subbundle  $\mathcal{L}$  of a real vector bundle  $\mathcal{P}$  over a manifold  $\Sigma$ , we denote by  $\text{Hom}(\mathcal{L}, \mathcal{P}/\mathcal{L})$  the vector bundle over  $\Sigma$  whose sections are bundle morphisms from  $\mathcal{P}$  into the quotient bundle  $\mathcal{P}/\mathcal{L}$ . The *fundamental tensor of  $\mathcal{L}$*  relative to any given connection  $\bar{\nabla}$  in  $\mathcal{P}$  is then defined to be the bundle morphism  $\Psi_{\mathcal{L}} : T\Sigma \rightarrow \text{Hom}(\mathcal{L}, \mathcal{P}/\mathcal{L})$  assigning to a vector field  $w$  on  $\Sigma$  the morphism  $\mathcal{L} \rightarrow \mathcal{P}/\mathcal{L}$  that sends any section  $v$  of  $\mathcal{L}$  to the image of  $\bar{\nabla}_w v$  under the quotient projection  $\mathcal{P} \rightarrow \mathcal{P}/\mathcal{L}$ . Note that  $\Psi_{\mathcal{L}}$  is well defined in view of the Leibniz rule, and  $\Psi_{\mathcal{L}}$  vanishes identically zero if and only if  $\mathcal{L}$  is  $\bar{\nabla}$ -parallel.

**Lemma 24.1.** *If  $\mathcal{L}$  is a line subbundle of a plane bundle  $\mathcal{P}$  over a manifold  $\Sigma$  and  $\bar{\nabla}$  is a connection in  $\mathcal{P}$  such that the fundamental tensor  $\Psi_{\mathcal{L}}$  is nonzero everywhere in  $\Sigma$ , then  $\text{Ker } \Psi_{\mathcal{L}}$  is a codimension-one distribution on  $\Sigma$ .*

**Proof.** In fact,  $\text{Hom}(\mathcal{L}, \mathcal{P}/\mathcal{L})$  then is a line bundle.  $\square$

**Lemma 24.2.** *For a traceless endomorphism  $\Phi$  of a two-dimensional real vector space,*

- (i)  $\Phi^2$  is a multiple of the identity,
- (ii)  $\Phi$  is diagonalizable and nonzero if and only if  $\text{tr } \Phi^2 > 0$ ,
- (iii) the kernel of  $\Phi$  equals the image of  $\Phi$  if and only if  $\Phi \neq 0$  and  $\text{tr } \Phi^2 = 0$ .

This is clear since, in some basis,  $\Phi$  is represented by one of the matrices

$$\begin{bmatrix} \kappa & 0 \\ 0 & -\kappa \end{bmatrix}, \quad \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{with } \kappa \in \mathbb{R}.$$

Given a plane bundle  $\mathcal{P}$  over a surface  $U$ , let  $\Omega$  be an  $\mathrm{SL}(2, \mathbb{R})$ -structure in  $\mathcal{P}$ , that is, a section of  $[\mathcal{P}^*]^{\wedge 2}$  without zeros. If  $\bar{R}$  is the curvature tensor of any connection  $\bar{\nabla}$  in  $\mathcal{P}$  such that  $\bar{\nabla}\Omega = 0$ , then at each point  $y \in U$  we have  $\mathrm{tr} \bar{R} = 0$  and one of the three relations  $\mathrm{tr} \bar{R}^2 > 0$ ,  $\mathrm{tr} \bar{R}^2 = 0$  and  $\mathrm{tr} \bar{R}^2 < 0$ , meaning that the trace (in)equality in question is satisfied by the endomorphism  $\bar{R}_y(w, w')$  of  $\mathcal{P}_y$  (cf. (3.5)) with some, or any, basis  $w, w'$  of  $T_y \Sigma$ . We are going to consider the following conditions:

- (24.1)            i)  $\bar{\nabla}\Omega = 0$  and  $\mathrm{tr} \bar{R}^2 > 0$  at each point of  $U$ ,  
                   ii)  $\bar{\nabla}\Omega = 0$ , while  $\mathrm{tr} \bar{R}^2 = 0$  and  $\bar{R} \neq 0$  everywhere in  $U$ ,  
                   iii)  $\bar{\nabla}\Omega = 0$  and  $\bar{\nabla}$  is flat, that is,  $\bar{R}$  vanishes identically.

