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Walker's theorem without coordinates

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We provide a coordinate-free version of the local classification, due to Walker [Q. J. Math. **1**, 69 (1950)], of null parallel distributions on pseudo-Riemannian manifolds. The underlying manifold is realized, locally, as the total space of a fiber bundle, each fiber of which is an affine principal bundle over a pseudo-Riemannian manifold. All structures just named are naturally determined by the distribution and the metric, in contrast with the noncanonical choice of coordinates in the usual formulation of Walker's theorem. © *2006 American Institute of Physics.*
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I. INTRODUCTION

In 1950, Walker[1] described the local structure of all pseudo-Riemannian manifolds with null parallel distributions. The present paper provides a coordinate-free version of Walker's theorem.

Many authors, beginning with Walker himself,[2] have invoked Walker's 1950 result, often to generalize it or derive other theorems from it. In our bibliography, which is by no means complete, Refs. 3–16 all belong to this category. They invariably cite Walker's result in its original, local-coordinate form (reproduced in the Appendix).

Such an approach, perfectly suited for the applications just mentioned, tends nevertheless to obscure the geometric meaning of Walker's theorem. In fact, Walker coordinates are far from unique; choosing them results in making noncanonical objects a part of the structure.

To keep the picture canonical, some authors[3,5] replace a single Walker coordinate system by a whole maximal atlas of them. What we propose here, instead, is to use only ingredients such as fiber bundles, widely seen as more directly "geometric" than a coordinate atlas (even though one may ultimately need atlases to define them).

In our description, the coordinate-independent content of Walker's theorem amounts to realizing the underlying manifold, locally, as a fiber bundle whose fibers are also bundles, namely, affine principal bundles over pseudo-Riemannian manifolds. The bundle structures are all naturally associated with the original null parallel distribution; the distribution and the metric can in turn be reconstructed from them.

II. PRELIMINARIES

Throughout this paper, all manifolds, bundles, sections, subbundles, connections, and mappings, including bundle morphisms, are assumed to be of class C^∞. A bundle morphism may operate only between two bundles with the same base manifold, and acts by identity on the base.

A bundle always means a C^∞ locally trivial bundle and the same symbol, such as M, is used both for a given bundle and for its total space; the bundle projection $M \to \Sigma$ onto the base manifold

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Σ is denoted by π (or, sometimes, p). We let M_y stand for the fiber $\pi^{-1}(y)$ over any $y\in\Sigma$, while $\operatorname{Ker} d\pi$ is the vertical distribution treated as a vector bundle (namely, a subbundle of the tangent bundle TM).

For real vector bundles $\mathcal{X},\mathcal{Y}$ over a manifold Σ and a real vector space V with $\dim V<\infty$, we denote by $\operatorname{Hom}(\mathcal{X},\mathcal{Y})$ the vector bundle over Σ whose sections are vector-bundle morphisms $\mathcal{X}\to\mathcal{Y}$, and by $\Sigma\times V$ the product bundle with the fiber V, the sections of which are functions $\Sigma\to V$. Thus, $\mathcal{X}^*=\operatorname{Hom}(\mathcal{X},\Sigma\times\mathbf{R})$ is the dual of $\mathcal{X}$.

We will say that a given fiberwise structure in a bundle M over a manifold Σ *depends C^∞-differentiably on* $y\in\Sigma$, or *varies C^∞-differentiably with* y, if suitable C^∞ local trivializations of M make the structure appear as constant (the same in each fiber).

The symbol ∇ will be used for various connections in vector bundles. Our sign convention about the curvature tensor $R=R^\nabla$ of a connection ∇ in a vector bundle $\mathcal{X}$ over a manifold Σ is

$$R(u,v)\psi=\nabla_v\nabla_u\psi-\nabla_u\nabla_v\psi+\nabla_{[u,v]}\psi, \tag{1}$$

for sections ψ of $\mathcal{X}$ and vector fields u,v tangent to Σ. By the Leibniz rule, when ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g and u,v,w are tangent vector fields, $2\langle\nabla_w v,u\rangle$ equals[17]

$$d_w\langle v,u\rangle+d_v\langle w,u\rangle-d_u\langle w,v\rangle+\langle v,[u,w]\rangle+\langle u,[w,v]\rangle-\langle w,[v,u]\rangle, \tag{2}$$

where d_v is the directional derivative and $\langle,\rangle$ stands for $g(,)$.

