

KÄHLER MANIFOLDS WITH GEODESIC HOLOMORPHIC GRADIENTS

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ABSTRACT.—A vector field on a Riemannian manifold is called *geodesic* if its integral curves are reparametrized geodesics. We prove a dichotomy theorem about compact Kähler manifolds admitting nontrivial real-holomorphic geodesic gradient vector fields: such a manifold must either satisfy an additional integrability condition implying, in particular, that it is biholomorphically isometric to a bundle of complex projective spaces with a bundle-like metric, or else through every zero of the real-holomorphic geodesic gradient vector field there passes an uncountable family of totally geodesic, holomorphically immersed complex projective spaces, each carrying a fixed multiple of the Fubini-Study metric.

Introduction

We say that a vector field on a Riemannian manifold is *geodesic* if its integral curves are reparametrized geodesics. The present paper discusses

(0.1) triples (M, g, τ) consisting of a compact complex manifold M , a Kähler metric g on M , and a nonconstant function $\tau : M \rightarrow \mathbb{R}$, the g -gradient of which is both geodesic and real-holomorphic.

We observe (Remark 12.1) that for $m = \dim_{\mathbb{C}} M$ and $d_{\pm} = \dim_{\mathbb{C}} \Sigma^{\pm}$, where Σ^{+} and Σ^{-} are the maximum and minimum level sets of τ , one then has

$$(0.2) \quad d_{+} + d_{-} \geq m - 1 \geq d_{\pm} \geq 0,$$

and every $(d_{+}, d_{-}, m) \in \mathbb{Z}^3$ satisfying (0.2) is realized by some (M, g, τ) with (0.1).

One of our three main results, Theorem 17.4, classifies the triples (0.1) such that

$$(0.3) \quad \text{Ker } d\pi^{+} \text{ and } \text{Ker } d\pi^{-} \text{ span an integrable distribution on } M'.$$

Here $M' = M \setminus (\Sigma^{+} \cup \Sigma^{-})$, while $\pi^{\pm} : M \setminus \Sigma^{\mp} \rightarrow \Sigma^{\pm}$ sends each $x \in M \setminus \Sigma^{\mp}$ to the unique point nearest x in $\Sigma^{\pm}$. (In case (0.1) $\pi^{\pm}$ always are disk-bundle projections, and their vertical distributions $\text{Ker } d\pi^{\pm}$ span a vector subbundle of TM' , cf. Section 11; however, (0.1) does not imply (0.3) – see Remark 20.5.)

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As a consequence of Theorem 17.4, in every triple with (0.1) and (0.3),

$$(0.4) \quad \begin{array}{l} M \text{ is biholomorphic to a bundle of (positive-dimensional) complex} \\ \text{projective spaces over some base manifold } B \text{ with } \dim_{\mathbb{C}} B \geq 0. \end{array}$$

The remaining two main results of the paper, Theorems 15.1 and 19.1, deal with the general case of (0.1), that is, do not assume (0.3).

According to Theorem 15.1, whenever $\Pi^{\pm}$ is a leaf of either (obviously integrable) vertical distribution $\text{Ker } d\pi^{\pm}$, the *other* projection $\pi^{\mp}$ maps $\Pi^{\pm} \cap M'$ onto the image $F(\mathbb{CP}^k)$ of some totally geodesic holomorphic immersion $F : \mathbb{CP}^k \rightarrow \Sigma^{\mp}$ inducing on $\mathbb{CP}^k$ a multiple of the Fubini-Study metric, with $k = k_{\pm} \geq 0$ given by $k_{\pm} = m - 1 - d_{\pm}$. Both $\Sigma^{\pm}$ are themselves (connected) totally geodesic compact complex submanifolds of M , cf. Remark 11.1(iii).

The third main result reveals a dichotomy involving the assignment

$$(0.5) \quad M' \ni x \mapsto d\pi_x^{\pm}(\text{Ker } d\pi_x^{\mp}) \in \text{Gr}_k(T_y \Sigma^{\pm}) \text{ for } y = \pi^{\pm}(x),$$

$\text{Gr}_k(T_y \Sigma^{\pm})$ being the complex Grassmannian, with $k = k_{\pm}$ defined as before.

Specifically, Theorem 19.1 states that one of the following two cases has to occur. First, (0.5) may be *constant* on every leaf of $\text{Ker } d\pi_x^{\pm}$ in M' , that is, on every fibre $\Pi^{\pm}$ of the projection $\pi^{\pm} : M \setminus \Sigma^{\mp} \rightarrow \Sigma^{\pm}$ restricted to M' , with either sign $\pm$. Otherwise, $l = k_{\mp}$ and $k = k_{\pm}$ are positive for both signs $\pm$, while (0.5) restricted to any such leaf $\Pi^{\pm}$ must be a composite mapping $\Pi^{\pm} \rightarrow \mathbb{CP}^l \rightarrow \text{Gr}_k(T_y \Sigma^{\pm})$ formed by a holomorphic bundle projection $\Pi^{\pm} \rightarrow \mathbb{CP}^l$, having the fibre $\mathbb{C} \setminus \{0\}$, and a *nonconstant holomorphic embedding* $\mathbb{CP}^l \rightarrow \text{Gr}_k(T_y \Sigma^{\pm})$.

The first case of Theorem 19.1 is equivalent to condition (0.3), and the immersions $\mathbb{CP}^k \rightarrow \Sigma^{\pm}$, mentioned in the above summary of Theorem 15.1, are then embeddings, for both signs $\pm$, while their images constitute foliations of $\Sigma^{\pm}$, both with the same leaf space B appearing in (0.4). See Remark 19.2.

In the second case (cf. Remark 19.3) the images of these immersions, rather than being pairwise disjoint, are totally geodesic, holomorphically immersed complex projective spaces, an uncountable family of which passes through each point of $\Sigma^{\pm}$.

Three special classes of the objects (0.1) have been studied before. One is provided by the gradient Kähler-Ricci solitons discovered by Koiso [16] and, independently, Cao [4], where τ is the soliton function; two more – by special Kähler-Ricci potentials τ on compact Kähler manifolds [8], and by triples with (0.1) such that M is a (compact) complex surface [6]. Each of these three classes satisfies (0.3).

The papers [8, 6] provide complete explicit descriptions of the classes discussed in them. Our Theorem 17.4 generalizes their classification results, namely, [8, Theorem 16.3] and [6, Theorem 6.1].

For more details on the preceding two paragraphs, see Remark 17.5.

Functions with geodesic gradients on arbitrary Riemannian manifolds, usually called *transnormal*, have been studied extensively as well [20, 17, 2].

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1. Preliminaries

Manifolds, mappings and tensor fields, including Riemannian metrics and functions, are by definition of class C^∞ . A (sub)manifold is always assumed connected.

Our sign convention about the curvature tensor $R = R^\nabla$ of a connection ∇ in a vector bundle E over a manifold M is

$$(1.1) \quad R(v, w)\xi = \nabla_w \nabla_v \xi - \nabla_v \nabla_w \xi + \nabla_{[v, w]}\xi$$

for any section ξ of E and vector fields v, w tangent to M . One may treat $R(v, w)$, the covariant derivative $\nabla\xi$, and any function f on M as bundle morphisms

$$(1.2) \quad R(v, w), f : E \rightarrow E, \quad \nabla\xi : TM \rightarrow E$$

sending ξ or v as above to $R(u, v)\xi$, $f\xi$ or, respectively, $\nabla_v\xi$. Notation of (1.2) is used in the next three displayed relations.

In the case of a Riemannian manifold (M, g) , the symbol ∇ will always stand for the Levi-Civita connection of g as well as the g -gradient. Given a function τ and vector fields w, w' on (M, g) , one has the Lie-derivative relation

$$(1.3) \quad [\mathcal{L}_v g](w, w') = 2g(Sw, w'), \quad \text{where } v = \nabla\tau \text{ and } S = \nabla v : TM \rightarrow TM,$$

due to the local-coordinate equalities $[\mathcal{L}_v g]_{jk} = v_{j,k} + v_{k,j} = 2v_{j,k}$. For vector fields v, u on a manifold M and a bundle morphism $B : TM \rightarrow TM$, the Leibniz rule gives $[\mathcal{L}_v B]u = [v, Bu] - B[v, u] = [\nabla_u B]u + B\nabla_u v - \nabla_{Bu}v$, and so

$$(1.4) \quad \mathcal{L}_v B = \nabla_u B + [B, \nabla v].$$

Next, let u be a Killing vector field on a Riemannian manifold (M, g) . The Ricci and Bianchi identities imply, as in [9, bottom of p. 572], the well-known relation

$$(1.5) \quad \nabla_v A = R(u, v), \quad \text{with } A = \nabla u.$$

Since the flow of a Killing field preserves the Levi-Civita connection, (1.5) also follows from the classical Lie-derivative equality $[\mathcal{L}_u \nabla]_v w = [\nabla_v A]w - R(u, v)w$, with $A = \nabla u$, cf. [18, formula (1.8) on p. 337], valid for any connection ∇ in TM .

Whenever $\tau : M \rightarrow \mathbb{R}$ is a function on a Riemannian manifold (M, g) , we have

$$(1.6) \quad \nabla Q = 2\nabla_v v, \quad \text{where } v = \nabla\tau \text{ and } Q = g(v, v),$$

as one sees noting that, in local coordinates, $(\tau_{,k}\tau^{,k})_{,j} = 2\tau_{,kj}\tau^{,k}$. Also, obviously

$$(1.7) \quad d_v \tau = g(v, \nabla\tau) = Q \quad \text{if } v = \nabla\tau \text{ and } Q = g(v, v).$$

Remark 1.1. Relation (1.3) becomes $d_v[g(w, w')] = 2g(Sw, w')$ if, in addition, v commutes with w and w' . Namely, $d_v = \mathcal{L}_v$ on functions, so that we may evaluate $d_v[g(w, w')]$ using the Leibniz rule for the Lie derivative with $\mathcal{L}_v w = \mathcal{L}_v w' = 0$.

Remark 1.2. Whenever the g -gradient $v = \nabla \tau$ of a function τ on a Riemannian manifold (M, g) is tangent to a submanifold Π with the submanifold metric g' , the restriction of v to Π obviously equals the g' -gradient of $\tau : \Pi \rightarrow \mathbb{R}$.

Remark 1.3. Given a manifold M and $\sigma, \tau : M \rightarrow \mathbb{R}$, we call σ a C^∞ function of τ if τ is nonconstant (so that its range $\tau(M)$ is an interval) and $\sigma = \chi \circ \tau$ for some C^∞ function $\chi : \tau(M) \rightarrow \mathbb{R}$. Note that χ is then uniquely determined by σ and τ . We will denote by σ both the original function $M \rightarrow \mathbb{R}$ and the function $\chi : I \rightarrow \mathbb{R}$ of the variable $\tau \in I$.

Let $(t, s) \mapsto x(t, s) \in M$ be a fixed *variation of curves* in a manifold M , that is, a C^∞ mapping in which the real variables t, s range independently over intervals. The *partial derivative* x_t (or, x_s) then assigns to each (t_0, s_0) the velocity vector at t_0 (or, s_0) of the curve $t \mapsto x(t, s_0)$ or, respectively, $s \mapsto x(t_0, s)$. (Thus, x_t and x_s are sections of a specific pullback bundle.) A connection ∇ on M allows us to define the *mixed second-order partial derivatives* x_{ts} and x_{st} of the variation, so that, for instance, the value of x_{ts} at (t_0, s_0) is the ∇ -covariant derivative, at the parameter s_0 , of the vector field $s \mapsto x_t(t_0, s)$ along the curve $s \mapsto x(t_0, s)$, and analogously for x_{st} . Obviously, $x_{st} = x_{ts}$ when ∇ is torsion-free, cf. [8, p. 101].

Remark 1.4. For a torsion-free connection ∇ on a manifold M and a smooth variation $(t, s) \mapsto x(t, s) = \exp_{y(s)}(t - t_0)\xi(s)$ of ∇ -geodesics, with (t, s) near (t_0, s_0) in $\mathbb{R}^2$ and a vector field $s \mapsto \xi(s) \in T_{y(s)}M$ along a curve $s \mapsto y(s) \in M$, let $t \mapsto \hat{w}(t)$ be the Jacobi vector field along the geodesic $t \mapsto x(t) = x(t, s_0)$ defined by $\hat{w}(t) = x_s(t, s_0)$ (notation of the last paragraph). Then $[\nabla_x \hat{w}](t_0) = [\nabla_y \xi](s_0)$, which is nothing else than $x_{st} = x_{ts}$ (see above) at $(t, s) = (t_0, s_0)$. Also, clearly, $\hat{w}(t_0) = \dot{y}(s_0)$. Note that $\nabla_y \xi$ may be nonzero even if $y(s) = y$ is a constant curve, as it then equals the ordinary derivative of $s \mapsto \xi(s) \in T_y M$ with respect to s .

Remark 1.5. Let N be a vector bundle over a manifold Σ . We use the same symbol N for its total space, which we identify, as a set, with $\{(y, \xi) : y \in \Sigma \text{ and } \xi \in N_y\}$. Given a connection D in N and vector fields v, w tangent to Σ ,

(1.8) the D -horizontal lifts of v and w commute if so do v, w and $R^D(v, w) = 0$

(notation of (1.2)). Namely, at any $x = (y, \xi) \in N$, the vertical (or, horizontal) component of the Lie bracket of the horizontal lifts of v and w equals $R_y^D(v_y, w_y)\xi$ (an easy exercise) or, respectively, the horizontal lift of $[v, w]_y$, cf. [15, p. 10].

Remark 1.6. Let D be the normal connection in the normal bundle $N\Sigma$ of a totally geodesic submanifold Σ in a Riemannian manifold (M, g) . We denote by $\text{Exp}^\perp : U \rightarrow M$ the normal exponential mapping of Σ , the domain of which is an open submanifold U of the total space $N\Sigma$ such that, for every normal space $N_y \Sigma$, where $y \in \Sigma^\pm$, the intersection $U \cap N_y \Sigma$ is nonempty and star-shaped (in the sense of being a union of line segments emanating from 0). Remark 1.4 leads to the following well-known description of the differential $d\text{Exp}_{(y, \xi)}^\perp$ of $\text{Exp}^\perp$ at

any $(y, \xi) \in U$, cf. Remark 1.5. Specifically, we may assume that $\xi \neq 0$ since, clearly, $d\text{Exp}_{(y, \xi)}^\perp = \text{Id}$ when $\xi = 0$, under the obvious isomorphic identification $T_{(y, 0)}[N\Sigma] = T_y\Sigma \oplus N_y\Sigma = T_yM$. The point $y \in \Sigma$ and the normal vector $\xi \in N_y\Sigma$ thus have the property that the nontrivial geodesic $r \mapsto x(r) = \exp_y r\xi$ is defined for all $r \in [0, 1]$. If $r > 0$, a vector tangent to $N\Sigma$ at $(y, r\xi)$ can be uniquely written as $r\eta + w_r^{\text{hrz}}$, where $\eta \in N_y\Sigma = T_{(y, r\xi)}[N_y\Sigma]$ is vertical and w_r^{hrz} denotes the D-horizontal lift of some $w \in T_y\Sigma$. Then, for the Jacobi field $r \mapsto \hat{w}(r)$ along our geodesic $r \mapsto x(r)$ such that $\hat{w}(0) = w$ and $[\nabla_{\dot{x}}\hat{w}](0) = \eta$,

$$(1.9) \quad d\text{Exp}_{(y, r\xi)}^\perp(r\eta + w_r^{\text{hrz}}) = \hat{w}(r) \quad \text{whenever } r \in [0, 1].$$

In fact, linearity of both sides in (η, w) allows us to consider two separate cases, $w = 0$ and $\eta = 0$. For s close to 0 in $\mathbb{R}$ and $r \in [0, 1]$, let us set $x(r, s) = \exp_{y(s)} r\xi(s)$, where in the former case $(y(s), \xi(s)) = (y, \xi + s\eta)$, and in the latter $s \mapsto \xi(s)$ is the D-parallel normal vector field with $\xi(0) = \xi$ along a fixed curve $s \mapsto y(s) \in \Sigma$ such that $y(0) = y$ and $\dot{y}(0) = w$. Thus, in both cases, the curve $s \mapsto (y(s), r\xi(s))$ in U has, at $s = 0$, the velocity $r\eta + w_r^{\text{hrz}}$. The velocity at $s = 0$ of its $\text{Exp}^\perp$ -image curve $s \mapsto x(r, s)$ therefore equals the left-hand side of (1.9). At the same time this last velocity is $\hat{w}(r) = x_s(r, 0)$ for $\hat{w}$ defined as in Remark 1.4 with the variable t and (t_0, s_0) replaced by r and $(0, 0)$. Now (1.9) follows since the two definitions of $\hat{w}$ agree: according to Remark 1.4, both Jacobi fields denoted by $\hat{w}$ satisfy the same initial conditions at $s = 0$.

Remark 1.7. Every Killing vector field u on a Riemannian manifold is a Jacobi field along any geodesic $t \mapsto x(t)$. In fact, the local flow of u , applied to the geodesic, yields a variation of geodesics. (Equivalently, one may note that (1.5) with $v = \dot{x}$, evaluated on $\dot{x}$, is precisely the Jacobi equation.)

Remark 1.8. Let $\Psi : \Pi \rightarrow M$ be a totally geodesic immersion of a manifold Π in a Riemannian manifold (M, g) . If $\Psi(A) \subseteq \Sigma$ and $\Psi(\Pi \setminus A) \subseteq M \setminus \Sigma$ for submanifolds A of Π and Σ of M , such that Σ is totally geodesic in (M, g) , then, for Σ endowed with the submanifold metric, $\Psi : A \rightarrow \Sigma$ is a totally geodesic immersion.

In fact, every point of A has a neighborhood U in Π on which Ψ is an embedding with a totally geodesic image $\Psi(U)$. Our claim now follows since the submanifold $\Psi(A \cap U)$ of Σ , being the intersection of the totally geodesic submanifolds $\Psi(U)$ and Σ , must itself be totally geodesic.

Remark 1.9. Let R, R' and $\hat{R}$ be the curvature tensors of connections ∇, ∇' in vector bundles E, E' over a fixed base manifold and, respectively, of the connection $\hat{\nabla}$ induced by them in the vector bundle $\text{Hom}(E, E')$. Then $\hat{R}$ is given by the commutator-type formula $\hat{R}(v, w)\Theta = [R'(v, w)]\Theta - \Theta[R(v, w)]$, cf. (1.2), for any section Θ of $\text{Hom}(E, E')$ (that is, any vector-bundle morphism $\Theta : E \rightarrow E'$) and vector fields v, w tangent to the base. This trivially follows from (1.1) and the fact that $[\hat{\nabla}_v\Theta]\xi = \nabla'_v(\Theta\xi) - \Theta\nabla_v\xi$ whenever ξ is a section of E .

2. Projectability of distributions

As usual, whenever $\pi : M \rightarrow B$ is a mapping between manifolds, we say that a vector field w (or, a distribution $\mathcal{E}$) on M is π -projectable if

$$(2.1) \quad d\pi_x w_x = u_{\pi(x)} \quad \text{or, respectively,} \quad d\pi_x(\mathcal{E}_x) = \mathcal{H}_{\pi(x)}$$

for some vector field u (or, some distribution $\mathcal{H}$) on B and all $x \in M$.

Remark 2.1. Let $\pi : M \rightarrow B$ be a bundle projection. A vector field w on M is π -projectable if and only if, for every section v of the vertical distribution $\mathcal{V} = \text{Ker } d\pi$, the Lie bracket $[v, w]$ is also a section of $\mathcal{V}$. This is easily verified in local coordinates for M that make π appear as a Cartesian-product projection.

Remark 2.2. For $\pi, M, B, \mathcal{V}$ as in Remark 2.1, a π -projectable vector field w on M , and $x \in M$ such that $w_x \neq 0$, every prescribed value $u_x \in \mathcal{V}_x$ is realized by a local section u of $\mathcal{V}$ commuting with w . Namely, we may first prescribe such u along a fixed codimension-one submanifold containing x , which is transverse to w at x , and then use the local flow of w to spread u over a neighborhood of x .