**Lemma 24.3.** *For  $U, \mathcal{P}, \Omega, \bar{\nabla}$  and  $\bar{R}$  as above, vector fields  $w, w'$  on  $U$  linearly independent at every point, and the corresponding bundle morphism  $\bar{R}(w, w') : \mathcal{P} \rightarrow \mathcal{P}$ , cf. (3.5),*

- (a) *condition (24.1.i) holds if and only if  $\mathcal{P}$  is the direct sum of two line subbundles  $\mathcal{L}^\pm$ , which are the eigenspace bundles of  $\bar{R}(w, w')$ ,*  
 (b) *condition (24.1.ii) is equivalent to the existence of a line subbundle  $\mathcal{L}$  of  $\mathcal{P}$  which is both the kernel and the image of  $\bar{R}(w, w')$ .*

**Proof.** The assertion is immediate from Lemma 24.2. □

In our subsequent discussion, (24.1.ii) will be coupled with the additional condition

- (24.2)            the fundamental tensor of  $\mathcal{L}$  relative to  $\bar{\nabla}$  is nonzero everywhere in  $U$ ,

where the fundamental tensor is defined at the beginning of this section, and  $\mathcal{L}$  stands for the line subbundle of  $\mathcal{P}$  characterized by Lemma 24.3(b).

In the next lemma, *general position* means the same as in the third paragraph of the Introduction. Given a connection  $\bar{\nabla}$  in a plane bundle  $\mathcal{P}$  over a surface  $U$ , with a fixed  $\mathrm{SL}(2, \mathbb{R})$ -structure  $\Omega$  in  $\mathcal{P}$  such that  $\bar{\nabla}\Omega = 0$ , local coordinates  $y^j$  in  $U$ , and local trivializing sections  $e_k$  of  $\mathcal{P}$  satisfying the condition  $\Omega(e_1, e_2) = 1$ , we denote by  $\Gamma_{jk}^l$  the corresponding component functions of  $\bar{\nabla}$ , characterized by  $\bar{\nabla}_w e_k = w^j \Gamma_{jk}^l e_l$  (summation over repeated indices), for any vector field  $w$  on  $\Sigma$  with the component functions  $w^j$ . We continue using subscripts for partial derivatives, the only exceptions being the symbols  $e_k, \Gamma_{jk}^l$  and  $\bar{R}_{12k}^l$ .

**Lemma 24.4.** *Suppose that a connection  $\bar{\nabla}$  in a plane bundle  $\mathcal{P}$  with an  $\mathrm{SL}(2, \mathbb{R})$ -structure  $\Omega$  over a surface  $U$  satisfies (24.1.i), or (24.1.ii) along with (24.2), or (24.1.iii). Then, at points in general position, locally, for some  $y^j$  and  $e_k$  as above,  $\bar{\nabla}$  has one of the following descriptions, in which  $\psi, \chi$  are functions of  $(y^1, y^2)$  and  $p$  is a function of one real variable:*

- (I)  $\Gamma_{12}^1 = e^{2\chi}$ ,  $\Gamma_{22}^2 = -\Gamma_{21}^1 = \chi_2$ ,  $\Gamma_{22}^1 = 0$  and either  
 (a)  $\Gamma_{11}^1 = -\Gamma_{12}^2 = \psi_1$ ,  $\Gamma_{11}^2 = 0$  and  $\Gamma_{21}^2 = e^{2\psi}$ , or  
 (b)  $\Gamma_{11}^1 = -\Gamma_{12}^2$  is arbitrary,  $\Gamma_{11}^2 = p(y^1)e^{-2\chi}$  and  $\Gamma_{21}^2 = 0$ , or  
 (c)  $\Gamma_{11}^1 = -\Gamma_{12}^2 = p(y^1) - \chi_1$ ,  $\Gamma_{11}^2$  is arbitrary, and  $\Gamma_{21}^2 = 0$ ,  
 (II)  $\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^2 = \Gamma_{22}^1 = 0$ , while  $\Gamma_{11}^1 = -\Gamma_{12}^2$  and  $\Gamma_{22}^2 = -\Gamma_{21}^1$  are arbitrary,  
 (III)  $\Gamma_{jk}^l = 0$  for all  $j, k, l$ .

*Conversely, (I) – (III) easily imply (24.1.i), or (24.1.ii) and (24.2), or (24.1.iii).*

*The descriptions in (I) – (III) and the three cases in (24.1) are related as follows: (24.1.i) is realized by (I-a), (I-b) and (II); (24.1.ii) with (24.2) by (I-c); and (24.1.iii) by (III).*

**Proof.** Obviously, (III) corresponds to (24.1.iii). From now on we assume (24.1.i), or (24.1.ii) with (24.2). Let  $\Psi^\pm$  (or,  $\Psi$ ) be the fundamental tensor, relative to  $\bar{\nabla}$ , of the line subbundle  $\mathcal{L}^\pm$  (or,  $\mathcal{L}$ ) appearing in Lemma 24.3.