Remark 2.1: Let $\pi:M\to\Sigma$ be a bundle projection. A vector field w on the total space M is π-projectable onto the base manifold Σ if and only if, for every vertical vector field u on M, the Lie bracket $[w,u]$ is also vertical. This well-known fact is easily verified in local coordinates for M which make π appear as a standard Euclidean projection.

III. AFFINE PRINCIPAL BUNDLES

All principal bundles discussed below have Abelian structure groups G, so one need not decide whether G acts from the left or right.

Let N be a G-principal bundle over a base manifold L, where G is an Abelian Lie group. By the *N-prolongation* of the tangent bundle TL we mean the vector bundle $\mathcal{F}$ over L whose fiber $\mathcal{F}_c$ over $c\in L$ is the space of all G-invariant vector fields tangent to N along N_c (and defined just on N_c), with N_c denoting, as usual, the fiber of N over c. A vector subbundle $\mathcal{G}\subset\mathcal{F}$ now can be defined by requiring $\mathcal{G}_c$, for any $c\in L$, to consist of all G-invariant vector fields defined just on N_c which are vertical (i.e., tangent to N_c). Since each $\mathcal{G}_c$ is canonically isomorphic to the Lie algebra $\mathfrak{g}$ of G, the vector bundle $\mathcal{G}$ is naturally trivialized, that is, identified with the product bundle $L\times\mathfrak{g}$. Therefore

$$L\times\mathfrak{g}=\mathcal{G}\subset\mathcal{F}. \tag{3}$$

The quotient bundle $\mathcal{F}/\mathcal{G}$ is in turn naturally isomorphic to TL, via the differential of the bundle projection $N\to L$.

An *affine space* is a set A with a simply transitive action on A of the additive group of a vector space V. One calls V the *vector space of translations* of the affine space A.

An *affine bundle* M over a manifold Σ is a bundle with fibres M_y, $y\in\Sigma$, carrying the structures of affine spaces whose vector spaces $\mathcal{X}_y$ of translations form a vector bundle $\mathcal{X}$ over Σ, called the *associated vector bundle* of M. We also require the affine-space structure of M_y to vary C^∞-differentiably with $y\in\Sigma$, in the sense of Sec. II.

If, in addition, $\mathcal{X}=\Sigma\times V$, that is, the associated vector bundle of M happens to be a product bundle, then M is also a V-principal bundle, with the obvious action of the additive group of the vector space V. Such *affine principal bundles* are distinguished from arbitrary affine bundles by having a structure group that, instead of general affine transformations of a model fiber, contains only translations.

IV. PARTIAL METRICS AND EXTENSIONS

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be real vector bundles over a manifold Q. By an $\mathcal{E}$-valued *pairing* of $\mathcal{C}$ and $\mathcal{D}$ we mean any vector-bundle morphism $\beta:\mathcal{C}\otimes\mathcal{D}\to\mathcal{E}$. This amounts to a C^∞ assignment of a bilinear mapping $\beta(z):\mathcal{C}_z\times\mathcal{D}_z\to\mathcal{E}_z$ to every $z\in Q$. An $\mathcal{E}$-valued *partial pairing* of $\mathcal{C}$ and $\mathcal{D}$ consists, by definition, of two vector subbundles $\mathcal{C}'\subset\mathcal{C}$ and $\mathcal{D}'\subset\mathcal{D}$, of some codimensions k and l, along with pairings $\gamma:\mathcal{C}\otimes\mathcal{D}'\to\mathcal{E}$ and $\gamma:\mathcal{C}'\otimes\mathcal{D}\to\mathcal{E}$ which coincide on the subbundle $\mathcal{C}'\otimes\mathcal{D}'$ (and so may be represented by the same symbol γ without risk of ambiguity). One can obviously restrict a given pairing $\beta:\mathcal{C}\otimes\mathcal{D}\to\mathcal{E}$ to $\mathcal{C}\otimes\mathcal{D}'$ and $\mathcal{C}'\otimes\mathcal{D}$, so that a partial pairing γ is obtained; we will then say that β is a *total-pairing extension* of γ.