Remark 2.3. Given a vector field v and a distribution $\mathcal{E}$ on a manifold, the local flow $t \mapsto e^{tv}$ of v preserves $\mathcal{E}$ if and only if, whenever w is a local section of $\mathcal{E}$, so is $[v, w]$. Namely, $[v, w] = \mathcal{L}_v w$, while, denoting by $\Theta \mapsto (de^{tv})\Theta$ the push-forward action of e^{tv} on tensor fields Θ of any type, we have

$$(2.2) \quad d[(de^{tv})\Theta]/dt = -(de^{tv})\mathcal{L}_v\Theta.$$

In fact, when $t = 0$, (2.2) is just the definition of $\mathcal{L}_v\Theta$, while, for arbitrary t , it follows from the group-homomorphic property of $t \mapsto e^{tv}$.

Remark 2.4. We say that a vector field w (or, a distribution $\mathcal{H}$) on a manifold M is *projectable along an integrable distribution* $\mathcal{V}$ on M , or $\mathcal{V}$ -projectable, if it is π -projectable, as in (2.1), when restricted to any open submanifold of M on which $\mathcal{V}$ forms the vertical distribution $\text{Ker } d\pi$ of a bundle projection π . For w this amounts to invariance of $\mathcal{V}$ under the local flow of w , cf. Remarks 2.1 and 2.3.

Remark 2.5. For an integrable distribution $\mathcal{Z}$ on a Riemannian manifold (M, g) , the following two conditions are equivalent.

- (i) $d_w[g(v, v)] = 0$ for all local sections v of $\mathcal{Z}$ and w of $\mathcal{Z}^\perp$ such that w is nonzero, $\mathcal{Z}$ -projectable, and $[v, w] = 0$.
- (ii) Every leaf (maximal integral manifold) of $\mathcal{Z}$ is totally geodesic in (M, g) .

In fact, let b be the second fundamental form of the leaves of $\mathcal{Z}$, with $b(v, v)$ equal to the $\mathcal{Z}^\perp$ component of $\nabla_v v$. If v and w commute, $d_w[g(v, v)] = 2g(\nabla_w v, v) = 2g(\nabla_v w, v) = -2g(\nabla_v v, w) = -2g(b(v, v), w)$, while b is symmetric and v, w as in (i) realize, at any $x \in M$, any given elements of $\mathcal{Z}_x$ and $\mathcal{Z}_x^\perp$ (see Remark 2.2).

Remark 2.6. Clearly, (ii) in Remark 2.5 also follows when $d_w[g(v, v)] = 0$ for all v, w satisfying specific further conditions besides (i), as long as the last line of Remark 2.5 still applies.

Lemma 2.7. *For two integrable distributions $\mathcal{E}^\pm$ on a manifold M such that the span $\mathcal{E}$ of $\mathcal{E}^+$ and $\mathcal{E}^-$ has constant dimension, the following conditions are all equivalent.*

- (a) $\mathcal{E}$ is integrable.
- (b) $\mathcal{E}^+$ is projectable along $\mathcal{E}^-$.
- (c) $\mathcal{E}^-$ is projectable along $\mathcal{E}^+$.

If (a) – (c) hold, the distributions that $\mathcal{E}^\pm$ locally project onto are integrable as well.

Proof. We may assume that $\mathcal{E}^+$ is the vertical distribution of a bundle projection $\pi : M \rightarrow B$ with connected fibres. First, let $\mathcal{E}$ be integrable. Since $\mathcal{E}$ contains $\mathcal{E}^+ = \text{Ker } d\pi$, its leaves are unions of fibres and so their π -images form a foliation of B , tangent to a distribution $\mathcal{H}$ satisfying (2.1), which proves projectability of $\mathcal{E}$, and hence of $\mathcal{E}^-$, along $\mathcal{E}^+$. In other words, (a) implies (c). Conversely, assuming (c), we obtain $d\pi_x(\mathcal{E}_x) = d\pi_x(\mathcal{E}_x^-) = \mathcal{H}_{\pi(x)}$ for all $x \in M$ and some distribution $\mathcal{H}$ on B , which is necessarily integrable: its leaves are π -images of the leaves of $\mathcal{E}^-$. Integrability of $\mathcal{E}$ now follows, as its leaves are the π -preimages of those of $\mathcal{H}$.

Finally, as (a) involves $\mathcal{E}^+$ and $\mathcal{E}^-$ symmetrically, it is also equivalent to (b). $\square$

3. Kähler manifolds

For Kähler manifolds we use symbols such as (M, g) , where M stands for the underlying complex manifold. Generally, in complex manifolds,

(3.1) J always denotes the complex-structure tensor.

Let v be a vector field on a Kähler manifold (M, g) . Since $\nabla J = 0$, one has

(3.2) $A = JS$ if one sets $S = \nabla v$ and $A = \nabla u$, for $u = Jv$,

J, S, A being viewed as bundle morphisms $TM \rightarrow TM$, cf. (1.2). For the curvature tensor R of a Kähler manifold (M, g) and any vector fields u, v on M ,

(3.3) $R(u, v) = R(Ju, Jv) : TM \rightarrow TM$ and $J : TM \rightarrow TM$ commute.

In fact, the condition $\nabla J = 0$ turns ∇ into a connection in TM treated as a *complex* vector bundle, g being the real part of a ∇ -parallel Hermitian fibre metric, that is,

(3.4) $g(Jw, Jw') = g(w, w')$

for all vector fields w, w' on M , and so the curvature operators $R(u, v)$ are all complex-linear and skew-Hermitian. The former property now amounts to commutation in (3.3), the latter to the equality $g(R(u, v)w, w') = g(R(w, w')u, v) = g(R(w, w')Ju, Jv) = g(R(Ju, Jv)w, w')$, with any vector fields w, w' .

Real-holomorphic vector fields v on Kähler manifolds will always be briefly referred to as *holomorphic*. Since they are characterized by $\mathcal{L}_v J = 0$, formula (1.4) for $B = J$ implies that, given a vector field v on a Kähler manifold (M, g) ,

(3.5) v is holomorphic if and only if $S = \nabla v$ commutes with J ,

where $J, S : TM \rightarrow TM$ as in (1.2). For any holomorphic vector field v ,

$$(3.6) \quad \begin{array}{l} Jv \text{ must be holomorphic as well, while } v \text{ is locally} \\ \text{a gradient if and only if } u = Jv \text{ is a Killing field.} \end{array}$$

In fact, for $S = \nabla v$ and $A = \nabla u$, (3.2) – (3.5) give $A = JS = SJ$, and so $A + A^* = J(S - S^*)$, while the local-gradient property of v amounts to $S - S^* = 0$, and the Killing condition for u reads $A + A^* = 0$.

Remark 3.1. As shown by Kobayashi [11], if u is a Killing vector field on a Riemannian manifold (M, g) , the connected components of the zero set of u are mutually isolated totally geodesic submanifolds of even codimensions.

Lemma 3.2. *If a complex manifold M admits a Kähler metric g , with the Kähler form $\omega = g(J\cdot, \cdot)$, and $\varepsilon : \mathbb{CP}^k \rightarrow M$ is a nonconstant holomorphic mapping, then $\varepsilon^*\omega$ represents a nonzero de Rham cohomology class in $H^2(\mathbb{CP}^k, \mathbb{R})$.*

Whether a holomorphic mapping $\varepsilon : \mathbb{CP}^k \rightarrow M$ is constant, or not, the same is the case for all holomorphic mappings $\mathbb{CP}^k \rightarrow M$ sufficiently close to ε in the C^0 topology.

Proof. Clearly, ε remains nonconstant (and holomorphic) when restricted to a suitable projective line $\mathbb{CP}^1 \subseteq \mathbb{CP}^k$. In addition to being positive semidefinite everywhere, the restriction h of ε^*g to $\mathbb{CP}^1$ must also be positive definite somewhere (or else h , being Hermitian, would vanish identically, making ε constant on $\mathbb{CP}^1$). The integral of $\varepsilon^*\omega$ over $\mathbb{CP}^1$ is thus positive, proving our first claim. The second one follows since nearby continuous mappings are, obviously, homotopic to ε . $\square$

Remark 3.3. We need the following well-known fact, valid both in the C^∞ and complex (holomorphic) categories: any integrable distribution with compact simply connected leaves constitutes the vertical distribution of a bundle projection.

The required local trivializations are provided by the – necessarily trivial – holonomy of the underlying foliation; see, for instance, [3, p. 71].

Remark 3.4. We need two more well-known facts; cf. [19, Example 1 of Sect. 2.2].

- (a) A continuous function $U \rightarrow \mathbb{C}$ on an open set $U \subseteq \mathbb{C}$, holomorphic on $U \setminus A$, where $A \subseteq U$ is discrete, is necessarily holomorphic everywhere in U .
- (b) The only injective holomorphic mappings $\mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ are biholomorphisms.

Remark 3.5. If $\Psi : \Pi \rightarrow M$ is a continuous mapping between complex manifolds, and a codimension-one complex submanifold A of Π , closed as a subset of Π , has the property that the restrictions of Ψ to Π and to the complement $\Pi \setminus A$ are both holomorphic, then Ψ is holomorphic on Π .

In fact, let $p = \dim_{\mathbb{C}} \Pi$. When $p = 1$, our claim is obvious from Remark 3.4(a). Generally, in local holomorphic coordinates $z^1, \dots, z^p$ for Π such that $z^2 = \dots = z^p = 0$ on the intersection of A with the coordinate domain, the complex partial derivatives of the components of Ψ (relative to any local holomorphic coordinates in M) all clearly exist: for $\partial/\partial z^1$ this follows from the case $p = 1$.

Remark 3.6. As usual, we call a differential 2-form ω on a complex manifold *positive* if it equals the Kähler form $g(J\cdot, \cdot)$ of some Kähler metric g . This amounts to requiring closedness of ω along with symmetry and positive definiteness of the twice-covariant tensor field $-\omega(J\cdot, \cdot)$.

In any complex manifold, $d\omega = 0$ and $\omega(J\cdot, \cdot)$ symmetric whenever $\omega = i\partial\bar{\partial}f$ or, equivalently, $2\omega = -d[J^*df]$ for a real-valued function f , with the 1-form J^*df , also denoted by $(df)J$, which sends any tangent vector field v to $d_{Jv}f$. Clearly,

$$(3.7) \quad 2i\partial\bar{\partial}f = 2if'\partial\bar{\partial}\chi - f''d\chi \wedge J^*d\chi, \quad \text{with } f' = df/d\chi,$$

if f is a C^∞ function of a function χ on the same manifold (cf. Remark 1.3). The exterior-derivative and exterior-product conventions used here, for any 1-forms ι, κ and vector fields u, v , are $(d\kappa)(u, v) = d_u[\kappa(v)] - d_v[\kappa(u)] - \kappa([u, v])$ and $(\iota \wedge \kappa)(u, v) = \iota(u)\kappa(v) - \iota(v)\kappa(u)$. When, in addition, v is real-holomorphic, one has

$$(3.8) \quad 2\omega(Jv, \cdot) = -d(d_v f) - J^*[d(d_{Jv} f)] \quad \text{for } \omega = i\partial\bar{\partial}f.$$

See [5, Lemma 2]; the Kähler metric used in [5] always exists locally.

Remark 3.7. For the real part $\langle \cdot, \cdot \rangle$ of a Hermitian inner product in a finite-dimensional complex vector space $\mathcal{N}$, let $\rho : \mathcal{N} \rightarrow [0, \infty)$ and $\mathcal{V}$ be the *norm function* and *complex radial distribution* on $\mathcal{N} \setminus \{0\}$, so that $\rho(\xi) = \langle \xi, \xi \rangle^{1/2}$ and $\mathcal{V}_\xi = \text{Span}_\mathbb{C}(\xi)$.

- (a) $d\rho^2$ is obviously given by $\xi \mapsto 2\langle \xi, \cdot \rangle$.
- (b) $i\partial\bar{\partial}\rho^2$ coincides with twice the Kähler form $\langle J\cdot, \cdot \rangle$ of the constant metric $\langle \cdot, \cdot \rangle$.
- (c) $d\rho^2 \wedge J^*d\rho^2$, on $\mathcal{N} \setminus \{0\}$, equals $-4\rho^2$ times the restriction of $\langle J\cdot, \cdot \rangle$ to $\mathcal{V}$.

In fact, (b) – (c) are immediate from (a) and Remark 3.6.

Remark 3.8. Let $\pi : M \rightarrow B$ be a surjective submersion between manifolds.

- (a) If the preimages $\pi^{-1}(y)$, $y \in B$, are all compact, then π can be factored as $M \rightarrow \Pi \rightarrow B$, with a bundle projection $M \rightarrow \Pi$ having compact (connected) fibres, and a finite covering projection $\Pi \rightarrow B$.
- (b) In the case where $\dim B = \dim \Pi$ and M is compact, π must necessarily be a (finite) covering projection.

Namely, (a) is a well-known fact, easily verified using parallel transports corresponding to a fixed vector subbundle $\mathcal{H}$ of TM for which $TM = \mathcal{V} \oplus \mathcal{H}$ cf. [10, Remark 1.1], or derived as in Remark 3.3, since the foliation with the leaves $\pi^{-1}(y)$ has trivial holonomy. Part (b) – in which the dimension equality means that π is locally-diffeomorphic – easily follows from (a).

Remark 3.9. For a Kähler manifold (Π, h) with $\dim_\mathbb{C} \Pi = l$, any holomorphic mapping $F : \mathbb{CP}^l \rightarrow \Pi$ such that F^*h is a positive constant multiple of the Fubini-Study metric on $\mathbb{CP}^l$ (cf. Remark 5.4) must be a biholomorphism.

In fact, F is then a covering projection (Remark 3.8(b)) and our claim follows since, due to a result of Kobayashi [12], Π has to be simply connected.

4. Geodesic-gradient Kähler triples

Given a manifold M endowed with a fixed connection ∇ , we refer to a vector field v on M as *geodesic* if the integral curves of v are reparametrized ∇ -geodesics. Equivalently, for some function ψ on the open set $M' \subseteq M$ on which $v \neq 0$,

$$(4.1) \quad \nabla_v v = \psi v \quad \text{everywhere in } M'.$$

A function τ on a Riemannian manifold (M, g) is said to *have a geodesic gradient* if its gradient v is a geodesic vector field relative to the Levi-Civita connection ∇ .

Functions with geodesic gradients are also called *transnormal* [20, 17, 2].

Lemma 4.1. *For a function τ on a Riemannian manifold (M, g) , the gradient of τ is a geodesic vector field if and only if $Q = g(\nabla\tau, \nabla\tau)$ is, locally in M' , a function of τ .*

Proof. By (1.6), condition (4.1) is equivalent to $dQ \wedge d\tau = 0$. $\square$

Definition 4.2. A *geodesic-gradient Kähler triple* (M, g, τ) consists of any Kähler manifold (M, g) and a nonconstant real-valued function τ on M such that the g -gradient $v = \nabla\tau$ is both geodesic and real-holomorphic.

Speaking of *compactness* of (M, g, τ) , or its *dimension*, we always mean those of the underlying complex manifold M , and we call two such triples (M, g, τ) , $(\hat{M}, \hat{g}, \hat{\tau})$ *isomorphic* if $\tau = \hat{\tau} \circ \Phi$ and $g = \Phi^*\hat{g}$ for some biholomorphism $\Phi : M \rightarrow \hat{M}$.

For (M, g, τ) as above, whenever the extrema of τ exist, we will also consider

$$(4.2) \quad \text{the } \tau\text{-preimages } \Sigma^+ \text{ and } \Sigma^- \text{ of } \tau_+ = \max \tau \text{ and } \tau_- = \min \tau.$$

Remark 4.3. A geodesic-gradient Kähler triple (M, g, τ) can be *trivially modified* to yield $(M, g, p\tau + q)$, with any real constants $p \neq 0$ and q . (Clearly, $\Sigma^\pm$ in (4.2) then become switched if $p < 0$.) Any such (M, g, τ) and any complex submanifold Π of M , tangent to $v = \nabla\tau$ (that is, forming a union of integral curves of v), and not contained in a single level set of τ , give rise (cf. Remark 1.2) to the new geodesic-gradient Kähler triple (Π, g', τ') , where g', τ' are the restrictions of g and τ to Π .

As shown next, geodesic-gradient Kähler triples naturally arise from suitable co-homogeneity-one isometry groups.

Lemma 4.4. *Let a connected Lie group G acting by holomorphic isometries on a Kähler manifold (M, g) , and having some orbits of real codimension 1, preserve a nontrivial holomorphic Killing field u with zeros. If $H^1(M, \mathbb{R}) = \{0\}$, then (M, g, τ) is a geodesic-gradient Kähler triple and $u = J(\nabla\tau)$ for some G -invariant function τ on M .*

Proof. Since $H^1(M, \mathbb{R}) = \{0\}$, (3.6) implies both the existence of a function τ with $u = J(\nabla\tau)$, and the fact that its gradient $v = \nabla\tau = -Ju$ is holomorphic. Thus, elements of G preserve τ up to additive constants. Let Σ now be a fixed connected component of the zero set of u , so that G , being connected, leaves Σ invariant, while τ is constant on Σ (cf. Remark 3.1). The additive constants just mentioned are therefore equal to 0. Due to their G -invariance, the functions τ

and $Q = g(\nabla\tau, \nabla\tau)$ are constant along codimension-one orbits of G and, consequently, functionally dependent. (Note that the union of such orbits is dense in M .) Consequently, by Lemma 4.1, the gradient $v = \nabla\tau$ is a geodesic vector field. $\square$

Remark 4.5. The assumptions about triviality of $H^1(M, \mathbb{R})$ and holomorphicity of u in Lemma 4.4 are well-known to be redundant when M is compact [14, p. 95, Corollary 4.5]; see also [6, formula (A.2c) and Theorem A.1].

The following fact will be used in the proof of Theorem 10.1.

Lemma 4.6. *If a vector field w on a Riemannian manifold (M, g) is orthogonal to a geodesic gradient v and commutes with v , then w is a Jacobi field along every integral curve of $v/|v|$ in the set M' where $v \neq 0$.*

Proof. Fix $\tau : M \rightarrow \mathbb{R}$ with $v = \nabla\tau$. For $Q = g(v, v) : M \rightarrow \mathbb{R}$, (1.6) and (4.1) give $dQ \wedge d\tau = 0$, so that $|v|$ is, locally in M' , a C^∞ function of τ (cf. Remark 1.3). As $\mathcal{L}_w\tau = 0$ due to the orthogonality assumption, and $\mathcal{L}_wv = 0$, we now have $\mathcal{L}_w(v/|v|) = 0$ on M' . The local flow of w , applied to any integral curve of $v/|v|$, thus yields a variation of unit-speed geodesics, and our claim follows. $\square$

Remark 4.7. A compact geodesic-gradient Kähler triple of complex dimension 1 is essentially, up to isomorphisms, nothing else than the sphere S^2 with a rotationally invariant metric. In fact, necessity of rotational invariance is due to (3.6), while its sufficiency follows from Lemma 4.4 with $G = S^1$, for the sphere treated as $\mathbb{CP}^1$ with a Kähler metric. (Once the S^1 action is chosen, the required function τ becomes unique up to trivial modifications, cf. Remark 4.3.)

5. Examples: Grassmannian and CP triples

In this section *vector spaces* are complex (except for Remark 5.7) and finite-dimensional. By *k-planes* in a vector space V we mean k -dimensional vector subspaces of V . When $k = 1$, they will also be called *lines* in V .