If (24.1.i) holds and  $\Psi^+ = \Psi^- = 0$  everywhere in  $U$ , we obtain (II) by choosing  $e_1$  and  $e_2$  to be sections of  $\mathcal{L}^+$  and  $\mathcal{L}^-$ . Conversely, it is clear that (II) gives  $\Psi^\pm = 0$ .

The remaining case, in general position, means that we have either (24.1.i) and  $\Psi^\pm \neq 0$  everywhere for some fixed sign  $\pm$ , or (24.1.ii) and (24.2). If we unify the notation by setting  $\Psi = \Psi^\pm$  and  $\mathcal{L} = \mathcal{L}^\pm$  for this sign  $\pm$ , then in both cases we may, locally, choose  $e_k$  and  $y^j$  such that  $e_2$  is a section of  $\mathcal{L}$  and  $\text{Ker } dy^1 = \text{Ker } \Psi$ , cf. Lemma 24.1. The coordinate vector field  $\partial_2$  thus spans the distribution  $\text{Ker } \Psi$ , and so  $\Gamma_{22}^1 = 0$ , while  $\Gamma_{12}^1 \neq 0$ . Let us change the sign of  $y^1$  (or  $e_1$ , or  $e_2$ ) if necessary, so as to make  $\Gamma_{12}^1$  positive. Hence  $\Gamma_{12}^1 = e^{2\chi}$  for a function  $\chi$ . By (3.4) with  $\Gamma_{22}^1 = 0$  and  $\Gamma_{21}^1 = -\Gamma_{22}^2$ , we now get  $0 = e^{-2\chi} \bar{R}_{122}^1 = 2\chi_2 + \Gamma_{21}^1 - \Gamma_{22}^2 = 2(\chi_2 - \Gamma_{22}^2)$ , which yields the first line in (I).

So far,  $e_1$  and  $y^2$  have been completely arbitrary except for the relations  $\Omega(e_1, e_2) = 1$  and  $dy^1 \wedge dy^2 \neq 0$ . Each of the following four paragraphs begins with a specific general-position assumption, and uses some special choice of one or both of  $e_1$  and  $y^2$ .

If (24.1.ii) and (24.2) hold, choosing  $e_1$  so that it spans a line subbundle which is  $\bar{\nabla}$ -parallel along the  $y^2$  coordinate direction, we get  $\Gamma_{21}^2 = 0$ . Since  $e_2$  spans both the kernel and the image of  $\bar{R}(\partial_1, \partial_2)$  (see Lemma 24.3(b)), one has  $0 = \bar{R}_{121}^1 = (\Gamma_{11}^1)_2 - (\Gamma_{21}^1)_1$ , which yields (I-c) as  $\Gamma_{21}^1 = -\Gamma_{22}^2 = -\chi_2$ .

For the remainder of the proof, we assume (24.1.i) and choose  $e_1$  to be a section of  $\mathcal{L}^\mp$ , with the sign  $\mp$  opposite to  $\pm$  that was fixed earlier. If  $\Psi^\mp = 0$  identically, we obviously obtain (I-b) with  $p(y^1) = 0$ .

If, on the other hand,  $\Psi^\mp \neq 0$  everywhere and  $\text{Ker } \Psi^\mp = \text{Ker } \Psi^\pm$ , so that  $\Gamma_{21}^2 = 0$ , we get, as before,  $\Gamma_{11}^2 = e^{2\psi}$  for a function  $\psi$  and  $0 = e^{-2\psi} \bar{R}_{121}^2 = 2\psi_2 + \Gamma_{22}^2 - \Gamma_{21}^1 = 2(\psi_2 + \Gamma_{22}^2)$ . Hence  $\psi_2 = -\Gamma_{22}^2 = -\chi_2$ , that is,  $2\psi = -2\chi + \log p(y^1)$  for some positive function  $p$  of one variable, which again gives (I-b).