Lemma 4.1: *For any fixed partial pairing γ, with $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{C}'$, $\mathcal{D}'$, k, l, and Q as above, and with m denoting the fiber dimension of $\mathcal{E}$, the total-pairing extensions of γ coincide with sections of a specific affine bundle of fiber dimension klm over Q, whose associated vector bundle is* $\mathrm{Hom}(\mathcal{C}/\mathcal{C}'\otimes\mathcal{D}/\mathcal{D}',\mathcal{E})$.

Proof: Our γ is nothing else than a vector-bundle morphism $\mathcal{X}\to\mathcal{E}$, where $\mathcal{X}\subset\mathcal{C}\otimes\mathcal{D}$ is the subbundle spanned by $\mathcal{C}\otimes\mathcal{D}'$ and $\mathcal{C}'\otimes\mathcal{D}$. The affine bundle in question is the preimage of the section γ under the (surjective) restriction morphism $\mathrm{Hom}(\mathcal{C}\otimes\mathcal{D},\mathcal{E})\to\mathrm{Hom}(\mathcal{X},\mathcal{E})$. □

As usual,[5] by a *pseudo-Riemannian fiber metric g* in a vector bundle $\mathcal{T}$ over a manifold M we mean any family of nondegenerate symmetric bilinear forms $g(x)$ in the fibers $\mathcal{T}_x$ that constitutes a C^∞ section of the symmetric power $(\mathcal{T}^*)^{\odot 2}$. Equivalently, such g is a pairing of $\mathcal{T}$ and $\mathcal{T}$ valued in the product bundle $M\times\mathbf{R}$, symmetric and nondegenerate at every point of M.

Let $\mathcal{T}$ again be a vector bundle over a manifold M. We define a *partial fiber metric* in $\mathcal{T}$ to be a triple $(\mathcal{P},\mathcal{P}',\alpha)$ formed by vector subbundles $\mathcal{P}$ and $\mathcal{P}'$ of $\mathcal{T}$ along with a pairing $\alpha:\mathcal{P}'\otimes\mathcal{T}\to M\times\mathbf{R}$, valued in the product bundle $M\times\mathbf{R}$, such that

(i) $\mathcal{T},\mathcal{P}$, and $\mathcal{P}'$ are of fiber dimensions n,r and, respectively, $n-r$ for some n,r with $0\le r\le n/2$, while $\mathcal{P}\subset\mathcal{P}'$,

(ii) at every $x\in M$ the bilinear mapping $\alpha(x):\mathcal{P}'_x\times\mathcal{T}_x\to\mathbf{R}$ has the rank $n-r$, its restriction to $\mathcal{P}'_x\times\mathcal{P}'_x$ is symmetric, and its restriction to $\mathcal{P}'_x\times\mathcal{P}_x$ equals 0.

By a *total-metric extension* of $(\mathcal{P},\mathcal{P}',\alpha)$ we then mean any pseudo-Riemannian fiber metric in $\mathcal{T}$ whose restriction to $\mathcal{P}'\otimes\mathcal{T}$ is α.

Lemma 4.2: *The total-metric extensions g of any partial fiber metric $(\mathcal{P},\mathcal{P}',\alpha)$, with r,M as above, coincide with the sections of a specific affine bundle of fiber dimension $r(r+1)/2$ over M. For every such g the subbundle $\mathcal{P}$ is g-null and $\mathcal{P}'$ is its g-orthogonal complement.*

Proof: For any fixed point $x\in M$, let us choose a basis $e_1,\ldots,e_n$ of $\mathcal{T}_x$ such that $e_1,\ldots,e_r\in\mathcal{P}_x$ and $e_1,\ldots,e_{n-r}\in\mathcal{P}'_x$. The matrix of $g(x)$, for any total-metric extension g of our partial fiber metric, then is the matrix appearing in Walker's original theorem (see the Appendix), with $\det A\neq 0$, and with the two occurrences of I replaced by some nonsingular $r\times r$ matrix C and its transpose C'. The submatrices A,H,C (and H',C') are prescribed, while the freedom in choosing $g(x)$ is represented by an arbitrary symmetric $r\times r$ matrix B. □

V. WALKER'S THEOREM

Suppose that the following data are given:

(a) Integers n and r with $0\le r\le n/2$.
(b) An r-dimensional manifold Σ.
(c) A bundle over Σ with some total space M, whose every fiber M_y, $y\in\Sigma$, is a $T^*_y\Sigma$-principal bundle over a $(n-2r)$-dimensional manifold Q_y (cf. the last paragraph of Sec. III).
(d) A pseudo-Riemannian metric h_y on each $Q_y, y\in\Sigma$.