Given a vector space V and $k \in \{0, 1, \dots, \dim_{\mathbb{C}}V\}$, the Grassmannian Gr_kV is the set of all k -planes in V . Each Gr_kV naturally forms a compact complex manifold (see Remark 5.5), and $PV = \text{Gr}_1V$ is the projective space of V , provided that $\dim_{\mathbb{C}}V > 0$. We will use the standard identification

$$(5.1) \quad P(\mathbb{C} \times V) = V \cup PV,$$

of $P(\mathbb{C} \times V)$ with the disjoint union of an open subset biholomorphic to V and a complex submanifold biholomorphic to PV via the biholomorphism sending $v \in V$, or the line $\mathbb{C}v$ spanned by $v \in V \setminus \{0\}$, to the line $\mathbb{C}(1, v)$ or, respectively, $\mathbb{C}(0, v)$. The *projectivization* of a holomorphic vector bundle N over a complex manifold Σ is, as usual, the holomorphic bundle PN of complex projective spaces over Σ with

$$(5.2) \quad \text{the fibres } [PN]_y = PV \text{ for } V = N_y, \text{ whenever } y \in \Sigma.$$

For a subspace L of a vector space V such that $\dim_{\mathbb{C}}V \geq 2$, let G be the group of all complex-linear automorphisms of V preserving both L and a fixed Hermitian

inner product in V . We now define a compact complex manifold M by

$$(5.3) \quad \begin{aligned} &\text{i)} \quad M = \text{Gr}_k V, \text{ where } 0 < k < \dim_{\mathbb{C}} V \text{ and } \dim_{\mathbb{C}} L = 1, \text{ or} \\ &\text{ii)} \quad M = \text{PV}, \text{ allowing } \dim_{\mathbb{C}} L \in \{1, \dots, \dim_{\mathbb{C}} V - 1\} \text{ to be arbitrary.} \end{aligned}$$

Then the hypotheses, and hence conclusions, of Lemma 4.4 are satisfied by these M, G , any G -invariant Kähler metric g on M , and some u . Specifically, u is a vector field arising from the central circle subgroup S^1 of G formed by all unimodular elements of G acting in both $L, L^\perp$ as multiples of Id . See the remarks below.

The triples (M, g, τ) arising via Lemma 4.4 in cases (5.3.i) and (5.3.ii) will from now on be called *Grassmannian triples* and, respectively, *CP triples*.

Since G as above contains all unit complex multiples of Id , its action on M is not effective. Lemma 4.4 does not require effectiveness of the action.

Remark 5.1. The cohomogeneity-one assumption of Lemma 4.4 follows here from the fact that the orbits of G coincide with the levels of a nonconstant real-analytic function $f : M \rightarrow \mathbb{R}$. Specifically, in case (5.3.i), $f(W) = |\text{pr}(X, W)|^2$, where

$$(5.4) \quad \text{pr}(X, W) \text{ denotes the orthogonal projection of } X \text{ onto } W,$$

and X is some/any unit vector spanning L , which yields G -invariance of f . Conversely, if $W, \tilde{W} \in M$ and $f(W) = f(\tilde{W})$, an element of G sending W to $\tilde{W}$ is provided by any linear isometry mapping the quadruple $W, W^\perp, \text{pr}(X, W), \text{pr}(X, W^\perp)$ onto $\tilde{W}, \tilde{W}^\perp, \text{pr}(X, \tilde{W}), \text{pr}(X, \tilde{W}^\perp)$. (Such an isometry will preserve $X = \text{pr}(X, W) + \text{pr}(X, W^\perp)$.) In case (5.3.ii) we may use f given by $f(W) = |\text{pr}(Y_W, L)|^2$, with Y_W standing for some/any unit vector that spans the line W .

Remark 5.2. For a Grassmannian or CP triple (M, g, τ) , critical points of τ , that is, the zeros of u or, equivalently, the fixed points of the central circle subgroup S^1 mentioned above, form the disjoint union of two (connected) compact complex submanifolds, which – since τ is clearly constant on either of them – must be the same as $\Sigma^\pm$ in (4.2). With $\approx$ denoting biholomorphic equivalence, these $\Sigma^\pm$ are

$$(5.5) \quad \begin{aligned} &\text{a)} \quad \{W \in M : L \subseteq W\} \approx \text{Gr}_{k-1}[V/L], \quad \text{b)} \quad \{W \in M : W \subseteq L^\perp\} \approx \text{Gr}_k L^\perp, \\ &\text{c)} \quad \{W \in M : W \subseteq L\} \approx \text{PL}, \quad \text{d)} \quad \{W \in M : W \subseteq L^\perp\} \approx \text{PL}^\perp, \end{aligned}$$

where (5.5.a) – (5.5.b) correspond to (5.3.i), and (5.5.c) – (5.5.d) to (5.3.ii). In fact, each of the four sets clearly consists of fixed points of S^1 . Conversely, suppose that $W \in M$ does not lie in the union of the sets (5.5.a) – (5.5.b) (or, (5.5.c) – (5.5.d)), and $\Xi \in S^1 \setminus \{\text{Id}, -\text{Id}\}$. Then $\Xi(W) \neq W$. Namely, if Ξ preserved W , the equalities $\Xi(L) = L$ and $\Xi(L^\perp) = L^\perp$, along with $\Xi(W) = W$ and $\Xi(W^\perp) = W^\perp$, would imply analogous equalities for the lines spanned by $\text{pr}(X, W), \text{pr}(X, W^\perp)$ (case (5.3.i)), or by $\text{pr}(X, L), \text{pr}(X, L^\perp)$ (case (5.3.ii)), cf. (5.4), X being any unit vector in the line L (or, respectively, in the line W). In either case, the plane K spanned by the two Ξ -invariant lines would contain a third such line, in the form of L (or, respectively, W). Thus, Ξ restricted to K would be a multiple of Id , leading to a contradiction: in both cases, K contains nonzero vectors from L and from $L^\perp$, which are eigenvectors of Ξ for two distinct eigenvalues.

Remark 5.3. All compact geodesic-gradient Kähler triples of complex dimension 1 are obviously isomorphic to CP triples constructed from the data (5.3.ii) with $m = 2$ and $\dim_{\mathbb{C}} \mathbf{L} = 1$. See Remark 4.7.

Remark 5.4. Given the real part $\langle, \rangle$ of a Hermitian inner product in a vector space $\mathbf{V}$, the *Fubini-Study metric* on $\mathbf{PV}$ associated with $\langle, \rangle$ is, as usual, uniquely characterized by requiring that the restriction of the projection $\xi \mapsto \mathbb{C}\xi$ to the unit sphere of $\langle, \rangle$ be a Riemannian submersion. Another such real part $\langle, \rangle'$ yields the same Fubini-Study metric as $\langle, \rangle$ only if $\langle, \rangle'$ is a constant multiple of $\langle, \rangle$. In fact, a $\mathbb{C}$ -linear automorphism of $\mathbf{V}$ taking $\langle, \rangle$ to $\langle, \rangle'$ descends to an isometry $\mathbf{PV} \rightarrow \mathbf{PV}$ and hence equals a $\langle, \rangle$ -unitary automorphism of $\mathbf{V}$ followed by $re^{i\theta}$ times Id for some $r, \theta \in \mathbb{R}$, which gives $\langle, \rangle = r\langle, \rangle'$.

Remark 5.5. Given a vector space $\mathbf{V}$ and $k \in \{1, \dots, \dim_{\mathbb{C}} \mathbf{V}\}$, we denote by $\text{St}_k \mathbf{V}$ and $\pi : \text{St}_k \mathbf{V} \rightarrow \text{Gr}_k \mathbf{V}$ the Stiefel manifold of all linearly independent ordered k -tuples of vectors in $\mathbf{V}$ (forming an open submanifolds of the k th Cartesian power of $\mathbf{V}$) and, respectively, the projection mapping sending each $\mathbf{e} \in \text{St}_k \mathbf{V}$ to $\pi(\mathbf{e}) = \text{Span}(\mathbf{e})$. Then $\text{Gr}_k \mathbf{V}$ has a unique structure of a compact complex manifold of complex dimension $(n - k)k$, where $n = \dim_{\mathbb{C}} \mathbf{V}$, such that π is a holomorphic submersion.

Remark 5.6. We will use the canonical isomorphic identification

$$(5.6) \quad T_{\mathbf{W}}[\text{Gr}_k \mathbf{V}] = \text{Hom}(\mathbf{W}, \mathbf{V}/\mathbf{W}) \quad \text{whenever } k \in \{0, 1, \dots, \dim_{\mathbb{C}} \mathbf{V}\},$$

for the tangent space of the complex manifold $\text{Gr}_k \mathbf{V}$ at any k -plane $\mathbf{W}$, where Hom means ‘the space of linear operators’ and $\mathbf{V}$ is any vector space.

Namely, let $\text{St}_k \mathbf{V}$ and $\pi : \text{St}_k \mathbf{V} \rightarrow \text{Gr}_k \mathbf{V}$ be as in Remark 5.5. Under the identification (5.6), $H \in \text{Hom}(\mathbf{W}, \mathbf{V}/\mathbf{W})$ corresponds to $d\pi_{\mathbf{e}}(\tilde{H}\mathbf{e}) \in T_{\mathbf{W}}[\text{Gr}_k \mathbf{V}]$ for any linear lift $\tilde{H} : \mathbf{W} \rightarrow \mathbf{V}$ of H and any basis $\mathbf{e}$ of $\mathbf{W}$ which, clearly, does not depend on how such $\tilde{H}$ and $\mathbf{e}$ were chosen.

Remark 5.7. Given a real (or, complex) manifold U and real (or, complex) vector spaces $\mathcal{T}, \mathcal{Y}$, let $F : U \rightarrow \text{Hom}(\mathcal{T}, \mathcal{Y})$ be a C^∞ (or, holomorphic) mapping giving rise to a *constant* function $U \ni \xi \mapsto \text{rank } F(\xi)$ or, equivalently, leading to the same value of $k = \dim \text{Ker } F(\xi)$ for all $\xi \in U$. Then the mapping

$$(5.7) \quad U \ni \xi \mapsto \text{Ker } F(\xi) \in \text{Gr}_k \mathcal{T}$$

is of class C^∞ (or, holomorphic) and its differential $T_\xi U \rightarrow \text{Hom}(\mathbf{W}, \mathcal{T}/\mathbf{W})$ at any $\xi \in U$, with $\mathbf{W} = \text{Ker } F(\xi)$, cf. (5.6), sends $\eta \in T_\xi U$ to the unique $H : \mathbf{W} \rightarrow \mathcal{T}/\mathbf{W}$ having a linear lift $\tilde{H} : \mathbf{W} \rightarrow \mathcal{T}$ such that

$$(5.8) \quad F(\xi) \circ \tilde{H} \text{ equals the restriction of } -dF_\xi \eta \text{ to } \mathbf{W}.$$

Here $dF_\xi : T_\xi U \rightarrow \text{Hom}(\mathcal{T}, \mathcal{Y})$, so that $F(\xi)$ and $dF_\xi \eta$ are linear operators $\mathcal{T} \rightarrow \mathcal{Y}$.

In fact, let ‘regular’ mean C^∞ or holomorphic. Given $\xi \in U$, we may select a subspace $\mathbf{V}$ of $\mathcal{T}$ so that $\mathcal{T} = \mathbf{V} \oplus \mathbf{W}$, where $\mathbf{W} = \text{Ker } F(\xi)$. For all η near ξ in U , the restriction of $F(\eta)$ to $\mathbf{V}$ is clearly an isomorphism onto the image

of $F(\eta)$. Denoting by F_η^{-1} its inverse isomorphism, we see that $F_\eta^{-1} \circ F(\eta)$ and $\text{pr}_\eta = \text{Id} - F_\eta^{-1} \circ F(\eta)$ coincide with the direct-sum projections of $\mathcal{T} = \mathbf{V} \oplus \text{Ker } F(\eta)$ onto $\mathbf{V}$ and $\text{Ker } F(\eta)$. A fixed basis $e_1, \dots, e_k$ of $\mathbf{W}$ thus gives rise to the basis $\text{pr}_\eta e_1, \dots, \text{pr}_\eta e_k$ of any such $\text{Ker } F(\eta)$, depending regularly on η , which constitutes a regular local lift of (5.7) valued in the Stiefel manifold $\text{St}_k \mathcal{T}$ (see Remark 5.5), proving regularity of (5.7).

Replacing our ξ with $\xi(0)$ for a curve $t \mapsto \xi(t) \in U$, and letting $e_1(0), \dots, e_k(0)$ be a fixed basis of $\mathbf{W} = \text{Ker } F(\xi(0))$, we set $e_j(t) = \text{pr}_{\xi(t)}[e_j(0)]$ and $\eta(t) = \xi(t)$. Suppressing from the notation the dependence on t , one thus gets $[F(\xi)]e_j = 0$ and, by differentiation, $[dF_\xi \eta]e_j + [F(\xi)]\dot{e}_j = 0$. The operators $\tilde{P} = \tilde{P}(t) : \text{Ker } F(\xi) \rightarrow \mathbf{V}$ defined by $\tilde{P}e_j = \dot{e}_j$, $j = 1, \dots, k$, yield (5.8) at $t = 0$ for $\tilde{P}$ instead of $\tilde{H}$. On the other hand, $\mathbf{e} = \mathbf{e}(t)$ with $\mathbf{e} = (e_1, \dots, e_k)$ is a regular lift of $\text{Ker } F(\xi(t))$ to $\text{St}_k \mathcal{T}$, showing that $d\pi_{\mathbf{e}}(\tilde{P}\mathbf{e}) = d\pi_{\mathbf{e}}\dot{\mathbf{e}}$ is the image of $\eta = \dot{\xi}$ under the differential of (5.7) at $\xi = \xi(t)$. This equality, at $t = 0$, uniquely determines $P : \mathbf{W} \rightarrow \mathcal{T}/\mathbf{W}$ for which our $\tilde{P} : \mathbf{W} \rightarrow \mathcal{T}$ is a linear lift realizing the image just mentioned as in the final paragraph of Remark 5.6), so that $P = H$, and our claim follows.

6. Some relevant types of data

We will repeatedly consider quadruples τ_-, τ_+, a, Q formed by

$$(6.1) \quad \begin{aligned} & \text{a nontrivial closed interval } [\tau_-, \tau_+], \text{ a constant } a \in (0, \infty), \\ & \text{and a } C^\infty \text{ function } Q \text{ of the variable } \tau \in [\tau_-, \tau_+], \text{ positive} \\ & \text{on } (\tau_-, \tau_+), \text{ such that } Q = 0 \text{ and } dQ/d\tau = \mp 2a \text{ at } \tau = \tau_\pm, \end{aligned}$$

$\mp$ being the opposite sign of $\pm$. As explained below, we may then also choose

$$(6.2) \quad \begin{aligned} & \text{a sign } \pm \text{ (equal to } + \text{ or } -), \text{ and a } C^\infty \text{ diffeomorphism} \\ & (\tau_-, \tau_+) \ni \tau \mapsto \rho \in (0, \infty) \text{ such that } d\rho/d\tau = \mp a\rho/Q, \end{aligned}$$

a function $(0, \infty) \ni \rho \mapsto f \in \mathbb{R}$, unique up to an additive constant, with

$$(6.3) \quad a\rho \, df/d\rho = 2|\tau - \tau_\pm|,$$

and the unique increasing diffeomorphism $(0, \infty) \ni \rho \mapsto \sigma \in (0, \delta)$ such that

$$(6.4) \quad a\rho \, d\sigma/d\rho = Q^{1/2} \quad \text{and} \quad \sigma \rightarrow 0 \text{ as } \rho \rightarrow 0,$$

the inverse of $\tau \mapsto \rho$ in (6.2) being used to treat τ, Q as functions of ρ , for

$$(6.5) \quad \delta \in (0, \infty) \text{ equal to the integral of } Q^{-1/2} d\tau \text{ over } (\tau_-, \tau_+).$$

In fact, one easily verifies that $\tau \mapsto \rho$ and $\rho \mapsto \sigma$ with the stated properties exist, while δ is finite. See [8, Remark 5.1], [6, Theorem 10.2(iii)], [6, p. 1661].

Lemma 6.1. *Any f satisfying (6.3) has a C^∞ extension to $[0, \infty)$, which is also a C^∞ function of $\rho^2 \in [0, \infty)$ in a sense analogous to Remark 1.3. The first and second derivatives of f with respect to ρ^2 obviously are $|\tau - \tau_\pm|/(a\rho^2)$ and $(Q - 2a|\tau - \tau_\pm|)/(2a^2\rho^4)$.*

Proof. Since $a\rho^2 df/d\rho^2 = |\tau - \tau_\pm|$, it clearly suffices to establish C^∞ -extensibility of $\rho^2 \mapsto (\tau - \tau_\pm)/\rho^2$ to $[0, \infty)$. To this end, note that, according to [8, top line on p. 83 and Remark 4.3(ii)], $(\tau - \tau_\pm)/\rho^2$ (or, Q/ρ^2) may be extended to a C^∞ function of $\tau \in [\tau_-, \tau_+] \setminus \{\tau_\mp\}$ (or, of $\rho^2 \in [0, \infty)$) with a nonzero value at $\tau_\pm$ (or, respectively, at 0). However, by (6.2), $2a d\tau/d\rho^2 = \mp Q/\rho^2$, so that the variables τ and ρ^2 just mentioned depend diffeomorphically on each other. $\square$

Remark 6.2. It is the limit condition in (6.4) that makes σ unique; by contrast, ρ with (6.2) is only unique up to a positive constant factor.

Remark 6.3. In (6.2), the increasing function $\mp\rho$ of the variable τ clearly tends to 0 as $\tau \rightarrow \tau_\pm$, and to ∞ as $\tau \rightarrow \tau_\mp$.

Remark 6.4. The composite $(0, \delta) \ni \sigma \mapsto \rho \mapsto \tau \in (\tau_-, \tau_+)$ of the inverses of the above two diffeomorphisms is the unique solution of the autonomous equation $d\tau/d\sigma = \mp Q^{1/2}$, with the sign $\pm$ fixed as in (6.2), such that $\tau \rightarrow \tau_\pm$ as $\sigma \rightarrow 0$. (We say ‘autonomous’ since (6.1) makes Q a function of τ .) In fact, any two solutions of the equivalent equation $d\sigma/d\tau = \mp Q^{-1/2}$ differ by a constant.

Remark 6.5. With (6.1) – (6.3) fixed as above, suppose that ρ simultaneously denotes a positive function on a complex manifold M' , which also turns f into a function $M' \rightarrow \mathbb{R}$. Now (3.7) for $\chi = \rho^2$ and the formulae in Lemma 6.1 give

$$(6.6) \quad 4ia\rho^4 \partial \bar{\partial} f = 4ia\rho^2 |\tau - \tau_\pm| \partial \bar{\partial} \rho^2 + (2a|\tau - \tau_\pm| - Q) d\rho^2 \wedge J^* d\rho^2.$$

Remark 6.6. If a C^∞ function $\tau \mapsto \zeta(\tau)$ on an interval I having an endpoint c vanishes at c , then $\zeta(\tau) = (\tau - c)\chi(\tau)$ for a C^∞ function χ on I equal, at c , to the derivative ζ' of ζ . This is the Taylor formula, with $\chi(\tau) = \int_0^1 \zeta'(c + s(\tau - c)) ds$.

Remark 6.7. Given a C^∞ function $t \mapsto \gamma(t)$ on an interval I having the endpoint 0 such that $(\gamma(0), \dot{\gamma}(0)) = (0, 1)$, where $(\cdot)^\cdot = d/dt$, and $\gamma > 0$ on $t \in I \setminus \{0\}$, there exists a C^∞ function $\theta : I \rightarrow \mathbb{R}$, unique up to multiplications by positive constants, for which $\dot{\theta} > 0$ and $\gamma\dot{\theta} = \theta$ everywhere in I , while $\theta(0) = 0$. Namely, Remark 6.6 implies that some C^∞ functions β (without zeros) and α on I satisfy the conditions $\gamma(t) = t/\beta(t)$ and $\beta(t) = 1 + t\dot{\alpha}(t)$. Thus, $1/\gamma(t) = \dot{\alpha}(t) + 1/t$ has the antiderivative $t \mapsto \alpha(t) + \log|t|$ on $t \in I \setminus \{0\}$, and we may set $\theta(t) = te^{\alpha(t)}$.