Finally, if  $\Psi^\mp \neq 0$  and  $\text{Ker } \Psi^\mp$  is transverse to  $\text{Ker } \Psi^\pm$  everywhere in  $U$ , we choose  $y^2$  with  $\text{Ker } dy^2 = \text{Ker } \Psi^\mp$ , so that  $\Gamma_{11}^2 = 0$ , while  $\Gamma_{21}^2$ , being nonzero, may as before be assumed positive and hence equal to  $e^{2\psi}$  for some function  $\psi$ . Now  $0 = e^{-2\psi} \bar{R}_{121}^2 = \Gamma_{11}^1 - \Gamma_{12}^2 - 2\psi_1 = 2(\Gamma_{11}^1 - \psi_1)$ , and (I-a) follows.  $\square$

## 25. Solutions to the system (18.4)

We use the same assumptions and notations as at the beginning of Section 18 and in Theorem 18.1. The subscripts  $( )_j$ ,  $j = 1, 2$ , stand, again, for the directional derivatives in the directions of the constant vector fields  $\partial_j$  on  $\Sigma$  forming the basis of  $\dot{\Sigma}$  dual to the basis  $\xi, \tau$  of  $\dot{\Sigma}^*$ . In other words,  $( )_j = \partial/\partial y^j$  for the affine coordinates  $y^j$  with  $dy^1 = \xi$  and  $dy^2 = \tau$ . Here  $\dot{\Sigma}$  is the translation vector space of the affine plane  $\Sigma$ , and, as before,  $U$  denotes a nonempty connected open subset of  $\Sigma$ , while  $K$  is any given real constant.

The area form  $\Omega \in [\Pi^*]^{\wedge 2} \setminus \{0\}$  gives rise to an isomorphic identification between the space  $\mathfrak{sl}(\Pi)$  of all traceless endomorphisms of  $\Pi$  and the space  $[\Pi^*]^{\circ 2}$  of all symmetric bilinear forms on  $\Pi$ , which associates with  $\Phi \in \mathfrak{sl}(\Pi)$  the form  $b$  sending  $u, v \in \Pi$  to  $b(u, v) = \Omega(\Phi u, v)$ . (Symmetry of  $b$  is immediate from Remark 3.3.) Under this identification any function  $(q, \lambda, \mu) : U \rightarrow V$  corresponds to a function  $(q, \delta, \varepsilon) : U \rightarrow \Pi \times \mathfrak{sl}(\Pi) \times \mathfrak{sl}(\Pi)$ . (As in Section 18, we set  $V = \Pi \times [\Pi^*]^{\circ 2} \times [\Pi^*]^{\circ 2}$ .) The functions  $\delta$  and  $\varepsilon$ , taking values in

endomorphisms of  $\Pi$ , can be multiplied by each other and multiplied by functions  $U \rightarrow \Pi$ , in the sense of the valuewise operations of composing endomorphisms and evaluating them on vectors. Thus, for instance, the valuewise commutator  $[\varepsilon, \delta]$  is a function  $U \rightarrow \mathfrak{sl}(\Pi)$ .

**Lemma 25.1.** *A function  $(q, \lambda, \mu) : U \rightarrow V$  satisfies (18.4) if and only if*

$$(25.1) \quad \text{i) } 2\varepsilon_1 - \delta_2 = 2[\varepsilon, \delta] - K\Omega(c, \cdot)q - K\Omega(q, \cdot)c, \quad \text{ii) } \Omega(c, \delta_1c - 2q_2 - 4\varepsilon q) = 1$$

for  $(q, \delta, \varepsilon)$  corresponding to  $(q, \lambda, \mu)$  as above.

**Proof.** Let  $\Phi_L$  and  $\Phi_R$  be the left-hand and right-hand sides of (25.1.i). Also, let  $\Phi = \Omega(c, \cdot)q + \Omega(q, \cdot)c$ . Both  $\Phi_L$  and  $\Phi_R$  are functions  $U \rightarrow \mathfrak{sl}(\Pi)$ . (In fact,  $\Phi$  takes values in  $\mathfrak{sl}(\Pi)$ , since  $\Omega(\Phi u, v)$  is symmetric in  $u, v \in \Pi$ .) For the same reasons of symmetry, we have  $\Phi_L = \Phi_R$  if and only if  $\Omega(\Phi_L X, X) = \Omega(\Phi_R X, X)$ , where  $X$  denotes, as usual, the radial vector field on  $\Pi$ . Clearly,  $\Omega(\Phi_L X, X) = 2\mu_1(X, X) - \lambda_2(X, X)$ , and, in view of (14.5.i),  $\Omega(\Phi X, X) = 2\phi\Omega(X, q)$ . On the other hand, by (18.5), the expression  $\lambda(c, X)\mu(X, X) - \mu(c, X)\lambda(X, X)$ , depending skew-symmetrically on  $\lambda$  and  $\mu$ , equals  $\phi\Omega(\varepsilon\delta X, X)$ . In view of skew-symmetry, this is further equal to  $\Omega([\varepsilon, \delta]X, X)\phi/2$ . Hence (25.1.i) is equivalent to the first equality in (18.4). Equivalence between (25.1.ii) and the second equality in (18.4) is in turn obvious from the definitions of  $\delta$  and  $\varepsilon$ .  $\square$