We assume that all y-dependent objects in (c) and (d), including the principal-bundle structure, vary C^∞-differentiably with $y \in \Sigma$ (in the sense of Sec. II) and, in particular, the Q_y are the fibers of a bundle over Σ with a total space Q of dimension $n-r$. When $r=n/2$, each h_y is the "zero metric" on the discrete space Q_y, cf. Sec. VIII.

Let $\mathcal{F}$ be the vector bundle over Q whose restriction to Q_y, for each $y \in \Sigma$, is the M_y-prolongation of the tangent bundle TQ_y (see Sec. III) for the $T_y^*\Sigma$-principal bundle M_y over Q_y. Relation (3) now yields $\mathrm{p}^*(T^*\Sigma) \subset \mathcal{F}$, where $\mathrm{p}: Q \to \Sigma$ denotes the bundle projection. In other words, $\mathrm{p}^*(T^*\Sigma)$ may be treated as a vector subbundle of $\mathcal{F}$.

Furthermore, the quotient-bundle identification following formula (3) yields $\mathcal{F}/\mathrm{p}^*(T^*\Sigma) = \mathrm{Ker}\, d\mathrm{p}$ (the vertical subbundle of T^Q, for the projection $\mathrm{p}: Q \to \Sigma$).

We define a partial pairing γ of $\mathcal{F}$ and TQ valued in the product bundle $Q \times \mathbf{R}$, as in Sec. IV, for our Q along with $\mathcal{C}=\mathcal{F}$, $\mathcal{D}=TQ$, $\mathcal{E}=Q\times\mathbf{R}$, $\mathcal{C}'=\mathrm{p}^*(T^*\Sigma)$ and $\mathcal{D}'=\mathrm{Ker}\, d\mathrm{p}$. Namely, given $z \in Q$, we set $\gamma(\xi,\zeta)=\xi(d\mathrm{p}_z\zeta)$ for $\xi \in T_y^*\Sigma = [\mathrm{p}^*(T^*\Sigma)]_z$ and $\zeta \in T_zQ$, with $y=\mathrm{p}(z) \in \Sigma$, as well as $\gamma(u,\psi)=h_y([u],\psi)$ for $u \in \mathcal{F}_z$ and $\psi \in \mathrm{Ker}\, d\mathrm{p}_z$, where $u \mapsto [u]$ denotes the surjective vector-bundle morphism $\mathcal{F} \to \mathrm{Ker}\, d\mathrm{p}$ with the kernel $\mathrm{p}^*(T^*\Sigma)$.

Our construction has two steps involving arbitrary choices.

Step 1: We choose $\beta: \mathcal{F} \otimes TQ \to Q \times \mathbf{R}$ *to be any total-pairing extension of* γ.

According to Lemma 4.1, such β is just an arbitrary section of an affine bundle of fiber dimension $(n-2r)r$ over Q. For the meaning of the above discussion in Walker's original language, see the Appendix.

The remainder of our construction proceeds as follows. Using β, we define a partial metric $(\mathcal{P},\mathcal{P}',\alpha)$ in the tangent bundle TM. Specifically, $\mathcal{T},\mathcal{P},\mathcal{P}'$ and n,r with the properties listed in (i) and (ii) of Sec. IV are chosen so that $\mathcal{T}=TM$, while n,r are the integers in (a) above, $\mathcal{P}$ is the subbundle of TM whose restriction to $M_y \subset M$, for each $y \in \Sigma$, is the vertical distribution on the $T_y^*\Sigma$-principal bundle M_y over Q_y, and $\mathcal{P}'=\mathrm{Ker}\, d\pi$ is the vertical distribution of the bundle projection $\pi: M \to \Sigma$. We also set $\alpha(u',w)=\beta(u,\zeta)$ for any $x \in M$ and any vectors $w \in T_xM$, $u' \in \mathcal{P}'_x = T_xM_y$ with $y=\pi(x) \in \Sigma$, where u is the $T_y^*\Sigma$-invariant vector field tangent to M_y along the $T_y^*\Sigma$-orbit of x and having the value u' at x, while ζ is the image of w under the differential at x of the bundle projection $M \to Q$.