7. The Chern connection

Let $\langle, \rangle$ be the real part of a Hermitian fibre metric in a holomorphic complex vector bundle N over a complex manifold Σ . The *Chern connection* of $\langle, \rangle$, also called its *Hermitian connection*, is the unique connection D in N which makes $\langle, \rangle$ parallel and satisfies the condition $D^{0,1} = \bar{\partial}$, meaning that, for any section ξ of N , the complex-antilinear part of the real vector-bundle morphism $D\xi : T\Sigma \rightarrow N$ equals $\bar{\partial}\xi$, the image of ξ under the Cauchy-Riemann operator. Cf. [13, Sect. 1.4].

The following five properties of the Chern connection D are well known – (e) is obvious; for (a) – (d) see [1, p. 32], [13, Propositions 1.3.5, 1.7.19 and 1.4.18].

- (a) D depends on N and $\langle, \rangle$ functorially with respect to all natural operations, including Hom, direct sums, and pullbacks under holomorphic mappings.
- (b) $R^D(Jw, Jw') = R^D(w, w')$, with the notation of (1.2) and (3.1), where w, w', R^D are any vector fields on Σ and, respectively, the curvature tensor of D .
- (c) D is the Levi-Civita connection of $\langle, \rangle$ if $N = T\Sigma$ and $\langle, \rangle$ is a Kähler metric.
- (d) D coincides with the normal connection in the normal bundle $N\Sigma$ for any totally geodesic complex submanifold Σ a Kähler manifold (M, g) and the Riemannian fibre metric $\langle, \rangle$ in N induced by g . (In addition, it follows then that N must be a holomorphic subbundle of TM .)
- (e) D -parallel sections of N are holomorphic.

Any given local holomorphic coordinates z^λ in Σ and local holomorphic trivializing sections e_b for N , on the same domain, associate with the Hermitian fibre metric $(,)$ having the real part $\langle, \rangle$, sections ξ of N , and any connection D , their component functions $\gamma_{b\bar{c}} = (e_b, e_c)$ and $\xi^b, \Omega_{\mu b}^c, \Omega_{\bar{\mu} b}^c$, the latter characterized by $\xi = \xi^b e_b$, as well as $D_\lambda e_b = \Omega_{\lambda b}^c e_c$ and $D_{\bar{\mu}} e_b = \Omega_{\bar{\mu} b}^c e_c$. Here D_λ and $D_{\bar{\mu}}$ denote the D -covariant differentiations in the direction of the complexified coordinate vector fields $\partial_\lambda = \partial/\partial z^\lambda$ and $\partial_{\bar{\mu}} = \partial/\partial \bar{z}^\mu$, repeated indices are summed over and, with $\bar{z}^\mu, \bar{\gamma}_{\bar{c}b}$ and $\bar{\eta}^{\bar{c}}$ standing for the complex conjugates of $z^\mu, \gamma_{c\bar{b}}$ and η^c , Hermitian symmetry of $(,)$ amounts to $\gamma_{b\bar{c}} = \gamma_{\bar{c}b}$, while $(\xi, \eta) = \gamma_{b\bar{c}} \xi^b \bar{\eta}^{\bar{c}}$ whenever ξ, η are sections of N .

The real coordinate vector fields corresponding to the real coordinates $\operatorname{Re} z^\mu, \operatorname{Im} z^\mu$ then are $\partial_\mu + \partial_{\bar{\mu}}, i(\partial_\mu - \partial_{\bar{\mu}})$, and so, given a complexified vector field $v = v^\mu \partial_\mu + v^{\bar{\mu}} \partial_{\bar{\mu}}$,

$$(7.1) \quad v \text{ is real if and only if each } v^{\bar{\mu}} \text{ equals the complex conjugate of } v^\mu.$$

For a connection D to make $(,)$ parallel it is clearly necessary and sufficient that $\partial_\mu \gamma_{b\bar{c}} = \Omega_{\mu b}^e \gamma_{e\bar{c}} + \Omega_{\mu \bar{c}}^{\bar{e}} \gamma_{b\bar{e}}$ and $\partial_{\bar{\mu}} \gamma_{b\bar{c}} = \Omega_{\bar{\mu} b}^e \gamma_{e\bar{c}} + \Omega_{\bar{\mu} \bar{c}}^{\bar{e}} \gamma_{b\bar{e}}$, where $\Omega_{\mu \bar{c}}^{\bar{e}}$ and $\Omega_{\bar{\mu} c}^e$ are defined to be the complex conjugates of $\Omega_{\mu c}^e$ and $\Omega_{\bar{\mu} b}^{\bar{e}}$. On the other hand, the condition $D^{0,1} = \bar{\partial}$ is obviously equivalent to $D_{\bar{\mu}} e_b = 0$, that is, $\Omega_{\bar{\mu} b}^c = 0$. Existence and uniqueness of the Chern connection D now follow, its component functions being

$$(7.2) \quad \Omega_{\bar{\mu} b}^c = 0 \quad \text{and} \quad \Omega_{\lambda b}^d \text{ characterized by } \Omega_{\lambda b}^d \gamma_{d\bar{c}} = \partial_\lambda \gamma_{b\bar{c}}.$$

Consequently, the Chern connection D has the curvature components

$$(7.3) \quad R_{\lambda\bar{\mu}b\bar{c}} = -R_{\bar{\mu}\lambda b\bar{c}} = \partial_\lambda \partial_{\bar{\mu}} \gamma_{b\bar{c}} - \Omega_{\lambda b}^d \partial_{\bar{\mu}} \gamma_{d\bar{c}}, \quad R_{\lambda\mu b\bar{c}} = R_{\bar{\lambda}\bar{\mu}b\bar{c}} = 0$$

(which implies (b) above). Here $R_{\lambda\bar{\mu}b\bar{c}} = (R^D(\partial_\lambda, \partial_{\bar{\mu}})e_b, e_c)$, and analogously for the other three pairs $\bar{\mu}\lambda, \lambda\mu, \bar{\lambda}\bar{\mu}$ of indices. We obtain (7.3) from (1.1) via differentiation by parts, noting that $[\partial_\lambda, \partial_\mu] = [\partial_\lambda, \partial_{\bar{\mu}}] = 0$ while, by (7.2), $D_{\bar{\mu}} e_b = 0$ and $(D_\lambda e_b, e_c) = \partial_\lambda \gamma_{b\bar{c}}$. For instance, $R_{\lambda\bar{\mu}b\bar{c}} = (D_{\bar{\mu}} D_\lambda e_b, e_c) = \partial_{\bar{\mu}} (D_\lambda e_b, e_c) - (D_\lambda e_b, D_{\bar{\mu}} e_c) = \partial_{\bar{\mu}} \partial_\lambda \gamma_{b\bar{c}} - \Omega_{\lambda b}^d \partial_{\bar{\mu}} \gamma_{d\bar{c}}$. Similarly, $(D_\mu D_\lambda e_b, e_c) = \partial_\mu (D_\lambda e_b, e_c) - (D_\lambda e_b, D_\mu e_c)$ equals $\partial_\mu (D_\lambda e_b, e_c) = \partial_\mu \partial_\lambda \gamma_{b\bar{c}}$, which is symmetric in λ, μ (and so $R_{\lambda\mu b\bar{c}} = 0$).

Let $f : \Sigma \rightarrow \mathbb{R}$. With the notational conventions of Remark 3.6,

$$(7.4) \quad i\partial\bar{\partial}f = [\partial_\lambda \partial_{\bar{\mu}} f] dz^\lambda \wedge d\bar{z}^{\bar{\mu}},$$

as $df = [\partial_\lambda f] dz^\lambda + [\partial_{\bar{\mu}} f] d\bar{z}^{\bar{\mu}}$ and, in the case where $f = z^\lambda$ (or, $f = \bar{z}^{\bar{\mu}}$), the complex-valued 1-form $J^*df = (df)J$ equals idz^λ or, respectively, $-id\bar{z}^{\bar{\mu}}$. Equivalently,

$$(7.5) \quad (i\partial\bar{\partial}f)(\partial_\lambda, \partial_{\bar{\mu}}) = \partial_\lambda \partial_{\bar{\mu}} f, \quad (i\partial\bar{\partial}f)(\partial_\lambda, \partial_\mu) = (i\partial\bar{\partial}f)(\partial_{\bar{\lambda}}, \partial_{\bar{\mu}}) = 0.$$

Thus, by (7.4) and (7.1), for $\omega = i\partial\bar{\partial}f$ and any real vector fields v, u on Σ ,

$$(7.6) \quad \omega(v, u) = 2\text{Im}(v^\lambda u^{\bar{\mu}} \partial_\lambda \partial_{\bar{\mu}} f).$$

Lemma 7.1. *With $\pi : N \rightarrow \Sigma$ and $\rho : N \rightarrow [0, \infty)$ denoting the bundle projection and the norm function of $\langle \cdot, \cdot \rangle$, for $N, \Sigma, \langle \cdot, \cdot \rangle$ as above, the Chern connection D of $\langle \cdot, \cdot \rangle$ and the 2-form $\omega = i\partial\bar{\partial}\rho^2$ satisfy the following conditions.*

- (i) *The horizontal distribution of D constitutes a complex vector subbundle of TN , and is ω -orthogonal, in an obvious sense, to the vertical distribution $\text{Ker } d\pi$.*
- (ii) *Part (b) of Remark 3.7 describes ω restricted to any fibre N_y of N , where $y \in \Sigma$.*
- (iii) *Whenever $x = (y, \xi) \in N$, cf. Remark 1.5, the restriction of ω_x to the horizontal space of D at x equals the $d\pi_x$ -pullback of the 2-form $\langle R_y^D(\cdot, \cdot)\xi, i\xi \rangle$ at $y \in \Sigma$.*
- (iv) *The Chern connection $\hat{D}$ of $e^\theta \langle \cdot, \cdot \rangle$, for any function $\theta : \Sigma \rightarrow \mathbb{R}$, is related to D by $\hat{D} = D + (\partial\theta) \otimes \text{Id}$, so that $\hat{\Omega}_{\bar{a}b}^c = 0$ and $\hat{\Omega}_{\lambda b}^c = \Omega_{\lambda b}^c + \delta_b^c \partial_\lambda \theta$. Also, $\hat{v}_x \in \text{Span}_{\mathbb{C}}(\tilde{v}_x, \xi)$, where $\tilde{v}_x$ and $\hat{v}_x$ denote the D -horizontal and $\hat{D}$ -horizontal lifts of any $v \in T_y \Sigma$ to $x = (y, \xi) \in N$.*

Proof. In terms of complexified coordinate vector fields $\partial_\lambda, \partial_{\bar{\mu}}$ in Σ and their analogs $\partial_\lambda, \partial_{\bar{\mu}}, \partial_b, \partial_{\bar{c}}$ corresponding to the local holomorphic coordinates z^λ, ξ^b in N , the D -horizontal lifts $\tilde{\partial}_\lambda, \tilde{\partial}_{\bar{\mu}}$ of the former are given by

$$(7.7) \quad \tilde{\partial}_\mu = \partial_\mu - \Omega_{\mu b}^e \xi^b \partial_e - \Omega_{\bar{\mu} \bar{e}}^{\bar{c}} \bar{\xi}^{\bar{e}} \partial_{\bar{c}}, \quad \tilde{\partial}_{\bar{\mu}} = \partial_{\bar{\mu}}.$$

Since $J\partial_\lambda = i\partial_\lambda$ and $J\partial_{\bar{\mu}} = -i\partial_{\bar{\mu}}$, (7.7) implies complex-linearity of the D -horizontal lift operation relative to the complex-structure tensors J , proving the first part of (i). Assertion (ii) is in turn obvious from naturality of the operator $i\partial\bar{\partial}$.

Next, applying (7.5) to $f = \rho^2$ and the coordinates z^λ, ξ^b rather than just z^λ , we obtain $\omega(\partial_\lambda, \partial_{\bar{\mu}}) = \xi^b \xi^{\bar{c}} \partial_\lambda \partial_{\bar{\mu}} \gamma_{b\bar{c}}$, $\omega(\partial_e, \partial_{\bar{\mu}}) = \xi^{\bar{c}} \partial_{\bar{\mu}} \gamma_{e\bar{c}}$, $\omega(\partial_{\bar{c}}, \partial_{\bar{\mu}}) = \omega(\partial_{\bar{c}}, \partial_{\bar{e}}) = 0$, $\omega(\partial_\lambda, \partial_{\bar{c}}) = \xi^b \partial_\lambda \gamma_{b\bar{c}}$, and $\omega(\partial_b, \partial_{\bar{c}}) = \gamma_{b\bar{c}}$. Thus, $\omega(\tilde{\partial}_\lambda, \partial_{\bar{c}}) = 0$ by (7.2), which amounts to the remaining claim in (i); similarly, (7.7) and (7.3) yield $\omega(\tilde{\partial}_\lambda, \tilde{\partial}_{\bar{\mu}}) = \xi^b \xi^{\bar{c}} (\partial_\lambda \partial_{\bar{\mu}} \gamma_{b\bar{c}} - \Omega_{\lambda b}^e \partial_{\bar{\mu}} \gamma_{e\bar{c}}) = \xi^b \xi^{\bar{c}} R_{\lambda \bar{\mu} b \bar{c}}$. Now (iii) follows: for the D -horizontal lifts $\tilde{v}, \tilde{u}$ of any real vectors $v, u \in T_y \Sigma$, (7.5) – (7.6) and the last equality give $\omega_x(\tilde{v}, \tilde{u}) = 2\text{Im}(v^\lambda u^{\bar{\mu}} R_{\lambda \bar{\mu} b \bar{c}} \xi^b \xi^{\bar{c}})$, while at the same time $(R_y^D(v, u)\xi, i\xi)$ is imaginary, and so $\langle R_y^D(v, u)\xi, i\xi \rangle = \text{Im}(R_y^D(v, u)\xi, i\xi) = (R_y^D(v, u)\xi, i\xi) = \xi^b \xi^{\bar{c}} (R_y^D(v, u)e_b, e_{\bar{c}})$ which, analogously, equals $2\text{Im}(v^\lambda u^{\bar{\mu}} R_{\lambda \bar{\mu} b \bar{c}} \xi^b \xi^{\bar{c}})$. Finally, (7.2) and (7.7) imply (iv). $\square$

8. Examples: Vector bundles

The geodesic-gradient Kähler triples constructed in this section are all noncompact. What makes them relevant is the fact that some of them serve as universal building blocks for compact geodesic-gradient Kähler triples. (See Theorem 14.2.)

We begin with data $\Sigma, h, N, \langle \cdot, \cdot \rangle, \tau_-, \tau_+, a, Q, \pm, \tau \mapsto \rho$ and $\rho \mapsto f$ consisting of

- (i) the real part $\langle \cdot, \cdot \rangle$ of a Hermitian fibre metric in a holomorphic complex vector bundle N of positive fibre dimension over a Kähler manifold (Σ, h) ,
- (ii) some objects $\tau_-, \tau_+, a, Q, \pm, \tau \mapsto \rho$ and $\rho \mapsto f$ satisfying (6.1) – (6.3).

Letting $\pi : N \rightarrow \Sigma$ stand for the bundle projection, D for the Chern connection of $\langle \cdot, \cdot \rangle$ (see Section 7), and ρ both for the variable in (ii) and for the norm function $N \rightarrow [0, \infty)$, we use the inverse mapping of $\tau \mapsto \rho$, cf. (6.2), to

$$(8.1) \quad \text{treat } \tau, Q \text{ and } f \text{ as functions } N \rightarrow \mathbb{R}, \text{ denoted here by } \hat{\tau}, \hat{Q} \text{ and } \hat{f}.$$

Denoting by $\hat{J}$ (rather than J) the complex-structure tensor of N , we define a Kähler metric $\hat{g}$ on N by requiring the Kähler forms $\hat{\omega} = \hat{g}(\hat{J} \cdot, \cdot)$ and $\omega^h = h(J \cdot, \cdot)$ to be related by $\hat{\omega} = \pi^* \omega^h + i \partial \bar{\partial} \hat{f}$, which amounts to

$$(8.2) \quad \hat{g} = \pi^* h - (i \partial \bar{\partial} \hat{f})(\hat{J} \cdot, \cdot).$$

As $\hat{\omega}$ should be positive (Remark 3.6), it is necessary to assume here that

$$(8.3) \quad \pi^* h - (i \partial \bar{\partial} \hat{f})(\hat{J} \cdot, \cdot) \text{ is positive-definite at every point of } N.$$

The above construction uses the objects (i) – (ii) with (8.3), and leads to what is shown below (Theorem 8.1) to be a geodesic-gradient Kähler triple $(N, \hat{g}, \hat{\tau})$.

It is convenient, however, to provide the following equivalent, though less concise, description of $\hat{g}$ and $\hat{J}$ restricted to the complement $N' = N \setminus \Sigma$ of the zero section in N . It uses the complex direct-sum decomposition

$$(8.4) \quad TN' = \hat{\mathcal{V}} \oplus \hat{\mathcal{H}}^\mp \oplus \hat{\mathcal{H}}^\bullet,$$

in which $\hat{\mathcal{H}}^\bullet$ is the horizontal distribution of D and $\hat{\mathcal{V}} \oplus \hat{\mathcal{H}}^\mp = \text{Ker } d\pi$ equals the vertical distribution, with the summands $\hat{\mathcal{V}}$ and $\hat{\mathcal{H}}^\mp$ forming, on each punctured fibre $N_y \setminus \{0\}$, the complex radial distribution (Remark 3.7) and, respectively, its $\langle \cdot, \cdot \rangle$ -orthogonal complement in $N_y \setminus \{0\}$. (The word ‘complex’ preceding (8.4) is justified by Lemma 7.1(i).) To describe $\hat{g}$ and $\hat{J}$, we declare that the three summands of (8.4) are $\hat{J}$ -invariant and mutually $\hat{g}$ -orthogonal, that $\hat{J}$ restricted to $\hat{\mathcal{V}}$ agrees, along each punctured fibre $N_y \setminus \{0\}$, with its standard complex-structure tensor of the complex vector space N_y , that the differential of π at every $(y, \xi) \in N_y \setminus \{0\}$, cf. Remark 1.5, maps $\mathcal{H}_{(y, \xi)}$ complex-linearly onto $T_y \Sigma$ and, with the constant $a \in (0, \infty)$ and

function $\hat{\tau}$ appearing in (i) and (8.1),

$$\begin{aligned}
 & \text{a) } \hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp \text{ and } \hat{\mathcal{H}}^\bullet \text{ in (8.4) are mutually } \hat{g}\text{-orthogonal,} \\
 (8.5) \quad & \text{b) } a^2 \rho^2 \hat{g} = \hat{Q} \langle \cdot, \cdot \rangle \text{ on } \hat{\mathcal{V}}, \quad a \rho^2 \hat{g} = 2|\hat{\tau} - \tau_\pm| \langle \cdot, \cdot \rangle \text{ on } \hat{\mathcal{H}}^\pm, \\
 & \text{c) } \hat{g}_x(w_x, w'_x) = h_y(w, w') - \frac{|\hat{\tau}(x) - \tau_\pm|}{a \rho^2} \langle R_y^D(w, J_y w') \xi, i \xi \rangle \text{ with } \rho = |\xi|,
 \end{aligned}$$

at any $x = (y, \xi) \in N_y \setminus \{0\}$, where w, w' are any two vectors in $T_y \Sigma$, and w_x, w'_x denote their D-horizontal lifts to x . The vertical vector fields $\hat{v}, \hat{u}$ with

$$(8.6) \quad \hat{v}_{(y, \xi)} = \mp a \xi, \quad \hat{u}_{(y, \xi)} = \mp a i \xi,$$

allow us to characterize the restrictions of $\hat{g}$ and $\hat{J}$ to $\hat{\mathcal{V}} = \text{Span}(\hat{v}, \hat{u})$ by

$$(8.7) \quad \hat{g}(\hat{v}, \hat{v}) = \hat{g}(\hat{u}, \hat{u}) = \hat{Q}, \quad \hat{g}(\hat{v}, \hat{u}) = 0, \quad \hat{u} = \hat{J} \hat{v}.$$

Note that the symmetry of $\hat{g}_x(w_x, w'_x)$ in w_x, w'_x reflects (b) in Section 7.