Treating  $U \times \Pi$  as the total space of a product vector bundle  $\mathcal{P}$  over  $U$ , we may view functions  $U \rightarrow \Pi$  (including the constant  $c$ ) as sections of  $\mathcal{P}$ , while  $\Omega$  then becomes an  $\text{SL}(2, \mathbb{R})$ -structure in  $\mathcal{P}$ , cf. Section 24.

**Lemma 25.2.** *Functions  $(q, \lambda, \mu) : U \rightarrow V$  are in a natural bijective correspondence with pairs  $(q, \bar{\nabla})$ , in which  $q$  is a section of the plane bundle  $\mathcal{P} = U \times \Pi$  with the  $\text{SL}(2, \mathbb{R})$ -structure  $\Omega$  and  $\bar{\nabla}$  is a connection in  $\mathcal{P}$  such that  $\bar{\nabla}\Omega = 0$ . The correspondence associates with  $(q, \lambda, \mu)$  the pair  $(q, \bar{\nabla})$  obtained by treating  $q : U \rightarrow \Pi$  as a section of  $\mathcal{P}$  and setting*

$$(25.2) \quad \bar{\nabla}_1 v = v_1 + \delta v, \quad \bar{\nabla}_2 v = v_2 + 2\varepsilon v$$

for functions  $v : U \rightarrow \Pi$ , also viewed as sections of  $\mathcal{P}$ , where  $(q, \delta, \varepsilon)$  is related to  $(q, \lambda, \mu)$  as in the lines preceding Lemma 25.1, and  $\bar{\nabla}_j$  denotes the  $\bar{\nabla}$ -covariant derivative in the direction of the coordinate vector field  $\partial_j$  on  $U$ .

**Proof.** This is obvious, as the condition  $\bar{\nabla}\Omega = 0$  accounts for the fact that  $\delta$  and  $\varepsilon$  take values in the space  $\mathfrak{sl}(\Pi)$  of traceless endomorphisms of  $\Pi$ .  $\square$

**Lemma 25.3.** *For a function  $(q, \lambda, \mu) : U \rightarrow V$ , condition (18.4) holds if and only if, in terms of the pair  $(q, \bar{\nabla})$  corresponding to  $(q, \lambda, \mu)$  in the sense of Lemma 25.2, the curvature tensor  $\bar{R}$  of  $\bar{\nabla}$ , and  $c, q$  treated as sections of  $\mathcal{P}$ ,*

$$(25.3) \quad \text{i) } \bar{R}(\partial_1, \partial_2) = K\Omega(c, \cdot)q + K\Omega(q, \cdot)c, \quad \text{ii) } \Omega(c, \bar{\nabla}_1\bar{\nabla}_1c - 2\bar{\nabla}_2q) = 1,$$

with  $\bar{\nabla}_j$  and  $\partial_j$  as in Lemma 25.2, so that  $\bar{R}(\partial_1, \partial_2)$  is a morphism  $\mathcal{P} \rightarrow \mathcal{P}$ , cf. (3.5).

**Proof.** By (3.4) and (25.2),  $\bar{R}(\partial_1, \partial_2) = \delta_2 - 2\varepsilon_1 - 2[\varepsilon, \delta]$ . Thus, (25.3.i) is equivalent to (25.1.i). Next, since  $c$  is constant, (25.2) gives  $\bar{\nabla}_1c = \delta c$ ,  $\bar{\nabla}_1\bar{\nabla}_1c = \delta_1c + \delta^2c$ , and  $\bar{\nabla}_2q = q_2 + 2\varepsilon q$ . As  $\delta^2$  takes values in multiples of the identity (see Lemma 24.2(i)), the left-hand side of (25.3.ii) coincides with that of (25.1.ii).  $\square$

**Remark 25.4.** Suppose that (25.3.i) is satisfied by a connection  $\bar{\nabla}$  in a plane bundle  $\mathcal{P}$  over a surface  $U$ , an  $\mathrm{SL}(2, \mathbb{R})$ -structure  $\Omega$  in  $\mathcal{P}$  with  $\bar{\nabla}\Omega = 0$ , the curvature tensor  $\bar{R}$  of  $\bar{\nabla}$ , sections  $c$  and  $q$  of  $\mathcal{P}$  such that  $c$  has no zeros, some vector fields  $\partial_j$  on  $U$ , linearly independent at every point, and a real constant  $K$ . Let  $y \in U$ .