Step 2: We select an arbitrary total-metric extension g of $(\mathcal{P},\mathcal{P}',\alpha)$ *restricted to U, where U is any fixed nonempty open subset of M.*

The construction just described gives a null distribution $\mathcal{P}$ of dimension r on the n-dimensional pseudo-Riemannian manifold (U,g). This is clear from Lemma 4.2, which also implies that such metrics g are just arbitrary sections of some affine bundle over U.

The reader is again referred to the Appendix for a description of what the above steps correspond to in Walker's formulation.

We can now state a coordinate-free version of Walker's theorem.

Theorem 5.1: *If g and* $\mathcal{P}$ *are obtained as above from any prescribed data* (a)–(d), *then g is a pseudo-Riemannian metric on the n-dimensional manifold U, and* $\mathcal{P}$ *is a g-null, g-parallel distribution of dimension r on U.*

Conversely, up to an isometry, every null parallel distribution $\mathcal{P}$ *on a pseudo-Riemannian manifold* (M,g) *is, locally, the result of applying the above construction to some data* (a)–(d). *The data themselves are naturally associated with g and* $\mathcal{P}$.

A proof of Theorem 5.1 is given in the next two sections.

VI. PROOF OF THE FIRST PART OF THEOREM 5.1

By Lemma 4.2, $\mathcal{P}$ is g-null and $\mathcal{P}'$ is its g-orthogonal complement. That $\mathcal{P}$ is g-parallel will be clear if we establish the relation $\langle \nabla_w v, u\rangle = 0$, where ∇ is the Levi-Civita connection of g and $\langle,\rangle$ stands for $g(,)$, while v,u,w are any vector fields tangent to M such that v is a section of $\mathcal{P}$ and u is a section of $\mathcal{P}'$. We may further require w to be projectable under both bundle projections $M \to Q$ and $\pi: M \to \Sigma$. Finally, we may also assume that v restricted to each $T_y^*\Sigma$-principal bundle space M_y is an infinitesimal generator of the action of $T_y^*\Sigma$, while u restricted to each M_y is $T_y^*\Sigma$-invariant, (Locally, such w,v,u span the vector bundles $TM,\mathcal{P}$ and $\mathcal{P}'$.)

First, $[w,v]$ is a section of $\mathcal{P}$ and $[u,w]$ is a section of $\mathcal{P}'$ (from Remark 2.1 applied to both bundle projections), while $[v,u]=0$ by $T_y^*\Sigma$-invariance of u. The last three terms in (2) thus all equal zero.

Our claim will follow if we show that the first three terms in (2) vanish as well. To this end, note that $d_w\langle v,u\rangle=0$ since $\langle v,u\rangle=0$. Next, $d_v\langle w,u\rangle=0$. Namely, $\langle w,u\rangle=\alpha(u,w)=\beta(u,\zeta)$, for α,β,ζ described in Sec. V, is constant in the direction of v (and, in fact, constant along each leaf of $\mathcal{P}$): at a point $x\in M_y\subset M$ we obtain ζ as the projection image of $w(x)$, while u is $T_y^*\Sigma$-invariant, so that, due to projectability of w, both u and ζ depend only on the image of x under the bundle projection $M\to Q$, rather than x itself. Finally, $d_u\langle w,v\rangle=0$ as $\langle w,v\rangle=\xi(\widetilde{w})$ is a function $\Sigma\to\mathbf{R}$, that is, a function $M\to\mathbf{R}$ constant along $\mathcal{P}'$. Here ξ is the section of $T^*\Sigma$ corresponding to v under the inclusion $\mathrm{p}^*(T^*\Sigma)\subset\mathcal{F}$ of Sec. V, while $\widetilde{w}$ is the vector field on Σ onto which w projects; therefore, $\langle w,v\rangle=\xi(\widetilde{w})$, since in Sec. V we set $\gamma(\xi,\zeta)=\xi(\mathrm{dp}_z\zeta)$.