Lemma 7.1 easily implies that the definition (8.5) of $\hat{g}$ is actually equivalent to (8.2), while condition (8.3) is nothing else than positivity of the right-hand side in (8.5.c) whenever $w = w' \neq 0$.

Theorem 8.1. *For any data (i) – (ii) with (8.3), let us define $\hat{g}, \hat{\tau}$ by (8.1) – (8.2).*

- (a) *(N, g, τ) is a geodesic-gradient Kähler triple.*
- (b) *The fibres $N_y = \pi^{-1}(y)$, $y \in \Sigma$, are totally geodesic complex submanifolds of (N, g) .*
- (c) *The zero section $\Sigma \subseteq N$ coincides with $\Sigma^\pm$, the $\tau_\pm$ level set of τ .*
- (d) *The $\hat{g}$ -gradient $\hat{v} = \hat{\nabla} \hat{\tau}$ and $\hat{S} = \hat{\nabla} \hat{v}$ satisfy (8.6) – (8.7) and the equality*

$$(8.8) \quad 2\hat{g}_x(\hat{S}_x w_x, w'_x) = \pm \frac{\hat{Q}(x)}{a \rho^2} \langle R_y^D(w, J_y w') \xi, i \xi \rangle, \quad \text{where } \rho = |\xi| > 0,$$

the assumptions being the same as in (8.5.c).

Proof. The $\hat{v}$ -directional derivatives of the norm squared ρ^2 and of ρ are, obviously, $\mp 2a \rho^2$ and $\mp a \rho$. As $d\tau/d\rho = \mp Q/(a\rho)$ in (6.2), we see using (8.1) and (8.7) that

- (e) $\hat{Q} = \hat{g}(\hat{v}, \hat{v})$ equals the $\hat{v}$ -directional derivative $\hat{g}(\hat{v}, \hat{\nabla} \hat{\tau})$ of $\hat{\tau}$.

Furthermore, D-parallel transports preserve the real fibre metric $\langle \cdot, \cdot \rangle$. Therefore,

- (f) $\rho, \hat{\tau}$ and $\hat{Q}$ are constant along every D-horizontal curve in N ,

due to (ii) and (8.1). The equality $\hat{v} = \hat{\nabla} \hat{\tau}$ now follows: $\hat{\mathcal{V}} = \text{Span}(\hat{v}, \hat{u})$, and $\hat{\tau}$ is a function of the norm ρ , so that $\hat{v} - \hat{\nabla} \hat{\tau}$ is $\hat{g}$ -orthogonal to $\hat{\mathcal{H}}^\bullet, \hat{\mathcal{H}}^\pm, \hat{u}$ and $\hat{v}$ by (f), (8.5.a), (8.7), and (e). Also, (8.6) clearly gives holomorphicity of $\hat{v}$, while closedness and positivity of the form $\hat{g}(\hat{J} \cdot, \cdot) = \pi^* \omega^h + i \partial \bar{\partial} \hat{f}$ imply that $\hat{g}$ is a Kähler metric, and $\hat{\tau}$ has a geodesic $\hat{g}$ -gradient $\hat{v}$, its $\hat{g}$ -norm squared $\hat{Q}$ being a function of $\hat{\tau}$, cf. (8.1) and Lemma 4.1. We have thus proved (a). Next, Remark 6.3 yields (c).

The π -projectable local sections of $\hat{\mathcal{H}}^\bullet$ are precisely the same as the D-horizontal lifts of local vector fields tangent to Σ , and their local flows act as D-parallel transports between the fibres. As the submanifold metrics of the fibres are defined

by (8.5.a) – (8.5.b), this last action consists – by (f) – of isometries which, being linear, also preserve the vertical vector field $\hat{v}$ with (8.6). Hence

(g) $\hat{v}$ commutes with all local D-horizontal lifts w ,

and, at the same time, applying Remark 2.5 to any such $w \neq 0$ we obtain (b).

Finally, by (g) and Remark 1.1, the left-hand side of (8.8) equals the $\hat{v}$ -directional derivative of the right-hand side in (8.5.c). To evaluate the latter, note that only the factor $-|\hat{\tau}(x) - \tau_{\pm}| = \pm(\hat{\tau}(x) - \tau_{\pm})$ in the second term needs to be differentiated, as the first term and the remaining factor of the second one are constant along $\hat{v}$ (due to constancy along $\hat{v}$ of $\xi/\rho = \xi/|\xi|$, obvious from (8.6)). Now (e) implies (8.8), completing the proof. $\square$

A *special Kähler-Ricci potential* [8] on a Kähler manifold (M, g) is any nonconstant function $\tau : M \rightarrow \mathbb{R}$ such that $v = \nabla \tau$ is real-holomorphic, while, at points where $v \neq 0$, all nonzero vectors orthogonal to v and Jv are eigenvectors of both ∇v and the Ricci tensor, with $\nabla v : TM \rightarrow TM$ as in (1.2). We then call (M, g, τ) an *SKRP triple*. All SKRP triples (M, g, τ) are geodesic-gradient Kähler triples, due to their easily-verified property [7, Remark 7.1] that $v = \nabla \tau$, wherever nonzero, is an eigenvector of ∇v . Cf. (4.1).

Compact SKRP triples (M, g, τ) have been classified in [8, Theorem 16.3]. They are divided into Class 1, in which M is the total space of a holomorphic $\mathbb{CP}^1$ bundle, and Class 2, with M biholomorphic to $\mathbb{CP}^m$ for $m = \dim_{\mathbb{C}} M$.

Lemma 8.2. *Up to isomorphisms, in the sense of Definition 4.2, compact SKRP triples of Class 2 are the same as CP triples constructed using (5.3.ii) with $\dim_{\mathbb{C}} L = 1$.*

Proof. See [8, Remark 6.2]. (Note that the case $\dim_{\mathbb{C}} L = m - 1$ in (5.3.ii) obviously leads to the same isomorphism type.) $\square$

In (i) above, $\dim_{\mathbb{C}} \Sigma \geq 0$, which allows the possibility of a one-point base manifold $\Sigma = \{y\}$, so that, as a complex manifold, N is a complex vector space, namely, the fibre N_y . According to [8, pp. 85-86], under the standard identification (5.1) for $V = N_y$, both $\hat{g}$ and $\hat{\tau}$ then can be extended to the projective space $P(\mathbb{C} \times N_y)$, giving rise to a Class 2 SKRP triple $(M, \hat{g}, \hat{\tau})$, where $M = P(\mathbb{C} \times N_y)$.

Lemma 8.3. *The SKRP triples (M, g, τ) just mentioned, with $M = P(\mathbb{C} \times N_y)$, represent all isomorphism types of compact SKRP triples of Class 2. Such types include all compact geodesic-gradient Kähler triples of complex dimension 1.*

Proof. For the first part, see [8, Remark 6.2]. The final clause is in turn immediate from Remark 5.3 and Lemma 8.2. $\square$

Remark 8.4. As a consequence of the second part of Remark 4.3, for (N, g, τ) as in Theorem 8.1, every fibre N_y is the underlying complex manifold of a geodesic-gradient Kähler triple, realizing a special case of Theorem 8.1: that of a one-point base manifold $\{y\}$. Its projective compactification $P(\mathbb{C} \times N_y)$ constitutes, for reasons mentioned above, the underlying complex manifold of an SKRP triple of Class 2. The

resulting submanifold metric on the complement of N_y in $P(\mathbb{C} \times N_y)$ (that is, on the projective hyperplane at infinity, identified via (5.1) with PN_y) equals $2(\tau_+ - \tau_-)/a$ times the Fubini-Study metric associated – as in Remark 5.4 – with $\langle, \rangle$.

Namely, let $\xi, \eta \in N_y$ have $\langle \xi, \xi \rangle = 1$ and $\langle \xi, \eta \rangle = \langle i\xi, \eta \rangle = 0$. The curve $t \mapsto t\eta$ of vectors $t\eta$ tangent to N_y at the points $t\xi$, satisfies, in view of (8.5.b) and Remark 6.3, the limit relation $\hat{g}_{(y, t\xi)}(t\eta, t\eta) \rightarrow 2(\tau_+ - \tau_-)\langle \eta, \eta \rangle/a$ as $t \rightarrow \infty$. At the same time, $t\xi$ (or, the tangent vector $t\eta$) tends, as $t \rightarrow \infty$, to the point $\mathbb{C}(0, \xi)$ of $P(\mathbb{C} \times N_y) \setminus N_y$, identified with $\mathbb{C}\xi \in PN_y$ or, respectively, to the vector tangent to PN_y at $\mathbb{C}\xi$ which is the image of η under

(*) the differential of the projection $\xi \mapsto \mathbb{C}\xi$ restricted to the unit sphere of $\langle, \rangle$.

The claim about the tangent vectors, which clearly implies our assertion, can be justified as follows. The vector $t\eta$ equals $x_s(t, 0)$ (notation preceding Remark 1.4) with $x(t, s) = t(\xi + s\eta) \in N_y$, so that $|x(t, s)|^2 = t^2(1 + |s\eta|^2)$ and, setting $\zeta(t, s) = [1 + t^2(1 + |s\eta|^2)]^{-1/2}(1, x(t, s)) \in \mathbb{C} \times N_y$, we get $|\zeta(t, s)| = 1$ for the direct-sum Euclidean norm. Identifying $x(t, s)$ with $\mathbb{C}(1, x(t, s)) = \mathbb{C}\zeta(t, s) \in P(\mathbb{C} \times N_y)$, we see that $t\eta$, treated as tangent to $P(\mathbb{C} \times N_y)$ at $\mathbb{C}(1, t\xi)$, is the image, under the analog of (*) for $\mathbb{C} \times N_y$, of the vector $\zeta_s(t, 0) = (1 + t^2)^{-1/2}(0, x_s(t, 0)) = (0, t(1 + t^2)^{-1/2}\eta)$ tangent to the unit sphere of $\mathbb{C} \times N_y$ at the point $\zeta(t, 0) = (1 + t^2)^{-1/2}(1, t\xi)$ and having the required limit $(0, \eta)$ as $t \rightarrow \infty$, which we identify with η .

Remark 8.5. The construction summarized in Theorem 8.1 has an obvious generalization, arising when, in (6.1), $\tau \mapsto Q$ is assumed to be only defined on the half-open interval $[\tau_-, \tau_+] \setminus \{\tau_\pm\}$, and $dQ/d\tau = \mp 2a$ at $\tau = \tau_\pm$ just for one fixed sign $\pm$. Our discussion focuses on a narrower case since this is the case relevant to the study of *compact* geodesic-gradient Kähler triples.

9. Local properties

Throughout this section (M, g, τ) is a fixed geodesic-gradient Kähler triple (Definition 4.2). We use the symbols

$$(9.1) \quad J, v, u, M', \psi, Q, \mathcal{V}, \mathcal{V}^\perp, S, A$$

for the complex-structure tensor $J : TM \rightarrow TM$ of the underlying complex manifold M , the gradient $v = \nabla \tau$, its J -image $u = Jv$, the open set M' where $v \neq 0$, the function ψ on M' with (4.1), the function $Q = g(v, v)$ on M , the distribution $\mathcal{V} = \text{Span}(v, u)$ on M' , its orthogonal complement, as well as the endomorphisms

$S = \nabla v$ and $A = \nabla u$ of TM , cf. (1.2). Under the above hypotheses,

- (9.2) a) v, u are both holomorphic, $|v| = |u| = Q^{1/2}$, and $A = JS = SJ$,
 b) $u = Jv$ is a Killing field commuting with v , and orthogonal to v ,
 c) $\nabla_w A = R(u, w)$ and $\nabla_w S = -J[R(u, w)]$ for any vector field w ,
 d) S is self-adjoint and J, A are skew-adjoint at every point of M ,
 e) $g([w, w'], u) = -2g(Aw, w')$ for any local sections w, w' of $\mathcal{V}^\perp$,
 f) $\nabla_v v = \psi v = -\nabla_u u$ and $\nabla_u v = \nabla_v u = \psi u$ everywhere in M' ,
 g) Q is, locally in M' , a function of τ , and $2\psi = dQ/d\tau$,
 h) J, S, A and the local flows of u and v leave $\mathcal{V}$ and $\mathcal{V}^\perp$ invariant.

In (9.2.c), R denotes the curvature tensor of g , and the notation of (1.2) is used.

In fact, holomorphicity of v (cf. Definition 4.2) combined with (3.2) – (3.5) gives (9.2.a), u being holomorphic due to (3.5), as $A = JS = SJ$ commutes with J . Next, (9.2.b) follows from (3.6) and the Lie-bracket equality $[u, v] = \nabla_u v - \nabla_v u = Su - Av = Su - SJv = 0$, obvious in view of (9.2.a), while (9.2.c) (or, (9.2.d)) is a direct consequence of (1.5) and (9.2.a) or, respectively, of (9.2.b) combined with the fact that v is a gradient. We now obtain (9.2.e) from (9.2.d), noting that $g(\nabla_w w', u) = -g(w', \nabla_w u) = -g(w', Aw)$. On the other hand, (9.2.b), (9.2.a) and (4.1) yield $\nabla_u v = \nabla_v u = \nabla_v(Jv) = J\nabla_v v = \psi Jv = \psi u$ and so $\nabla_u u = \nabla_u(Jv) = J\nabla_u v = \psi Ju = -\psi v$, establishing (9.2.f), while Lemma 4.1, (1.6) and (9.2.f) imply (9.2.g). That J, S, A all leave $\mathcal{V} = \text{Span}(v, u)$ invariant is clear as $Jv = u$ and $Ju = -v$ while, by (9.2.f), Sv, Su, Av, Au are sections of $\mathcal{V}$. The same conclusion for $\mathcal{V}^\perp$ is now immediate from (9.2.d). By (9.2.b), the local flows of v and u preserve v, u and $\mathcal{V} = \text{Span}(v, u)$. The u -invariance of $\mathcal{V}^\perp$ now follows from (9.2.b). Finally, let w be a section of $\mathcal{V}^\perp$. Writing $\langle \cdot, \cdot \rangle$ for g , we get $\langle [v, w], v \rangle = \langle \nabla_v w - \nabla_w v, v \rangle = -\langle w, \nabla_v v \rangle - \langle Sw, v \rangle = -\langle Sw, v \rangle = -\langle w, Sv \rangle = 0$, cf. (9.2.d) and (9.2.f). Similarly, $\langle [v, w], u \rangle = \langle \nabla_v w - \nabla_w v, u \rangle = -\langle w, \nabla_v u \rangle - \langle Sw, u \rangle = -\langle Sw, u \rangle = -\langle w, Su \rangle = 0$. Thus, $[v, w]$ is a section of $\mathcal{V}^\perp$ as well. In view of Remark 2.3, this completes the proof of (9.2.h). For easy reference, note that, by (9.2.a) – (9.2.b),

$$(9.3) \quad g(v, v) = g(u, u) = Q, \quad g(v, u) = 0, \quad u = Jv.$$

Lemma 9.1. *Under the assumptions preceding (9.2), on M' ,*

- (a) *the distribution $\mathcal{V} = \text{Span}(v, u)$ is integrable and has totally geodesic leaves,*
 (b) *a local section of $\mathcal{V}^\perp$ is projectable along $\mathcal{V}$ if and only if it commutes with u and v ,*
 (c) *if local sections w and w' of $\mathcal{V}^\perp$ commute with u and v , then*
 (9.4) i) $d_v[g(w, w')] = 2g(Sw, w')$, ii) $d_v[g(Sw, w')] = 2\psi g(Sw, w')$,
 iii) $d_v[Q^{-1}g(Sw, w')] = 0$,
 (d) $d_v Q = 2\psi Q$ and $d_u[g(w, w')] = d_u[g(Sw, w')] = d_u Q = 0$ for any w, w' as in (c),
 (e) $[\nabla_v S]w = 2(\psi - S)Sw$ whenever w is a local section of $\mathcal{V}^\perp$.

Proof. Assertions (a) – (b) are obvious from (9.2.b) and, respectively, Remark 2.1 combined with (9.2.h). Next, let $\mathcal{L}_v w = \mathcal{L}_v w' = \mathcal{L}_u w = \mathcal{L}_u w' = 0$. Since $\mathcal{L}_v$ and $\mathcal{L}_u$ act on functions as d_v and d_u , (1.3) implies (9.4.i), and $d_u[g(w, w')] = 0$ as $\mathcal{L}_u g = 0$

by (9.2.b). For similar reasons, $d_u[g(Sw, w')] = \mathcal{L}_u[g(Sw, w')] = 0$. (Namely, (9.2.c) gives $\nabla_u S = 0$, so that (9.2.a) and (1.4), with u, S rather than v, B , yield $\mathcal{L}_u S = 0$.) On the other hand, by (9.3), $g(v, v) = Q$. Now (1.6), (9.2.f) and (9.2.b) imply that $d_u \tau = d_u Q = 0$ and $d_v Q = 2\psi Q$, establishing (d).

Using (9.2.a) we get $g(Sw, w') = g(JSw, Jw') = g(Aw, Jw')$ which, by (9.2.e), is nothing else than $-g([w, Jw'], u)/2$. Hence $2d_v[g(Sw, w')] = 2\mathcal{L}_v[g(Sw, w')] = -\mathcal{L}_v[g(u, [w, Jw'])] = -[\mathcal{L}_v g](u, [w, Jw'])$. (Our assumption that $\mathcal{L}_v w = \mathcal{L}_v w' = 0$ gives $\mathcal{L}_v(Jw') = 0$, as v is holomorphic, which in turn yields $\mathcal{L}_v[w, Jw'] = 0$, while $\mathcal{L}_v u = 0$, cf. (9.2.b).) From (1.3), (9.2.f) and (9.2.a) we now obtain $d_v[g(Sw, w')] = -[\mathcal{L}_v g](u, [w, Jw'])/2 = -g(Su, [w, Jw']) = -2g(\psi u, [w, Jw']) = 2\psi g(Aw, Jw') = -2\psi g(JAw, w') = 2\psi g(Sw, w')$, that is, (9.4.ii), which, since $d_v Q = 2\psi Q$ by (d), also proves (9.4.iii).

Finally, (9.2.h) and the equality $\nabla_v S = -J[R(u, v)]$, cf. (9.2.c), combined with (a), imply that $\nabla_v S - (2\psi - S)S$ leaves $\mathcal{V}^\perp$ invariant. To obtain (e), it now suffices to show that $[\nabla_v S]w - (2\psi - S)Sw$ is orthogonal to w' for any local sections w, w' of $\mathcal{V}^\perp$. We are free to assume here that $w = w'$ (due to self-adjointness of $S = \nabla v$) and that w commutes with u and v (see (b)). Differentiation by parts gives, by (9.4.iii) and (9.2.d), $g([\nabla_v S]w, w) = d_v[g(Sw, w)] - g(S\nabla_v w, w) - g(Sw, \nabla_v w) = 2\psi g(Sw, w) - 2g(Sw, Sw)$, as required, with $\nabla_v w = Sw$ since $[v, w] = 0$. $\square$

10. Horizontal Jacobi fields

In addition to using the assumptions and notations of Section 9, we now let Γ stand for the underlying one-dimensional manifold of a fixed maximal integral curve of v in M' . We restrict the objects in (9.1) to Γ without changing the notation, and select a unit-speed parametrization $t \mapsto x(t)$ of the geodesic Γ such that

$$(10.1) \quad \dot{x} = v/|v| = Q^{-1/2}v \quad \text{along } \Gamma, \quad \text{where } v = \nabla \tau.$$

As an obvious consequence of (10.1), (1.7) and Lemma 9.1(d),

$$(10.2) \quad \dot{\tau} = Q^{1/2}, \quad \dot{Q} = 2\psi Q^{1/2}, \quad \text{with } (\cdot)' = d/dt = d_{\dot{x}}.$$

Any constant $c \in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\}$, where $\tau(\Gamma)$ is the range of τ on Γ , gives rise to the function $\lambda_c : \Gamma \rightarrow \mathbb{R}$ defined by

$$(10.3) \quad \lambda_c = Q/[2(\tau - c)],$$

the convention being that λ_c is identically zero when $c = \infty$. We denote by $\mathcal{W}$ the set of all $\mathcal{V}^\perp$ -valued vector fields $t \mapsto w(t) \in \mathcal{V}_{x(t)}^\perp$ along Γ satisfying the equation

$$(10.4) \quad \nabla_{\dot{x}} w = Q^{-1/2}Sw.$$

Of particular interest to us are c such that

$$(10.5) \quad \begin{aligned} \text{a) } & c \in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\} \quad \text{and } \mathcal{W}[c] \neq \{0\}, \quad \text{where} \\ \text{b) } & \mathcal{W}[c] = \{w \in \mathcal{W} : Sw = \lambda_c w\}. \end{aligned}$$

About projectability along $\mathcal{V}$ in (i) below, see Remark 2.4 and Lemma 9.1(a).