- (a) If  $c$  and  $Kq$  are linearly independent at  $y$ , they are eigenvectors of  $\bar{R}(\partial_1, \partial_2)$ , acting in the fibre  $\mathcal{P}_y$ , for the nonzero eigenvalues  $K\Omega(q, c)$  and  $-K\Omega(q, c)$  (at  $y$ ).
- (b) If  $Kq$ , at  $y$ , is a nonzero multiple of  $c$ , then  $c$  spans, at  $y$ , the kernel of  $\bar{R}(\partial_1, \partial_2)$  acting in  $\mathcal{P}_y$  (which is at the same time its image).
- (c) If  $Kq = 0$ , at  $y$ , then  $\bar{R}(\partial_1, \partial_2) = 0$ , at  $y$ .
- (d) In case (a), or (b), or (c), we have at  $y$ , respectively,  $\mathrm{tr} \bar{R}^2 > 0$ , or  $\mathrm{tr} \bar{R}^2 = 0$  and  $\bar{R} \neq 0$ , or  $\bar{R} = 0$ .

In fact, (a) – (c) are obvious, and (d) is immediate from Lemma 24.2.

**Lemma 25.5.** *Given  $U, \mathcal{P}, \Omega, \bar{\nabla}, \bar{R}, c, q, \partial_j$  and  $K$  as in Remark 25.4, if conditions (24.1.ii) and (25.3) are both satisfied, and  $\mathcal{L}$  is the line subbundle of  $\mathcal{P}$  appearing in Lemma 24.3(b), then the fundamental tensor of  $\mathcal{L}$  relative to  $\bar{\nabla}$ , defined in Section 24, is nonzero at all points of a dense open subset of  $U$ .*

**Proof.** By (25.3.i),  $c$  and  $q$  span the image of  $\bar{R}(\partial_1, \partial_2)$ , so that they are sections of  $\mathcal{L}$  (cf. Lemma 24.3(b)). If  $\mathcal{L}$  were  $\bar{\nabla}$ -parallel on some nonempty open set  $U' \subset U$ , then  $\bar{\nabla}_1 \bar{\nabla}_1 c$  and  $\bar{\nabla}_2 q$  would be sections of  $\mathcal{L}$  as well, and so  $\Omega(c, \bar{\nabla}_1 \bar{\nabla}_1 c - 2\bar{\nabla}_2 q)$  would vanish on  $U'$ , contradicting (25.3.ii).  $\square$

We will now describe all  $U, \mathcal{P}, \Omega, \bar{\nabla}, \bar{R}, c, q, \partial_j$  and  $K$  with the properties listed in Remark 25.4, for which (25.3) holds. According to Lemma 25.3, this is equivalent to solving (18.4). (Since a quadruple  $(U, \mathcal{P}, \Omega, c)$ , having those of the properties just named that pertain just to  $U, \mathcal{P}, \Omega$  and  $c$ , is, locally, unique up to structure-preserving bundle isomorphisms and diffeomorphisms of the base, it makes no difference whether we treat  $(U, \mathcal{P}, \Omega, c)$  as fixed, or allow it to vary.)

First, let  $K = 0$ . Condition (25.3.i) now amounts to flatness of  $\bar{\nabla}$ . If we fix an arbitrary flat connection  $\bar{\nabla}$  in  $\mathcal{P}$  and choose, locally, a section  $a$  of  $\mathcal{P}$  with  $\Omega(a, c) = 1$ , (25.3.ii) is equivalent to requiring that  $2\bar{\nabla}_2 q = a + \chi c + \bar{\nabla}_1 \bar{\nabla}_1 c$  for some function  $\chi$ . With prescribed  $c$  (as well as  $a$  and  $\chi$ ), this is a system of linear ordinary differential equations with parameters, and our task has been reduced to solving it for  $q$ , which is well understood.

Therefore, from now on,  $K \neq 0$ . We also make a general-position assumption:

- (25.4) i)  $c$  and  $q$  are linearly independent at each point of  $U$ , or,
- ii)  $q$  is, at every point, a nonzero multiple of  $c$ , or, finally,
- iii)  $q$  vanishes identically on  $U$ .