VII. PROOF OF THE SECOND PART OF THEOREM 5.1

For any null parallel distribution $\mathcal{P}$ of dimension r on an $n-$dimensional pseudo-Riemannian manifold (M,g), the g-orthogonal complement $\mathcal{P}^\perp$ is a parallel distribution of dimension $n-r$. If the sign pattern of g has i_- minuses and i_+ pluses, it follows that

$$r\leq\min(i_-,i_+),\tag{4a}$$

$$\mathcal{P}\subset\mathcal{P}^\perp,\tag{4b}$$

$$r\leq n/2.\tag{4c}$$

In fact, $\mathcal{P}$ is null, which gives (4b) and $r\leq n-r$, that is (4c), while (4a) follows since, in a pseudo-Euclidean space with the sign pattern as above, i_- (or, i_+) is the maximum dimension of a subspace on which the inner product is negative (or, positive) semidefinite.

Every null parallel distribution $\mathcal{P}$ satisfies the curvature relations

$$R(\mathcal{P},\mathcal{P}^\perp,-,-)=0,\tag{5a}$$

$$R(\mathcal{P},\mathcal{P},-,-)=0,\tag{5b}$$

$$R(\mathcal{P}^\perp,\mathcal{P}^\perp,\mathcal{P},-)=0,\tag{5c}$$

(5a) meaning that $R(v,u,w,w')=0$ whenever v,u,w,w' are vector fields, v is a section of $\mathcal{P}$, and u is a section of $\mathcal{P}^\perp$. [Similarly for (5b) and (5c).] In fact, for such v,u,w,w', (1) implies that $R(w,w')v$ is a section of $\mathcal{P}$, and so it is orthogonal to u. This proves (5a); (5a) and (4b) yield (5b), while (5a) and the first Bianchi identity give (5c).

We now show how a null parallel distribution $\mathcal{P}$ on a pseudo-Riemannian manifold (M,g) gives rise to objects (a)–(d) in Sec. V.

First, n and r are the dimensions of M and $\mathcal{P}$. By (4c), $r\leq n/2$.

Being parallel, the distribution $\mathcal{P}^\perp$ is integrable. Since our discussion is local, we will assume, from now on, that M is the total space of a bundle over some r-dimensional base manifold Σ, whose fibers M_y, $y\in\Sigma$, are all contractible and coincide with the leaves of $\mathcal{P}^\perp$. As $\mathcal{P}$ is parallel, the Levi-Civita connection ∇ induces a connection in the vector bundle obtained by restricting $\mathcal{P}$ to any given submanifold N of M. In the case where $N=M_y$ is a leaf of $\mathcal{P}^\perp$, we have, for each $y\in\Sigma$, the following conclusion.

$$T_y^*\Sigma \text{ is naturally isomorphic to the space } V_y \text{ of those sections of the restriction of } \mathcal{P} \text{ to } M_y \text{ which are parallel (along } M_y). \tag{6}$$

Instead of establishing (6) directly, we will show that *sections of* $T^*\Sigma$ *can be naturally identified with sections of* $\mathcal{P}$ *parallel along* $\mathcal{P}^\perp$, using an identification which is clearly valuewise, i.e., consists of operators $V_y \to T_y^*\Sigma, y \in \Sigma$. To this end, we denote by π the bundle projection $M \to \Sigma$. Every vector field on Σ is the π-image $(\mathrm{d}\pi)w$ of some π-projectable vector field w on M. Let v now be a section of the vector bundle $\mathcal{P}$ over M, parallel in the direction of $\mathcal{P}^\perp$. Our identification associates with v the cotangent vector field ξ on Σ that sends each vector field $(\mathrm{d}\pi)w$ to $g(v,w)$ treated as a function $\Sigma \to \mathbf{R}$. Note that ξ is well defined: two π-projectable vector fields w on M with the same π-image $(\mathrm{d}\pi)w$ differ by a section of $\mathcal{P}^\perp = \mathrm{Ker}\,\mathrm{d}\pi$, necessarily orthogonal to v, so that $g(v,w)$ is the same for both choices of w. Also, $g(v,w): M \to \mathbf{R}$ actually descends to a function $\Sigma \to \mathbf{R}$, i.e., is constant along the fibers M_y (leaves of $\mathcal{P}^\perp$). In fact, $d_u[g(v,w)]=0$ for any section u of $\mathcal{P}^\perp$, as $\nabla_u v=0$ in view of the assumption about v, and $\nabla_u w=[u,w]+\nabla_w u$, while $[u,w]$ (or $\nabla_w u$) is a section of $\mathcal{P}^\perp$ by Remark 2.1 (or, since $\mathcal{P}^\perp$ is parallel).