Theorem 10.1. *Under the above hypotheses, the following conclusions hold.*

- (i) $\mathcal{V}^\perp$ -valued solutions w to (10.4) are nothing else than restrictions to Γ of those local sections of $\mathcal{V}^\perp$ with domains containing Γ which are projectable along $\mathcal{V}$.
- (ii) All w as in (i), that is, all elements of $\mathcal{W}$, are Jacobi fields along Γ .
- (iii) Every vector in $\mathcal{V}_{x(t)}^\perp$ equals $w(t)$ for some unique $w \in \mathcal{W}$.
- (iv) $\mathcal{W}$ is a complex vector space of complex dimension $\dim_{\mathbb{C}} M - 1$, and the direct sum of all $\mathcal{W}[c]$ for c in (10.5.a), with $w \mapsto Jw$ serving as the multiplication by $i \in \mathbb{C}$.
- (v) A function $t \mapsto \lambda(t)$ on the parameter interval of $t \mapsto x(t)$ satisfies the equation $d\lambda/dt = 2(\psi - \lambda)\lambda Q^{-1/2}$, with ψ, Q evaluated at $x(t)$, if and only if $\lambda(t) = \lambda_c(x(t))$, cf. (10.3), for some $c \in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\}$ and all t .
- (vi) At any $x = x(t) \in \Gamma$, the eigenvalues of $S_x : \mathcal{V}_x^\perp \rightarrow \mathcal{V}_x^\perp$, cf. (9.2.h), are precisely the values $\lambda_c(x)$ for all c in (10.5.a). The eigenspace of $S_x : \mathcal{V}_x^\perp \rightarrow \mathcal{V}_x^\perp$ corresponding to $\lambda_c(x)$ is $\{w(t) : w \in \mathcal{W}[c]\}$.
- (vii) $R(w, u)u = R(w, v)v = (\psi - S)Sw = R(v, u)Jw/2$ on M' for sections w of $\mathcal{V}^\perp$.
- (viii) If $\tau(\Gamma) = (\tau_-, \tau_+)$ is bounded, then $Q/(\tau - \tau_+) \leq 2S \leq Q/(\tau - \tau_-)$ on $\mathcal{V}^\perp$.

Proof. Any w as in the second line of (i), restricted to Γ , becomes both a Jacobi field (by Lemmas 4.6 and 9.1(b)) and a $\mathcal{V}^\perp$ -valued solution to (10.4) (since $S = \nabla v$, so that (10.1) and Lemma 9.1(b) give $\nabla_x w = Q^{-1/2} \nabla_v w = Q^{-1/2} \nabla_w v = Q^{-1/2} Sw$). With Γ replaced by suitable shorter subgeodesics covering all points of Γ , the inclusion just established between the two vector spaces appearing in (i) is actually an equality: in either class, the vector field in question is uniquely determined by its initial value at any given point $x \in \Gamma$. This proves (i) – (ii) as well as (iii) – (iv), the latter in view of the fact that $JS = SJ$, cf. (9.2.a).

For a C^1 function λ defined on the parameter interval of $t \mapsto x(t)$, one has

$$(10.6) \quad \dot{\lambda} = 2(\psi - \lambda)\lambda Q^{-1/2} \quad \text{with } (\cdot)' = d/dt$$

if and only if either $\lambda = 0$ identically, or $\lambda \neq 0$ everywhere and the function c characterized by $2c = 2\tau - Q/\lambda$ is constant. (In fact, the either-or claim about vanishing of λ is due to uniqueness of solutions of initial-value problems, while (10.2) yields $2\dot{c} = Q\lambda^{-2}[\dot{\lambda} - 2(\psi - \lambda)\lambda Q^{-1/2}]$.) Now (v) easily follows, all nonzero initial conditions for (10.6) at fixed t being realized by suitably chosen constants $c \in \mathbb{R} \setminus \tau(\Gamma)$ (and $\lambda = 0$ satisfying (v) with $c = \infty$).

On the other hand, from (10.1) and Lemma 9.1(e),

$$(10.7) \quad [\nabla_x S]w = 2Q^{-1/2}(\psi - S)Sw, \quad \text{if } w \text{ is a } \mathcal{V}^\perp\text{-valued vector field along } \Gamma.$$

Next, we fix $x = x(t) \in \Gamma$ and express any prescribed eigenvalue-eigenvector pair for $S_x : \mathcal{V}_x^\perp \rightarrow \mathcal{V}_x^\perp$ as $\lambda_c(x)$ and $w(t)$, with some unique $c \in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\}$ and $w \in \mathcal{W}$. By (v), $\lambda = \lambda_c$ satisfies (10.6), so that, in view of (10.4) and (10.7), the vector field $\hat{w} = Sw - \lambda w$ is a solution of the linear homogeneous differential equation $\nabla_x \hat{w} = Q^{-1/2}(2\psi - 2\lambda - S)\hat{w}$. Since $\hat{w}$ vanishes at $x = x(t)$, it must vanish identically, which establishes (vi).

Now let $w \in \mathcal{W}$. As $\dot{Q} = 2\psi Q^{1/2}$ (see the lines following (10.6)), the Jacobi equation and (10.4) give, by (ii) and (10.7), $R(w, \dot{x})\dot{x} = \nabla_{\dot{x}}\nabla_{\dot{x}}w = \nabla_{\dot{x}}[Q^{-1/2}Sw] = Q^{-1}(\psi - S)Sw$, that is, $R(w, v)v = (\psi - S)Sw$, the second equality in (vii). Also, Lemma 9.1(e), (9.2.c) and (3.3) yield $2(\psi - S)Sw = [\nabla_v S]w = -J[R(u, v)w] = -R(u, v)Jw = R(v, u)Jw = R(v, Jv)Jw$, the last equality in (vii). Combining the two relations, and repeatedly using (3.3), we get $2R(w, v)v = R(v, Jv)Jw$, that is, $R(w, v)v = R(v, w)v + R(v, Jv)Jw = R(Jv, Jw)v + R(v, Jv)Jw$. Thus, from the Bianchi identity, $R(w, v)v = R(v, Jw)Jv = R(Jv, JJw)Jv = R(w, u)u$, which proves (vii). Finally, (viii) is an easy consequence of (vi) and (9.2.d). $\square$

11. Consequences of compactness

Let (M, g, τ) be a fixed geodesic-gradient Kähler triple (Definition 4.2). We use the notation of (9.1), (4.2) and – in (i) below – the terminology of Remark 1.3.

Remark 11.1. According to [6, Lemmas 11.1, 11.2 and Remark 2.1], the following holds for any (M, g, τ) as above with compact M , the objects (4.2), and $v = \nabla\tau$.

- (i) $Q = g(v, v)$ is a C^∞ function of τ , leading to data τ_-, τ_+, a, Q with (6.1).
- (ii) The flow of the Killing vector field $u = Jv$ is periodic.
- (iii) $\Sigma^\pm$ are (connected) totally geodesic compact complex submanifolds of M .
- (iv) $\Sigma^+ \cup \Sigma^-$ is the zero set of v , that is, the set of critical points of τ .

(Conclusion (iv) is a special case of a result due to Wang [20, Lemma 3].) Furthermore, restricting $\tau \mapsto Q$ in (i) to the open interval (τ_-, τ_+) we have

$$(11.1) \quad dQ/d\tau = 2\psi, \quad \text{and so } \psi \rightarrow \mp a \text{ as } \tau \rightarrow \tau_\pm,$$

ψ being the function with (4.1) on the open set M' on which $v \neq 0$ (so that ψ is also a C^∞ function of τ). This is clear as (1.6) and (4.1) give $dQ = 2\psi d\tau$ on M' . Finally, by [20, Lemma 1] (see also [6, Example 8.1 and Lemma 8.4(iv)]),

$$(11.2) \quad v \text{ is tangent to every geodesic normal to } \Sigma^\pm.$$

Remark 11.2. Under the assumptions of Remark 11.1, for a as in Remark 11.1(i), $\mp a$ is the unique nonzero eigenvalue of the Hessian of τ (that is, of $S = \nabla v$) at any critical point $y \in \Sigma^\pm$. The $\mp a$ -eigenspace of S_y is the normal space $N_y\Sigma^\pm$, and $\text{Ker } S_y = T_y\Sigma^\pm$ (which thus constitutes the 0-eigenspace of S_y unless $\Sigma^\pm = \{y\}$).

In fact, as τ is a Morse-Bott function [6, Example 8.1], applying [6, Lemma 8.4(i)] we see that $N_y\Sigma^\pm$ is the eigenspace of S_y for its unique nonzero eigenvalue, and so $\text{Ker } S_y = T_y\Sigma^\pm$ in view of self-adjointness of S_y . That the nonzero eigenvalue equals $\mp a$ is obvious from (11.1), since (4.1) amounts to $Sv = \psi v$. Cf. [17, Theorem 1.3].

Still assuming compactness of a geodesic-gradient Kähler triple (M, g, τ) , let $N^\delta\Sigma^\pm$ be the bundle of radius δ normal open disks around the zero section in the normal bundle $N\Sigma^\pm$, with δ characterized by (6.5). According to [6, Lemma 10.3],

δ is then the distance between Σ^+ and Σ^- , while, with $\text{Exp}^\perp$ as in Remark 1.6,

$$(11.3) \quad \begin{aligned} & \text{the restriction to } N^\delta \Sigma^\pm \text{ of the normal exponential mapping} \\ & \text{Exp}^\perp : N\Sigma^\pm \rightarrow M \text{ is a diffeomorphism } N^\delta \Sigma^\pm \rightarrow M \setminus \Sigma^\mp. \end{aligned}$$

Cf. [2], [20, Lemma 2], [17, Theorem 1.1]. Its inverse $M \setminus \Sigma^\mp \rightarrow N^\delta \Sigma^\pm$, composed with the projection $N^\delta \Sigma^\pm \rightarrow \Sigma^\pm$, yields a new disk-bundle projection

$$(11.4) \quad \pi^\pm : M \setminus \Sigma^\mp \rightarrow \Sigma^\pm.$$

Remark 11.3. Clearly, $\pi^\pm \circ \text{Exp}^\perp$ is the normal-bundle projection $N\Sigma^\pm \rightarrow \Sigma^\pm$. Also, according to the lines preceding (11.4),

$$(11.5) \quad \begin{aligned} & \text{the image } \pi^\pm(x) \text{ of any } x \in M' \text{ is the unique } y \in \Sigma^\pm \text{ that can} \\ & \text{be joined to } x \text{ by a (necessarily unique) geodesic segment } \Gamma_x \text{ of} \\ & \text{length less than } \delta \text{ emanating from } y \text{ in a direction normal to } \Sigma^\pm, \end{aligned}$$

which implies [6, Remark 4.6, Example 8.1 and Theorem 10.2(iii)–(vi)] that $\pi^\pm$ sends every $x \in M \setminus \Sigma^\mp$ to the unique point nearest x in $\Sigma^\pm$.

In the next lemma, by a *leaf* we mean – as usual – a maximal integral manifold.

Lemma 11.4. *Under the above hypotheses, $\mathcal{V} \subseteq \text{Ker } d\pi^\pm$ for the integrable distribution $\mathcal{V} = \text{Span}(v, u)$ on $M' = M \setminus (\Sigma^+ \cup \Sigma^-)$, cf. Lemma 9.1(a) and Remark 11.1(iv). If ξ is a unit vector normal to $\Sigma^\pm$ at a point y , then, with δ as in (11.3),*

- (a) *the punctured radius δ disk $\{z\xi : z \in \mathbb{C} \text{ and } 0 < |z| < \delta\}$ in $N_y \Sigma^\pm$ is mapped by exp_y diffeomorphically onto a leaf Λ of $\mathcal{V}$.*

Furthermore, every leaf $\Lambda \subseteq M'$ of $\mathcal{V}$ has the following properties.

- (b) *The closure of Λ in M is a totally geodesic complex submanifold, biholomorphic to $\mathbb{CP}^1$ and equal to $\Lambda \cup \{y_+, y_-\}$, where $y_\pm \in \Sigma^\pm$ are such that $\{y_\pm\} = \pi^\pm(\Lambda)$.*
- (c) *The leaf Λ arises from (a) for some unit normal vector ξ at the point $y = y_\pm$ corresponding to Λ as in (b), and then*

$$(11.6) \quad \pi^\mp(\text{Exp}^\perp(y_\pm, z\xi)) = y_\mp \quad \text{whenever } z \in \mathbb{C} \text{ and } 0 < |z| \leq \delta,$$

$\text{Exp}^\perp$ being the normal exponential mapping $N\Sigma \rightarrow M$.

Proof. Let us fix x, y and Γ_x as in (11.5). Due to Remark 11.1(iv), the Killing field $u = Jv$ vanishes along $\Sigma^\pm$, so that its infinitesimal flow at y preserves both $T_y \Sigma^\pm$ and $N_y \Sigma^\pm$. The images of Γ_x under the flow transformations of u thus are geodesic segments normal to $\Sigma^\pm$ emanating from y and, as a consequence of (11.5), $\pi^\pm$ maps them all onto $\{y\}$. In other words, the union of such segments, with the point y removed, is simultaneously a subset of the $\pi^\pm$ -preimage of y as well as – according to (11.2), (9.3) and parts (ii), (iv) of Remark 11.1 – a surface embedded in M' . This surface is, due to its very definition and (11.2), tangent to both u and v which, in view of (11.3), yields (a); note that, by (9.2.a) and Remark 11.2, the orbit of ξ under the flow of $A = \nabla u$ at y consists of all unit complex multiples of ξ .

What we just observed about the orbit of ξ clearly ensures smoothness of the closure of the leaf at y . By (11.3) and (11.2), the union of I_x and its analog for the same point x and the *other* projection $\pi^\mp$ is a length δ geodesic segment joining $y \in \Sigma^\pm$ to its other endpoint $y_\mp \in \Sigma^\mp$. The above discussion of the images of such a segment under the flow of u applies equally well to $y_\mp$, so that (b) – (c) follow from Lemma 9.1(a) and the fact that $x \in M'$ was arbitrary. $\square$

Remark 11.5. Let (M, g, τ) be a Grassmannian or CP triple, constructed as in Section 5 from some data (5.3.i) or (5.3.ii). We use the notation of (9.1) and (11.4).

- (a) We already know that the critical manifolds $\Sigma^\pm$ of τ are given by (5.5).
- (b) In the case of (5.5.c) (or, (5.5.b) and (5.5.d)), $\pi^\pm$ acts on W as the orthogonal projection into L (or, respectively, into $L^\perp$).
- (c) When $\Sigma^\pm$ has the form (5.5.a), $\pi^\pm$ sends W to $L \oplus (W \cap L^\perp)$.
- (d) The leaf of $\mathcal{V}$ through any $W \in M'$ consists
 - (d1) for (5.3.i) – of all $L' \oplus W'$, where $W' = W \cap L^\perp$ and L' is any line in the plane $L \oplus (W' \cap W^\perp)$ other than the lines L and $W' \cap W^\perp$ themselves,
 - (d2) for (5.3.ii) – of all lines other than W' and W'' in the plane $W' \oplus W''$, where W' and W'' denote the orthogonal projections of W into L and $L^\perp$.

Namely, in both cases, let G' be the complex Lie group of all complex-linear automorphisms of V preserving both L and $L^\perp$. The obvious action of G' on the Stiefel manifold $\text{St}_k \mathcal{T}$ (see Remark 5.5) descends to a holomorphic action on $\text{Gr}_k V$, which becomes one on $PV = G_1 V$ when $k = 1$. The elements of the center of G' , restricted to both subspaces L and $L^\perp$, are complex multiples of Id , and the action of the center on $\text{Gr}_k V$ includes the circle subgroup S^1 of G generated by the Killing field u , mentioned in the lines following (5.3). Holomorphicity of the action implies that the flow of the gradient $v = \nabla \tau$, related to u via $u = Jv$, also consists of transformations of $\text{Gr}_k V$ arising from the action of the center, and – for dimensional reasons – the orbits of the center coincide with the leaves of $\mathcal{V} = \text{Span}(v, u)$. This easily gives (d). Now (b) – (c) follow: by Lemma 11.4(b), the two $\pi^\mp$ -images of any leaf of $\mathcal{V}$ are the two points that, added to the leaf, yield its closure.

As $\mathcal{V} \subseteq \text{Ker } d\pi^\pm$ (Lemma 11.4), we may define vector subbundles $\mathcal{H}^\pm$ of TM' by

$$(11.7) \quad \mathcal{H}^\pm = \mathcal{V}^\perp \cap \text{Ker } d\pi^\mp, \quad \text{so that} \quad \text{Ker } d\pi^\pm = \mathcal{V} \oplus \mathcal{H}^\mp.$$

Theorem 11.6. *Given a geodesic-gradient Kähler triple (M, g, τ) with compact M , $v, M', Q, \mathcal{V}, S, \Sigma^\pm, \tau_\pm, \pi^\pm, \mathcal{H}^\pm$ be defined as in (9.1), (4.2) and (11.7). Then the bundle endomorphism $2(\tau - \tau_\pm)S - Q$ of TM , restricted to $\mathcal{V}^\perp$, has constant rank on M' , while*

$$(11.8) \quad \mathcal{H}^\mp = \mathcal{V}^\perp \cap \text{Ker } [2(\tau - \tau_\pm)S - Q],$$

and some subbundle $\mathcal{H}$ of TM' yields an S -invariant complex orthogonal decomposition

$$(11.9) \quad TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^- \oplus \mathcal{H}.$$

Furthermore, for any $\Gamma \subseteq M'$ chosen as at the beginning of Section 10, the closure of Γ in M admits a unit-speed C^∞ parametrization $[t_-, t_+] \ni t \mapsto x(t)$ which, restricted to (t_-, t_+) , is a parametrization of Γ satisfying (10.1) along with the following conditions.