In a dense open subset  $U'$  of  $U$ , one of (25.4.i), (25.4.ii) and (25.4.iii) will clearly hold if we replace  $U$  by a suitable neighborhood of any given point of  $U'$ .

Rather than dealing with the three possibilities in (25.4), we will consider those listed in (24.1). Namely, if (25.3) holds (which is what we want to achieve), each of the three lines in (25.4) implies, according to Remark 25.4 and Lemma 24.3, the corresponding line in (24.1). In case (24.1.ii), we also assume (24.2), which, by Lemma 25.5, is a general-position version of a condition necessary for (25.3).

To summarize, we now begin with a connection  $\bar{\nabla}$  in a plane bundle  $\mathcal{P}$  over a surface  $U$  and an  $\mathrm{SL}(2, \mathbb{R})$ -structure  $\Omega$  in  $\mathcal{P}$ , which satisfy (24.1.i), or (24.1.ii) and (24.2), or (24.1.iii),

but are otherwise arbitrary. Locally, at points in general position, all such  $\bar{\nabla}$  and  $\Omega$  are described by Lemma 24.4. We may therefore proceed with the next step: finding all sections  $c$  and  $q$  of  $\mathcal{P}$  with (25.3), such that  $c \neq 0$  everywhere.

In the cases (24.1.i) and (24.1.ii), let  $\mathcal{L}^\pm$  (or  $\mathcal{L}$ ) be as in Lemma 24.3. Choosing, locally, sections  $c$  of  $\mathcal{L}^+$  and  $q$  of  $\mathcal{L}^-$  (or, sections  $c$  and  $q$  of  $\mathcal{L}$ ) without zeros, we easily see that  $\bar{R}(\partial_1, \partial_2)$  equals some function without zeros times the right-hand side of (25.3.i). Multiplying  $c$  or  $q$  by a suitable function, we obtain (25.3.i) for the new  $c$  and  $q$ . The choice of  $c$  and  $q$  that satisfies (25.3.i) is not unique, as we are free to switch  $c$  with  $q$ , and/or use, instead of them,  $zc$  and  $z^{-1}q$ , for any function  $z : U \rightarrow \mathbb{R}$  without zeros. (In our discussion,  $U$  may be repeatedly replaced with smaller connected open sets, still denoted by  $U$ .) As (25.3.i) is already satisfied, we just need to determine which functions  $z : U \rightarrow \mathbb{R} \setminus \{0\}$  lead to (25.3.ii) for  $zc$  and  $z^{-1}q$  rather than  $c$  and  $q$ . Such  $z$  are easily seen to be characterized by (23.1) with  $\rho = \Omega(c, \bar{\nabla}_1 c)z^2$ ,  $\sigma = \Omega(c, q)$ ,  $\chi = [z - \Omega(c, \bar{\nabla}_1 \bar{\nabla}_1 c)]z^3/2$ ,  $I = (0, \infty)$  or  $I = (-\infty, 0)$ , and  $U$  treated as a connected open set in  $\mathbb{R}^2$ , for which  $y^1, y^2$  serve as the Cartesian coordinates.

Until now our discussion has covered both cases (24.1.i) and (24.1.ii). We now separate them, first assuming (24.1.i). As  $\sigma = \Omega(c, q)$  is nonzero everywhere, we may fix  $y \in U$  and choose the curve  $\Upsilon$  with (23.2) to be a line segment in  $U \subset \mathbb{R}^2$  containing  $y$  and parallel to the  $y^1$  coordinate direction. As stated at the end of Section 23, germs at  $y$  of functions  $z : U \rightarrow \mathbb{R} \setminus \{0\}$  leading to (25.3.ii) are in a natural bijective correspondence with germs at  $y$  of arbitrary nonzero functions on  $\Upsilon$ , which concludes our description of solutions to (18.4) in the case (24.1.i).

Next, suppose that (24.1.ii) and (24.2) are satisfied. Thus,  $\sigma = \Omega(c, q)$  vanishes identically, while  $\rho = \Omega(c, \bar{\nabla}_1 c)z^2$ , so that  $\rho \neq 0$  by (24.2), as the values of  $z$  range over  $\mathbb{R} \setminus \{0\}$ . The transversality condition (23.2) then holds for  $\Upsilon$  chosen to be a line segment through a given point  $y$ , parallel to the  $y^2$  coordinate direction. Once again, germs at  $y$  of functions  $z : U \rightarrow \mathbb{R} \setminus \{0\}$  that lead to (25.3.ii) may be bijectively identified with germs at  $y$  of arbitrary functions  $\Upsilon \rightarrow \mathbb{R} \setminus \{0\}$ .