Injectivity of the above assignment $v \mapsto \xi$ is obvious, since π-projectable vector fields w span TM. Surjectivity of the resulting operators $V_y \to T_y^*\Sigma$ now follows: both spaces have the same dimension, as the connections induced by ∇ in the restrictions of $\mathcal{P}$ to the leaves M_y are flat in view of (5c) [cf. (1)]. This proves (6).

Flatness of the induced connections also implies that the leaves of $\mathcal{P}$ contained in any given leaf M_y of $\mathcal{P}^\perp$ are the fibers of a V_y-principal bundle with the total space M_y over some base manifold Q_y. (Here M should be replaced with an open subset, if necessary.) Since each $T_y^*\Sigma$ is identified with V_y by (6), we thus obtain the data (c) of Sec. V.

Next, we define the metric h_y on each Q_y, required by (d) in Sec. V, so that it assigns the function $g(u,u')$ to two vector fields on Q_y which are images, under the $T_y^*\Sigma$-principal bundle projection $M_y \to Q_y$, of $T_y^*\Sigma$-invariant vector fields u,u' on M_y. Constancy of $g(u,u')$ along the $T_y^*\Sigma$-orbits, meaning that $d_v[g(u,u')]=0$ for any section v of $\mathcal{P}$ defined on M_y and parallel along $\mathcal{P}^\perp$, now follows: as v is $\mathcal{P}^\perp$-parallel and u is $T_y^*\Sigma$-invariant, we have $\nabla_u v=[v,u]=0$, cf. (6), so that $\nabla_v u=0$. For the same reason, $\nabla_v u'=0$.

Finally, a suitable version of the construction in Sec. V, applied to the data (a)–(d) defined above, leads to the original g and $\mathcal{P}$, which is a consequence of how the identification (6) and the definition of h_y use g. The choices of the total-pairing and total-metric extensions, required in Sec. V, are provided by g as well. For instance, β in Step 1 is given by $\beta(u,\zeta)=g(u,w)$, where u is a section of $\mathcal{P}^\perp$ commuting with every section v of $\mathcal{P}$ that is parallel along $\mathcal{P}^\perp$, and ζ is a vector field on Q (the union of all Q_y), while w is any vector field on M projectable onto ζ under the bundle projection $M \to Q$. That $g(u,w)$ depends just on u and ζ (but not on w) is clear: two choices of w differ by a section of $\mathcal{P}$. Also, $g(u,w)$ is constant in the direction of $\mathcal{P}$ (and so it may be treated as a function $Q \to M$). Namely, $d_v[g(u,w)]=0$ for any section v of $\mathcal{P}$ parallel along $\mathcal{P}^\perp$, which follows as $\nabla_v u=\nabla_u v=0$ (note that $[u,v]=0$), while $\nabla_v w=[v,w]+\nabla_w v$, and $[v,w]$ (or $\nabla_w v$) is a section of $\mathcal{P}$ by Remark 2.1 (or, respectively since $\mathcal{P}$ is parallel). This completes the proof of Theorem 5.1.

VIII. THE MID-DIMENSIONAL CASE

For an r-dimensional null parallel distribution $\mathcal{P}$ on a pseudo-Riemannian manifold (M,g) of dimension $n=2r$, the discussion in Sec. V amounts to nothing new: implicitly at least, it is already present in Sec. (6) of Walker's original paper.[1] See also Sec. 9 in Ref. 3. (A related global result is Theorem 5 in Ref. 5.) In this section we point out how the construction may be simplified when $n=2r$.