- (a) The endpoint $y_\pm = x(t_\pm)$ lies in $\Sigma^\pm$, and $\dot{x}(t_\pm)$ is normal to $\Sigma^\pm$ at $y_\pm$.
- (b) Every solution $(t_-, t_+) \ni t \mapsto w(t) \in \mathcal{V}_{x(t)}^\perp$ of (10.4) along Γ has a C^∞ extension to $[t_-, t_+]$ such that $d\pi_{x(t)}^\pm[w(t)] = w(t_\pm)$ whenever $t \in (t_-, t_+)$.
- (c) The bundle projection $\pi^\pm: M \setminus \Sigma^\mp \rightarrow \Sigma^\pm$ is holomorphic.
- (d) If $w \in \mathcal{W}[\tau_\pm]$, cf. (10.5), then, in (b), $w(t_\pm) = 0$, and $[\nabla_{\dot{x}} w](t_\pm)$ is normal to $\Sigma^\pm$ at $y_\pm = x(t_\pm)$ as well as orthogonal to $\dot{x}(t_\pm)$ and $J\dot{x}(t_\pm)$.
- (e) If w lies in the direct sum of spaces $\mathcal{W}[c] \neq \{0\}$ with $c \neq \tau_\pm$, for a fixed sign $\pm$, then $w(t_\pm)$ is tangent to $\Sigma^\pm$ at $y_\pm = x(t_\pm)$, and $[\nabla_{\dot{x}} w](t_\pm) = 0$.
- (f) Whenever $t \in (t_-, t_+)$ and $x = x(t)$, the assignment $w(t) \mapsto (w(t_\pm), [\nabla_{\dot{x}} w](t_\pm))$, with w as in (b), is a $\mathbb{C}$ -linear isomorphism $\mathcal{V}_x^\perp \rightarrow T_y \Sigma^\pm \times N'_y$, where $y = y_\pm$ and N'_y denotes the orthogonal complement of $\text{Span}(\dot{x}(t_\pm), J\dot{x}(t_\pm))$ in $N_y \Sigma^\pm$. At the same time, $w(t)$ then equals the image, under the differential of the normal exponential mapping $\text{Exp}^\perp: N\Sigma^\pm \rightarrow M$ at $(y, \xi) \in N\Sigma^\pm$ given by $y = x(t_\pm)$ and $\xi = (t - t_\pm)\dot{x}(t_\pm)$, of the vector tangent to $N\Sigma^\pm$ at (y, ξ) which equals the sum of the vertical vector $\eta = (t - t_\pm)[\nabla_{\dot{x}} w](t_\pm)$ and the D-horizontal lift of $w(t_\pm)$ to (y, ξ) , for the normal connection D in $N\Sigma^\pm$. Similarly, $u_{x(t)}$, for $u = Jv$, is the image, under the differential of $\text{Exp}^\perp$ at (y, ξ) , of the vertical vector $\eta = \mp a i \xi$.
- (g) For any $w, w' \in \mathcal{W}$, the function $Q^{-1}g(Sw, w')$ is constant on Γ and the restriction of $g(w, w')$ to Γ is an affine function of $\tau: \Gamma \rightarrow \mathbb{R}$ with the derivative $d[g(w, w')]/d\tau = 2Q^{-1}g(Sw, w')$.
- (h) Explicitly, in (g), with a as in Remark 11.1(i), either sign $\pm$, and $y = y_\pm = x(t_\pm)$,
 - (h1) $g(w, w') = (\tau_+ - \tau_-)^{-1}|\tau - \tau_\mp|g_y(w_\pm, w'_\pm)$ if $w \in \mathcal{W}[\tau_\mp]$ and $w' \in \mathcal{W}$,
 - (h2) $g(w, w') = g_y(w_\pm, w'_\pm) - a^{-1}|\tau - \tau_\pm|g_y(R_y(w_\pm, J_y w'_\pm)\dot{x}_\pm, J_y \dot{x}_\pm)$ if w, w' both satisfy the assumption of (e),
 - (h3) $g(w, w') = 2a^{-1}|\tau - \tau_\pm|g_y([\nabla_{\dot{x}} w]_\pm, [\nabla_{\dot{x}} w']_\pm)$ if $w, w' \in \mathcal{W}[\tau_\pm]$,

where the subscript $\pm$ next to $w, w', \nabla_{\dot{x}} w, \nabla_{\dot{x}} w'$ or $\dot{x}$ represents their evaluation at $t_\pm$.

Remark 11.7. Since $|\tau - \tau_\pm| = \mp(\tau - \tau_\pm)$ and $\pm(\tau_+ - \tau_-) = \tau_\pm - \tau_\mp$, applying $d/d\tau$ to the right-hand side in (h1), or (h2), or (h3), we get the three values

$$(\tau_\pm - \tau_\mp)^{-1}g_y(w_\pm, w'_\pm), \pm a^{-1}g_y(R_y(w_\pm, J_y w'_\pm)\dot{x}_\pm, J_y \dot{x}_\pm), \mp 2a^{-1}g_y([\nabla_{\dot{x}} w]_\pm, [\nabla_{\dot{x}} w']_\pm).$$

As a consequence of parts (g) – (h) of Theorem 11.6, this triple provides the three expressions for $2Q^{-1}g(Sw, w')$ in the cases (h1), (h2) and (h3), respectively.

Note that the three different formulae for $g(w, w')$ in (h1), (h2) and – with the reversed sign – in (h3), are all simultaneously valid when $w, w' \in \mathcal{W}[\tau_\mp]$.

Remark 11.8. Under the assumptions of Theorem 11.6,

- (i) the relation $\xi = (t - t_\pm)\dot{x}(t_\pm)$ in (f) clearly gives $\dot{x}_\pm = \mp \xi/|\xi|$ in (h2),

- (ii) by (d) – (f), the images under the differential of $\text{Exp}^\perp$ of vertical (or, horizontal) vectors tangent to $N\Sigma^\pm$ at the point (y, ξ) appearing in (f) have the form $(*)$ $w(t)$ for w satisfying the hypothesis of (d) (or, respectively, of (e)),
- (iii) the differential of $\pi^\pm$ at any $x \in M'$ maps the summands $\mathcal{H}_x^\pm$ and $\mathcal{H}_x$ in (11.9) isomorphically onto the images $d\pi_x^\pm(\mathcal{H}_x^\pm)$ and $d\pi_x^\pm(\mathcal{H}_x)$, orthogonal to each other in $T_y\Sigma^\pm$ for $y = \pi^\pm(x)$,
- (iv) one has $(\tau_+ - \tau_-)g_x(w, w') = |\tau(x) - \tau_\mp|g_y(d\pi_x^\pm w, d\pi_x^\pm w')$ whenever $w \in \mathcal{H}_x^\pm$ and $w' \in \mathcal{V}_x^\perp$ at any $x \in M'$, while $y = \pi^\pm(x)$,
- (v) (4.2) and (a) imply the inequality of Theorem 10.1(viii) everywhere in M' .

Only (iii) and (iv) require further explanations. For (iii), $d\pi_x^\pm$ is injective on the space $\mathcal{H}_x^\pm \oplus \mathcal{H}_x$, orthogonal, by (11.7) and (11.9), to its kernel $\mathcal{V}_x \oplus \mathcal{H}_x^\mp$. Orthogonality in (11.7) also shows, via (11.8), (10.5.b) and (9.2.d), that vectors in $\mathcal{H}_x^\pm$ (or, in $\mathcal{H}_x$) have the form $(*)$ with $x = x(t)$, cf. (f), and the former remain orthogonal to the latter as t varies, leading to (iii) as a consequence of the final clause of (b). Assertion (iv) is nothing else than (h1) for $w = w(t)$ at $x = x(t)$, cf. (f), where $y = \pi^\pm(x)$ and $d\pi_x^\pm w = w_\pm$ by (11.5) and (b).

Remark 11.9. As another immediate consequence of Theorem 11.6, the assignment $x \mapsto d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm)$ defines a holomorphic section of the bundle over M' arising via the pullback under $\pi^\pm$ from $\text{Gr}_k(T\Sigma^\pm)$, for a suitable integer $k = k_\pm$. Here $\text{Gr}_k(T\Sigma^\pm)$ is the Grassmannian bundle with the fibres $\text{Gr}_k(T_y\Sigma^\pm)$, $y \in \Sigma^\pm$ (cf. Section 5), holomorphicity and the equality $d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm)$ are clear from Theorem 11.6(c) (which also implies, due to (11.7), that $\mathcal{V} \oplus \mathcal{H}^\pm$ is a holomorphic subbundle of TM') and (11.9) (which, combined with (11.7), ensures constancy of the dimension $k = k_\pm$ of the spaces $d\pi_x^\pm(\mathcal{H}_x^\pm)$).

12. Proof of Theorem 11.6

We begin by establishing (a) – (f) under the stated assumptions about Γ .

Let $(t_-, t_+) \mapsto x(t)$ be a parametrization of Γ with (10.1). As τ then is clearly an increasing function of t , it has some limits $\hat{\tau}_\pm$ as $t \rightarrow t_\pm$, finite due to boundedness of τ . The length of Γ obviously equals the integral of $Q^{-1/2}$ over $(\hat{\tau}_-, \hat{\tau}_+) \subseteq (\tau_-, \tau_+)$, and so it is finite in view of (6.5). This implies the existence of limits $x(t_\pm)$ of $x(t)$ as $t \rightarrow t_\pm$. Furthermore, each $x(t_\pm)$ lies in $\Sigma^\pm$ since, if one $x(t_\pm)$ did not, Remark 11.1(iv) would yield $v \neq 0$ at $x(t_\pm)$, contradicting maximality of Γ . Thus, $[t_-, t_+] \mapsto x(t)$ parametrizes the closure of Γ . Next, $M \setminus (\Sigma^+ \cup \Sigma^-)$ is, by (11.3) and (11.2), a disjoint union of maximal integral curves of v , each of which has two limit points, one in Σ^- and one in Σ^+ , and the corresponding limit directions of the curve are normal to Σ^- and Σ^+ . Since Γ is one of these curves, (a) follows.

In (b), a C^∞ extension to $[t_-, t_+]$ must exist as w is a Jacobi field; see Theorem 10.1(ii). To obtain (d) – (e), we fix $w \in \mathcal{W}[c]$, so that, from (10.3) – (10.5),

$$(12.1) \quad \text{i) } Sw = Qw/[2(\tau - c)], \quad \text{ii) } \nabla_x w = Q^{1/2}w/[2(\tau - c)].$$

Let $y = x(t_\pm)$ and $w_y = w(t_\pm)$. By Remark 11.1(iv), $Q = \tau - \tau_\pm = 0$ on $\Sigma^\pm$ while, in view of (a) and (11.1), $Q/[2(\tau - \tau_\pm)]$ evaluated at $x(t)$ tends to $\mp a \neq 0$ as $t \rightarrow t_\pm$. If $c = \tau_\pm$, (12.1.ii) multiplied by $Q^{1/2}$ thus yields $w_y = 0$, and the relation $Sw' = Qw'/[2(\tau - c)]$ for $w' = \nabla_x w$, obvious from (12.1), implies that $[\nabla_x w](t_\pm)$ lies in the $\mp a$ -eigenspace of S_y . When $c \neq \tau_\pm$, (12.1.i) and (12.1.ii) give, respectively, $S_y w_y = 0$ and $[\nabla_x w](t_\pm) = 0$. Due to Remark 11.2, this proves (d) and (e): orthogonality in (d) follows since w and $\nabla_x w$ take values in $\mathcal{V}^\perp$, for $\mathcal{V} = \text{Span}(v, u)$ (so that $g(w, v) = g(w, u) = 0$), while $\dot{x} = v/|v|$ by (10.1), and $u = Jv$.

Furthermore, the assignment in (f) is well-defined, injective, complex-linear and $(T_y \Sigma^\pm \times N'_y)$ -valued due to parts (iii), (ii), (iv) of Theorem 10.1 and, respectively, (d) – (e). The first claim of (f) thus follows since both spaces have the same dimension. The second (or, third) one is in turn immediate from (1.9) applied, at $r = 1$, to any $w \in \mathcal{W}$, cf. Theorem 10.1(ii) (or, to $w = u$), with y, ξ, η as in (f), and $\hat{w}$ defined by $\hat{w}(r) = w(rt + (1 - r)t_\pm)$. (That $r \mapsto \hat{w}(r)$ then is a Jacobi field along the geodesic $r \mapsto x(rt + (1 - r)t_\pm)$ follows from Theorem 10.1(ii) or, respectively, (9.2.b) and Remark 1.7, while, in the latter case, due to (9.2.a) along with Remarks 11.1(iv) and 11.2, $w = u$ realizes the initial conditions $(u, \nabla_{dx/dr} u) = (0, \mp ai\xi)$ at $r = 0$.)

The remaining equality $d\pi_{x(t)}^\pm[w(t)] = w(t_\pm)$ in (b) now becomes an obvious consequence of the second part of (f) combined with the first line of Remark 11.3. This proves (b) and, combined with Theorem 10.1(iv), implies (c).

Next, for $t \mapsto x(t)$ as in (a) – (f), any $t \in (t_-, t_+)$, a fixed sign $\pm$, and $x = x(t)$, Theorem 10.1(iii), (11.7) and (b) give $\mathcal{H}_x^\mp = \{w(t) : w \in \mathcal{W} \text{ and } w(t_\pm) = 0\}$. Writing any $w \in \mathcal{W}$ as $w = w' + w''$, where $w' \in \mathcal{W}[\tau_\pm]$ and w'' lies in the direct sum of the spaces $\mathcal{W}[c] \neq \{0\}$ with $c \neq \tau_\pm$, cf. Theorem 10.1(iv), we see that, by (d) – (e), the isomorphism in (f) sends $w'(t)$ and $w''(t)$, respectively, to pairs of the form $(0, \cdot)$ and $(\cdot, 0)$. Thus, $w(t) \in \mathcal{H}_x^\mp$ if and only if $w'' = 0$, that is, $w \in \mathcal{W}[\tau_\pm]$. Combining Theorem 10.1(vi) with (10.3) and (10.5.b), one now obtains (11.8), so that (11.7) implies the constant-rank assertion preceding (11.8). On the other hand, $\mathcal{H}_x^+$ and $\mathcal{H}_x^-$ are mutually orthogonal at every $x \in M'$, being, by (11.8), contained in eigenspaces corresponding to different eigenvalues of the self-adjoint operator S_x , cf. (9.2.d), so that (11.9) follows.

Let $w, w' \in \mathcal{W}$. Constancy of $Q^{-1}g(Sw, w')$ along Γ trivially follows from (9.4.iii) and (11.2), cf. Lemma 9.1(b) and parts (i) – (ii) of Theorem 10.1. The operators $d/d\tau$ and d_v acting on functions $\Gamma \rightarrow \mathbb{R}$ are in turn related by $d_v = Q d/d\tau$, since (10.1) gives $d_v = Q^{1/2} d_{\dot{x}} = Q^{1/2} d/dt$, while $d/dt = Q^{1/2} d/d\tau$ due to (10.2). Now (g) is immediate from (9.4.ii).

In (h), all three right-hand sides are affine functions of τ with the correct values at $t = t_\pm$ (that is, limits at the endpoint $y_\pm = x(t_\pm)$). Proving (h) is thus reduced by (g) to showing that, in each case, $\chi = 2Q^{-1}g(Sw, w')$ coincides with the derivative of the right-hand side provided by Remark 11.7, which – even though χ is constant on Γ , cf. (g) – will be achieved via evaluating the limit of χ at $y_\pm \in \Gamma$ or, equivalently,

at $t_{\pm} \in [t_-, t_+]$. When $w \in \mathcal{W}[\tau_{\mp}]$, (10.5.b) and (10.3) imply that $2Q^{-1}Sw = (\tau - \tau_{\mp})^{-1}w$ and, consequently, $\chi = (\tau - \tau_{\mp})^{-1}g(w, w')$ has the value (and limit) $\pm(\tau_+ - \tau_-)^{-1}g_y(w_{\pm}, w'_{\pm})$ at $y = y_{\pm}$, as required in (h1).

Let w, w' now satisfy the hypotheses of (e). Consequently, along $\Gamma \setminus \{y_{\pm}\}$,

$$(12.2) \quad Q, \quad Sw', \quad Q^{-1}g(Sw, Sw') \quad \text{all tend to 0 at } y, \text{ where } y = y_{\pm}.$$

In fact, $Q(y) = 0$ by (a). Next, $Q^{-1}Sw$ is bounded near the endpoint y of $\Gamma \setminus \{y\}$ (and similarly for w'); to see this, we may assume that $w \in \mathcal{W}[c]$ with $c \neq \tau_{\pm}$, cf. (e), and then (10.5.b) and (10.3) give $2Q^{-1}Sw = (\tau - c)^{-1}w$, which is bounded as $\tau \rightarrow \tau_{\pm}$ since, due to (b), w has a limit at $t = t_{\pm}$. Now (12.2) follows.

In view of (12.2) and (b), we may now evaluate the limit of $\chi = 2Q^{-1}g(Sw, w')$ as $t \rightarrow t_{\pm}$ using l'Hôpital's rule: it coincides with the limit of $2d_{\dot{x}}[g(Sw, w')]/\dot{Q}$. By (9.2.c) for $\dot{x}$ rather than of w , (10.4), (9.2.d) and (10.2), this last expression is the sum of two terms, $\psi^{-1}Q^{-1/2}g(R(u, \dot{x})w, Jw')$ and $2\psi^{-1}Q^{-1}g(Sw, Sw')$. According to (12.2) and (11.1), only the first term contributes to the limit and, as it equals $\psi^{-1}g(R(J\dot{x}, \dot{x})w, Jw')$, cf. (9.3) and (11.2), relation (11.1) yields (h2).

Finally, suppose that $w, w' \in \mathcal{W}[\tau_{\pm}]$. It follows that

$$(12.3) \quad Q^{-1/2}w \rightarrow \mp a^{-1}[\nabla_{\dot{x}}w]_{\pm} \quad \text{at } y, \text{ where } y = y_{\pm},$$

and analogously for w' . Namely, Q and w vanish at y (see (a), (d)), while $(Q^{1/2})' = \psi$ by (10.2), and so $[\nabla_{\dot{x}}w]/(Q^{1/2})' = \psi^{-1}\nabla_{\dot{x}}w$. L'Hôpital's rule and (11.1) now imply (12.3). Since $S_y[\nabla_{\dot{x}}w]_{\pm} = \mp a[\nabla_{\dot{x}}w]_{\pm}$ by (d) and Remark 11.2, assertion (h3) is obvious from (12.3), completing the proof of Theorem 11.6.

Remark 12.1. With the same notations and assumptions as in Theorem 11.6, denoting by $k_{\pm}$ and q the complex fibre dimensions of the subbundles $\mathcal{H}^{\pm}$ and $\mathcal{H}$ of TM' , we have, for $m = \dim_{\mathbb{C}} M$ and $d_{\pm} = \dim_{\mathbb{C}} \Sigma^{\pm}$,

$$(12.4) \quad d_+ + d_- = m - 1 + q,$$

as one sees adding the equalities $d_{\pm} = m - 1 - k_{\pm}$ and $m = 1 + k_+ + k_- + q$ (the former due to (11.4) and (11.7), the latter to (11.9)). Consequently,

$$(12.5) \quad d_+ + d_- \geq m - 1,$$

with equality if and only if the distribution $\mathcal{H}$ in (11.9) is 0-dimensional, that is, if

$$(12.6) \quad TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-.$$

The explicit descriptions of $\Sigma^{\pm}$ in (5.5.c) – (5.5.d) clearly show that

$$(12.7) \quad d_+ + d_- = m - 1 \quad \text{for every CP triple } (M, g, \tau).$$

13. Examples: Nontrivial modifications

Remark 13.1. For any two functions $\tau \mapsto Q$ and $\hat{\tau} \mapsto \hat{Q}$ having the properties listed in (6.1), with the same $\tau_{\pm}$ and a , there must exist an increasing C^{∞} diffeomorphism $[\tau_{-}, \tau_{+}] \ni \tau \mapsto \hat{\tau} \in [\tau_{-}, \tau_{+}]$ which realizes

$$(13.1) \quad \text{the equality } \hat{Q} d/d\hat{\tau} = Q d/d\tau \text{ of vector fields on } [\tau_{-}, \tau_{+}] \text{ expressed in terms of the two diffeomorphically-related coordinates } \hat{\tau} \text{ and } \tau.$$

Such a diffeomorphism is unique up to compositions from the left (or, right) with transformations forming the flow of the first (or, second) vector field in (13.1).