Finally, let us assume (24.1.iii), so that  $\bar{\nabla}$  is flat. As  $K \neq 0$  and we want  $c$  to have no zeros, (25.3.i) can only be satisfied if  $q = 0$  identically. Condition (25.3.ii) with  $q = 0$  reads  $\Omega(c, \bar{\nabla}_1 \bar{\nabla}_1 c) = 1$ . Fixing, locally,  $\bar{\nabla}$ -parallel sections  $e_1, e_2$  of  $\mathcal{P}$  with  $\Omega(e_1, e_2) = 1$  and writing  $c = (\operatorname{Re} \chi)e_1 + (\operatorname{Im} \chi)e_2$  for some unspecified function  $\chi : U \rightarrow \mathbb{C} \setminus \{0\}$ , we get  $\Omega(c, \bar{\nabla}_1 \bar{\nabla}_1 c) = (\rho^2 \sigma_1)_1$ , where, locally,  $\chi = \rho e^{i\sigma}$  for real-valued functions  $\rho > 0$  and  $\sigma$ . The resulting equation  $(\rho^2 \sigma_1)_1 = 1$  can be solved by repeated integration in the  $y^1$  direction, with any prescribed  $\rho$ .

## 26. Final remarks

Let  $(M, g)$  be a strictly non-Walker self-dual neutral Einstein four-manifold of Petrov type III. The geometry of  $g$  gives rise, at least locally, to further structures in  $M$ , with properties arguably simpler than those assumed for  $g$ . One such structure is the basic octuple  $(M, \mathcal{V}, \mathcal{D}, h, \alpha, \beta, \theta, \zeta)$  determined by  $(M, g)$  as in Theorem 8.4. A part of it is the affine foliation  $(\mathcal{V}, \mathcal{D})$  on  $M$ , which allows us to identify  $M$ , locally, with the total space of an affine plane bundle  $\mathcal{A}$  over a surface  $\Sigma$  (see Section 7). As in the proof of Theorem 9.2, setting  $\mathcal{J} = \phi^{-1}(0)$ , with  $\phi$  chosen as at the end of Section 8, we obtain an affine-line subbundle  $\mathcal{J}$  of  $\mathcal{A}$ , which is well defined, since  $\phi$ , although not unique, is nevertheless unique up to multiplication by functions  $\Sigma \rightarrow (0, \infty)$ . Denoting by  $\mathcal{P}$  and  $\mathcal{L}$  the vector bundles over  $\Sigma$

associated with  $\mathcal{A}$  and  $\mathcal{J}$ , we may treat functions  $\phi$  mentioned above as positive sections of the canonically-oriented line bundle  $[\mathcal{P}/\mathcal{L}]^*$ . The conclusions about constancy along  $\mathcal{V}$  in (iii) at the end of Section 8 now allow us to treat  $\beta, \zeta$  and  $\theta$  as sections of the vector bundles  $T^*\Sigma \otimes [\mathcal{P}/\mathcal{L}]^{\otimes 2}$ ,  $[T^*\Sigma]^{\wedge 2} \otimes [\mathcal{P}/\mathcal{L}]$  and  $([\mathcal{P}]^{\wedge 2} \otimes [\mathcal{P}/\mathcal{L}]^{\otimes 2})^*$  over  $\Sigma$  (all three pulled back, via the bundle projection  $M \rightarrow \Sigma$ , to the corresponding pullback bundles over  $M$ ). Thus,  $\beta$  viewed as a  $[\mathcal{P}/\mathcal{L}]^{\otimes 2}$ -valued 1-form on  $\Sigma$  distinguishes the one-dimensional distribution  $\text{Ker } \beta$  on  $\Sigma$ .

Despite being natural geometric invariants of  $g$ , the objects just listed do not play a prominent role in our local-structure result (Theorem 22.1). On the contrary, most ingredients of the construction appearing in Theorem 22.1(a), such as the two-plane system  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$ , cannot be canonically recovered from the resulting metric  $g$ . (The extent to which  $(\Sigma, \xi, \tau, \Pi, c, \Omega)$  fails to be unique is made explicit in the proof of Theorem 9.2.)

The ideal form of a structure theorem for a class of metrics would be one involving a construction that uses only natural invariants of the metrics in question. As explained above, for metrics discussed in this paper such a goal still appears elusive.

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Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

e-mail: andrzej@math.ohio-state.edu