Let $\mathcal{P}$ and (M,g) be as above, with $n=2r \geq 2$. The relations $i_- + i_+ = n$ and (4a) imply that g has the *neutral* sign pattern: $i_- = i_+ = r = n/2$. In (c) and (d) of Sec. V, each Q_y is a 0-dimensional (discrete) manifold, and h_y is the "zero metric" on Q_y. Also, the choice of a total-pairing extension β in Step 1 of Sec. V is now unique: the affine bundle having β as a section is of fiber dimension 0. The construction in Sec. V can therefore be rephrased as follows. Given

(a) an even integer $n \geq 2$,
(b) a manifold Σ of dimension $r=n/2$,

(c) an affine bundle over Σ with some total space M, for which $T^*\Sigma$ is the associated vector bundle (Sec. III),

we define a partial metric $(\mathcal{P},\mathcal{P}',\alpha)$ in the tangent bundle TM by choosing $\mathcal{P}=\mathcal{P}'$ to be the vertical distribution $\operatorname{Ker} d\pi$ for the bundle projection $\pi:M\to\Sigma$, and setting $\alpha(\xi,w)=\xi(d\pi_x w)$ for any $x \in M, \xi\in\mathcal{P}_x=T_y^*\Sigma$, where $y=\pi(x)$, and $w\in T_xM$. Selecting any total-metric extension g of $(\mathcal{P},\mathcal{P}',\alpha)$ on a fixed nonempty open set $U\subset M$, we now obtain an n-dimensional pseudo-Riemannian manifold (U,g) on which $\mathcal{P}$ is a g-null, g-parallel distribution of dimension $r=n/2$.

Conversely, up to an isometry, every null parallel distribution $\mathcal{P}$ of dimension $r\geq 1$ on a pseudo-Riemannian manifold (M,g) with $\dim M=2r$ arises, locally, from the above construction applied to some data (a)–(c), themselves naturally determined by g and $\mathcal{P}$.

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APPENDIX: WALKER'S ORIGINAL STATEMENT

Walker stated his classification result as follows.[1]

Theorem 1: *A canonical form for the general V_n of class C^∞ (or C^ω) admitting a parallel null r-plane is given by the fundamental tensor*

$$(g_{ij})=\begin{pmatrix} O & O & I \\ O & A & H \\ I & H' & B \end{pmatrix},$$

where I is the unit $r\times r$ matrix and A,B,H,H' are matrix functions of the coordinates, of the same class as V_n, satisfying the following conditions but otherwise arbitrary:

(i) *A and B are symmetric, A is of order $(n-2r)\times(n-2r)$ and nonsingular, B is of order $r\times r$, H is of order $(n-2r)\times r$, and H' is the transpose of H.*

(ii) *A and H (and therefore H') are independent of the coordinates $x^1,\ldots,x^r$.*

A basis for the parallel null r-plane is the set of vectors $\delta_1^i,\delta_2^i,\ldots,\delta_r^i$.

Here is how the coordinates and matrix functions appearing above correspond to the objects used for the construction in Sec. V. Walker's coordinates $x^i, i=1,\ldots,n$, serve as a coordinate system for the manifold M of Sec. V. Coordinates for other manifolds appearing in Sec. V are obtained from x^i by restricting the range of the index i, to $i>n-r$ (for Σ), $i>r$ (for Q), $i\leq n-r$ (for each M_y) and $r<i\leq n-r$ (for each Q_y). The center submatrix A in Walker's matrix corresponds to the family $h_y, y\in\Sigma$, of pseudo-Riemannian metrics [(d) in Sec. V] and, consequently, also to the formula for $\gamma(u,\psi)$, while the last two matrices $O\ I$ in the first row represent the definition of $\gamma(\xi,\zeta)$. The Walker-matrix counterpart of the extension β chosen in Step 1 is the $(n-r)\times(n-r)$ submatrix with the rows $O\ I$ and $A\ H$, so that the freedom in choosing β amounts to arbitrariness in the selection of H (and H is independent of the coordinates $x^i, i=1,\ldots,r$, which translates into the fact that β is a morphism of vector bundles over the manifold Q with the coordinates $x^i, i>r$). Once chosen, β is used in Sec. V to define $\mathcal{P},\mathcal{P}'$ and α. In terms of Walker's coordinates and matrix functions, $\mathcal{P}$ (or, $\mathcal{P}'$) is spanned by the x^i coordinate directions with $i\leq r$ (or, respectively, $i\leq n-r$), while the analog of α is the $(n-r)\times n$ submatrix with the rows $O\ O\ I$ and $O\ A\ H$. Finally, the extension in Step 2 is nothing else than augmenting this last submatrix by a third row, $I\ H'B$, in which B is completely arbitrary.

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