To see this, apply Remark 6.7 to $t = \tau - \tau_{\pm}$ and $\gamma = \mp Q/(2a)$ (or, $t = \hat{\tau} - \tau_{\pm}$ and $\gamma = \mp \hat{Q}/(2a)$), obtaining a function θ (or, $\hat{\theta}$), unique up to a positive constant factor and vanishing at $\tau = \tau_{\pm}$ (or, $\hat{\tau} = \tau_{\pm}$), with $d\theta/d\tau = \theta/Q$ (or, $d\hat{\theta}/d\hat{\tau} = \hat{\theta}/\hat{Q}$), the derivative being positive everywhere in $[\tau_{-}, \tau_{+}]$. Adjusting the constant factor, we may require $\tau \mapsto \theta$ and $\hat{\tau} \mapsto \hat{\theta}$ to be increasing diffeomorphisms of $[\tau_{-}, \tau_{+}]$ onto *the same* interval having the endpoint 0, and then define $\tau \mapsto \hat{\tau}$ by declaring θ “equal to” $\hat{\theta}$, that is, letting $\tau \mapsto \hat{\tau}$ be $\tau \mapsto \theta$ followed by the inverse of $\hat{\tau} \mapsto \hat{\theta}$. Consequently, $d\hat{\tau}/d\tau = \hat{Q}/Q$ on (τ_{-}, τ_{+}) , which amounts to (13.1).

The uniqueness clause is obvious: the only self-diffeomorphisms ζ of (τ_{-}, τ_{+}) preserving a given vector field without zeros are its flow transformations, since ζ acts on an integral curve as a shift of the parameter.

Theorem 13.2. *For the data $\tau_{\pm}$ and $\tau \mapsto Q$ related via Remark 11.1(i) to a given compact geodesic-gradient Kähler triple (M, g, τ) , and any increasing C^{∞} diffeomorphism $[\tau_{-}, \tau_{+}] \ni \tau \mapsto \hat{\tau} \in [\tau_{-}, \tau_{+}]$, there exists a C^{∞} function $[\tau_{-}, \tau_{+}] \ni \tau \mapsto \phi \in \mathbb{R}$, unique up to additive constants, such that $\hat{\tau} = \tau + Q d\phi/d\tau$.*

With $\hat{\tau}, \phi$ treated, due to their dependence on τ , as functions on the complex manifold M , the formula $\hat{g} = g - 2(i\partial\bar{\partial}\phi)(J \cdot, \cdot)$ then defines another Kähler metric on M , and

(a) $(M, \hat{g}, \hat{\tau})$ is a new geodesic-gradient Kähler triple.

In addition, denoting by $\hat{\tau} \mapsto \hat{Q}$ the analog of $\tau \mapsto Q$ arising when Remark 11.1(i) is applied to $(M, \hat{g}, \hat{\tau})$, and by $\hat{\nabla}\hat{\tau}$ the $\hat{g}$ -gradient of $\hat{\tau}$, one has (13.1) and $\hat{\nabla}\hat{\tau} = \nabla\tau$.

Proof. As $\hat{\tau} = \tau + Q\phi'$, where $(\cdot)' = d/d\tau$, our assumption about $\tau \mapsto \hat{\tau}$ gives $\hat{\tau}' > 0$ and $\tau_{-} \leq \hat{\tau} \leq \tau_{+}$, leading to the inequalities

$$(13.2) \quad \tau_{-} \leq \tau + Q\phi' \leq \tau_{+} \quad (\text{strict except at } \tau = \tau_{\pm}) \text{ and } 1 + Q'\phi' + Q\phi'' > 0.$$

Note that ϕ exists since, by Remark 6.6 and (6.1), $\hat{\tau} - \tau$ and Q are smoothly divisible by $\tau - \tau_{\pm}$, their quotients being equal at $\tau_{\pm}$ to the value of $\hat{\tau}' - 1$ and $\mp 2a$, respectively, and so, as $\hat{\tau}' > 0$,

$$(13.3) \quad \mp 2a\phi' > -1 \quad \text{at } \tau = \tau_{\pm}.$$

For the self-adjoint bundle endomorphism K of TM with $\hat{g} = g(K \cdot, \cdot)$ one has

$$(13.4) \quad K = \text{Id} + 2\phi'S + \phi''[g(v, \cdot)v + g(u, \cdot)u],$$

where v, u, S are, as usual, given by $v = \nabla \tau$, $u = Jv$, and $S = \nabla v$.

We proceed to prove positivity of K at all points of M , considering two separate cases: $y \in \Sigma^\pm$ and $x \in M' = M \setminus (\Sigma^+ \cup \Sigma^-)$, cf. Remark 11.1(iv).

If y lies in either critical manifold $\Sigma^\pm$, the relations $v_y = u_y = 0$ and $\tau(y) = \tau_\pm$ imply positivity of K_y as a consequence of (13.3) since, by Remark 11.2, any eigenvalue of S_y must be equal to 0 or $\mp a$.

On M' , we use the S -invariant decomposition $TM' = \mathcal{V} \oplus \mathcal{V}^\perp$, cf. (11.9). In view of (9.2.f) – (9.2.g), the restriction of $2S = 2\nabla v$ to $\mathcal{V} = \text{Span}(v, u)$ equals Q' times Id. Using (13.4) and (9.3) we now see that K acts in $\mathcal{V}$ via multiplication by the function $1 + Q'\phi' + Q\phi''$, which is positive according to (13.2). Theorem 10.1(vi) states in turn that the eigenvalues of $S_x : \mathcal{V}_x^\perp \rightarrow \mathcal{V}_x^\perp$, for $x \in M'$, have the form $\lambda_c(x)$ with (10.3) and (10.5.a). Writing K, S, τ, Q, ϕ' instead of their values at x , we conclude from (13.4) that the corresponding eigenvalues of $(\tau - c)K$ are $\tau + Q\phi' - c$ and so, due to the (strict) first inequality of (13.2), they all lie in the interval $(\tau_- - c, \tau_+ - c)$. Positivity of K on $\mathcal{V}$ thus easily follows both when $c < \tau_- < \tau$ and when $\tau < \tau_+ < c$.

Consequently, $\hat{g}$ is a Kähler metric on M , with the Kähler form $\hat{\omega} = \hat{g}(\hat{J}\cdot, \cdot)$ related to $\omega = g(\hat{J}\cdot, \cdot)$ by $\hat{\omega} = \omega + 2i\partial\bar{\partial}\phi$. Applying (3.8) to $v = \nabla \tau$ and ϕ rather than f , we obtain $\hat{g}(v, \cdot) = g(v, \cdot) - 2\omega(Jv, \cdot) = d\tau + d(d_v\phi)$. (Note that $Jv = u$ and (9.3) gives $d_u\phi = d_u\tau = 0$, since ϕ is a function of τ .) As $d_v\tau = Q$, cf. (1.7), v is thus the $\hat{g}$ -gradient of $\tau + d_v\phi = \tau + Q\phi' = \hat{\tau}$. On the other hand, again from (1.7), $\hat{Q} = \hat{g}(v, v)$ equals $d_v\hat{\tau} = \hat{\tau}'d_v\tau = \hat{\tau}'Q$, which is a function of τ , and of $\hat{\tau}$, proving both (a) (see Lemma 4.1) and (b). $\square$

Remark 13.3. Let G be the group of all automorphisms (Definition 4.2) of a given compact geodesic-gradient Kähler triple (M, g, τ) . Then every quadruple $\tau_-, \tau_+, a, \hat{\tau} \mapsto \hat{Q}$ satisfying the analog of (6.1) arises when Remark 11.1(i) is applied to a suitably chosen G -invariant geodesic-gradient Kähler triple $(M, \hat{g}, \hat{\tau})$ with the same underlying complex manifold M .

In fact, a trivial modification (see Remark 4.3) followed by rescaling of the metric allows us to assume that $\tau_\pm$ and a are the same as those for (M, g, τ) . Our claim is now obvious from Remark 13.1 and Theorem 13.2.

Remark 13.4. As a special case of Remark 13.3, for the first triple using the Fubini-Study metric g and G as in the lines preceding (5.3), all quadruples $\tau_-, \tau_+, a, \tau \mapsto Q$ with (6.1) are realized, via Remark 11.1(i), by CP triples (M, g, τ) having arbitrarily fixed values of $m = \dim_{\mathbb{C}} M$ and $d_\pm = \dim_{\mathbb{C}} \Sigma^\pm$ that satisfy (12.7).

Remark 13.5. Conversely, we can apply Remark 13.1 and Theorem 13.2 to canonically modify any given CP triple, obtaining one with the Fubini-Study metric and the same group G .

We will not use the easily-verified fact that, for such a Fubini-Study CP triple, $(\tau_+ - \tau_-)Q = 2a(\tau - \tau_\pm)(\tau_+ - \tau)$ and, in (5.3.ii), the value of τ at $\mathbb{C}(\xi + \eta)$, where $\xi \in \mathbb{L}$ and $\eta \in \mathbb{L}^\perp$, equals $(\tau_\pm|\xi|^2 + \tau_\mp|\eta|^2)/(|\xi|^2 + |\eta|^2)$ for some sign $\pm$.

14. The normal-geodesic biholomorphisms

In this section (M, g, τ) is a fixed compact geodesic-gradient Kähler triple (Definition 4.2). We use the notation of (9.1), denote by τ_-, τ_+, a, Q the data (6.1) associated with (M, g, τ) (see Remark 11.1(i)), and choose for them the further data (6.2) – (6.5), so that a sign $\pm$ is fixed as well. We also let $\Sigma, N, h, \langle, \rangle$ and D stand for $\Sigma^\pm$, the normal bundle $N\Sigma^\pm$, the submanifold metric of Σ , the Riemannian fibre metric in N induced by g , and the Chern connection of $\langle, \rangle$ in N , cf. (d) in Section 7. We write $(y, \xi) \in N$ when $y \in \Sigma$ and $\xi \in N_y$, as in Remark 1.5.

Using the normal exponential diffeomorphism $\text{Exp}^\perp : N^\delta \Sigma^\pm \rightarrow M \setminus \Sigma^\mp$ in (11.3), we define $\Phi = \Phi^\pm : N \rightarrow M \setminus \Sigma^\mp$, depending on the sign $\pm$, to be the composite

$$(14.1) \quad \Phi = \text{Exp}^\perp \circ \Delta,$$

where $\Delta : N \rightarrow N^\delta \Sigma^\pm$ is given by $\Delta(y, \xi) = y$ if $\xi = 0$ and, otherwise,

$$(14.2) \quad \begin{aligned} \Delta(y, \xi) &= (y, t\xi), \text{ where } t = \sigma/\rho \text{ for } \rho = |\xi|, \text{ the function } \sigma \\ &\text{of the variable } \rho \in [0, \infty) \text{ being chosen as above, with (6.4).} \end{aligned}$$

Note that Δ is a homeomorphism and, restricted to the complement $N' = N \setminus \Sigma$ of the zero section, it becomes a diffeomorphism $N' \rightarrow N^\delta \Sigma^\pm \setminus \Sigma^\pm$. In fact, $t\xi$ with $t = \sigma/\rho$ determines ξ (smoothly if $\xi \neq 0$), since $|t\xi| = \sigma$ and σ determines ρ according to the line preceding (6.4). Consequently, $\Phi : N \rightarrow M \setminus \Sigma^\mp$ is a homeomorphism, and the restriction $\Phi : N' \rightarrow M'$ a diffeomorphism. In addition,

$$(14.3) \quad \pi^\pm \circ \Phi^\pm \text{ equals the normal-bundle projection } N\Sigma^\pm \rightarrow \Sigma^\pm$$

due to (14.1), the fibre-preserving property of Δ , and the first line of Remark 11.3.

Remark 14.1. Suppose that a vector field w on N' is

- (a) the D -horizontal lift of a vector field on Σ , or
- (b) a vertical vector field of the form $(y, \xi) \mapsto \Theta\xi$ for some complex-linear vector-bundle morphism $\Theta : N \rightarrow N$, skew-adjoint relative to $\langle, \rangle$ at every point.

Then Δ , restricted to N' , sends w onto its restriction to $N' \cap N^\delta \Sigma^\pm$.

In fact, let $r \mapsto (y(r), \xi(r))$ be an integral curve of w . Then the function $r \mapsto |\xi(r)|$ is constant, and so, by (14.2), $\Delta(y(r), \xi(r)) = (y(r), c\xi(r))$ with some real constant c . This proves our claim since, in case (b), w restricted to every fibre N_y , being a linear vector field on N_y , is invariant under multiplications by scalars.

Theorem 14.2. *For either critical manifold $\Sigma^\mp$ of τ in any compact geodesic-gradient Kähler triple (M, g, τ) , the triple $(M \setminus \Sigma^\mp, g, \tau)$ is isomorphic to one constructed in Section 8 from some data (6.1) – (6.2) and $\Sigma, h, N, \langle, \rangle$.*

The data consist of (6.1) associated with (M, g, τ) as in Remark 11.1(i), any choice of $\tau \mapsto \rho$ with (6.2) for (6.1) and our fixed sign $\pm$, the submanifold metric h and normal bundle $N = N\Sigma^\pm$ of $\Sigma = \Sigma^\pm$, and the fibre metric $\langle, \rangle$ in N induced by g . Furthermore,

- (i) *the required isomorphism $N \rightarrow M \setminus \Sigma^\mp$ is provided by the mapping $\Phi = \Phi^\pm$ with (14.1), which, in particular, must be biholomorphic,*

- (ii) Φ sends the horizontal distribution of the Chern connection D of $\langle, \rangle$ in N , cf. (d) of Section 7, onto the summand $\mathcal{V} \oplus \mathcal{H}^\pm$ in (11.9),
- (iii) the leaves of $\mathcal{V}$ are precisely the same as the Φ -images of all punctured complex lines through 0 in the normal spaces of Σ .

In the special case where $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$, that is, the summand distribution $\mathcal{H}$ in (11.9) is 0-dimensional, formula (8.5.c) used in the construction of Section 8 may also be replaced by the following equality, using the simplified notation of (8.5.c):

$$(14.4) \quad \hat{g}(w, w') = \frac{|\tau - \tau_\mp|}{\tau_+ - \tau_-} h(w, w').$$

Proof. It suffices to prove that the restriction of Φ to $N' = N \setminus \Sigma$ is an isomorphism between the geodesic-gradient Kähler triples $(N', \hat{g}, \hat{\tau})$ and (M', g, τ) , since the analogous conclusion about Φ itself then follows from [8, Lemma 16.1].

We start by establishing the equality

$$(14.5) \quad \tau \circ \Phi = \hat{\tau}.$$

Namely, $|\rho\xi| = \rho$ for any $\rho \in (0, \infty)$ and any $(y, \xi) \in N$ with $|\xi| = 1$, so that $\Phi(y, \rho\xi) = x_\sigma$, where $x_\sigma = \exp_y \sigma\xi$ and σ depends on ρ as in (6.4). Since $\sigma \mapsto x_\sigma$ is a unit-speed geodesic, (11.2) and (10.2) give $d[\tau(x_\sigma)]/d\sigma = \mp Q^{1/2}$, the sign factor being due to the relation $d(x_\sigma)/d\sigma = \mp v/|v|$ (immediate from (4.2) with $v = \nabla\tau$). Here $Q = g(v, v)$ depends on $\tau(x_\sigma)$ as in Remark 11.1(i). However, according to Remark 6.4 and the text preceding (8.5.a) – (8.5.b), the same autonomous equation $d[\hat{\tau}(y, \rho\xi)]/d\sigma = \mp Q^{1/2}$ holds when $\tau(x_\sigma)$ is replaced by $\hat{\tau}(y, \rho\xi)$, with *the same* dependence of Q on the unknown function. The uniqueness clause of Remark 6.4 thus gives $\tau(\Phi(y, \rho\xi)) = \tau(x_\sigma) = \hat{\tau}(y, \rho\xi)$, as required.

One has two complex direct-sum decompositions, $TM' = \mathcal{V} \oplus \mathcal{H}^\mp \oplus \mathcal{H}^\bullet$ and $TN' = \hat{\mathcal{V}} \oplus \hat{\mathcal{H}}^\mp \oplus \hat{\mathcal{H}}^\bullet$, orthogonal relative to g and, respectively, $\hat{g}$. The former arises from (11.9) if one sets $\mathcal{H}^\bullet = \mathcal{H}^\pm \oplus \mathcal{H}$. In the latter $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ are the distributions introduced in the lines following (8.4). First, for $\hat{u}$ as in (8.6) and our $u = Jv$, where $v = \nabla\tau$, we show that

$$(14.6) \quad \begin{aligned} &\text{i) } \Delta \text{ preserves } \hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet \text{ and } \hat{u}, \\ &\text{ii) } \text{Exp}^\perp \text{ sends } \hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet, \hat{u} \text{ to } \mathcal{V}, \mathcal{H}^\mp, \mathcal{H}^\bullet, u, \\ &\text{iii) both } \Delta \text{ and } \text{Exp}^\perp \text{ act complex-linearly on } \hat{\mathcal{H}}^\mp \text{ and } \hat{\mathcal{H}}^\bullet. \end{aligned}$$

More precisely, Δ (or, $\text{Exp}^\perp$) appearing in (14.2) (or, (11.3)), restricted to N' (or, $N' \cap N^\delta \Sigma^\pm$), sends $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet, \hat{u}$ onto their restrictions to $N' \cap N^\delta \Sigma^\pm$ (or, respectively, onto $\mathcal{V}, \mathcal{H}^\mp, \mathcal{H}^\bullet, u$). The claims about $\hat{\mathcal{V}}$ in (14.6.i) – (14.6.ii) follow as Δ clearly preserves each leaf of $\hat{\mathcal{V}}$, that is, each punctured complex line through 0 in the normal space $N_y \Sigma$ at any point $y \in \Sigma$, while, by Lemma 11.4(a), $\text{Exp}^\perp$ maps the leaves of $\hat{\mathcal{V}}$ intersected with $N' \cap N^\delta \Sigma^\pm$ onto leaves of $\mathcal{V}$. This also proves (ii). Next, the class of vertical vector fields of Remark 14.1(b) obviously includes $\hat{u}$ and, locally, some of them span $\hat{\mathcal{H}}^\mp$. Remark 14.1 thus yields the remainder

of (14.6.i), while (14.6.iii) for Δ follows from complex-linearity of the D-horizontal lift operation (due to Lemma 7.1(i)), and the fact that Δ acts on the vertical vector fields in Remark 14.1(b) as the identity operator. On the other hand, (14.6.ii) in the case of $\hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ (or, of $\hat{u}$) is an immediate consequence of the second (or, third) claim in Theorem 11.6(f). (To be specific, for $\hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ this is clear from Remark 11.8(ii) combined with (11.7) – (11.9).) Finally, the complex-linearity assertion of Theorem 11.6(f) implies (14.6.iii).

By (14.6), the diffeomorphism $\Phi = \text{Exp}^\perp \circ \Delta : N' \rightarrow M'$ maps $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ onto $\mathcal{V}, \mathcal{H}^\mp$ and $\mathcal{H}^\bullet$. Proving the theorem is thus reduced to showing that

$$(14.7) \quad \hat{J} \text{ and } \hat{g}, \text{ on each of the three summands } \hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp \text{ and } \hat{\mathcal{H}}^\bullet, \text{ correspond under the differential } d\Phi \text{ to } J \text{ and } g \text{ on } \mathcal{V}, \mathcal{H}^\mp \text{ and } \mathcal{H}^\bullet, \text{ respectively.}$$

To begin with, for $\hat{Q}$ as in Section 8, $\hat{v}$ given by (8.6), and our $v = \nabla \tau$,

$$(14.8) \quad \Phi \text{ pushes } \hat{Q}, \hat{u} \text{ and } \hat{v} \text{ forward onto } Q, u \text{ and } v.$$

In the case of $\hat{Q}$ this amounts to $Q \circ \Phi = \hat{Q}$, which is a trivial consequence of (14.5) and the fact that $\hat{Q}$ was defined in Section 8 to be the same function of $\hat{\tau}$ as Q is of τ . For $\hat{u}$, (14.8) follows from (14.6) and (14.1). Next, any integral curve of $\hat{v}$ in $N_y \setminus \{0\}$ has, up to a shift of the parameter, the form $r \mapsto (y, e^{\mp ar} \xi)$ with a unit vector $\xi \in N_y$, so that $\Delta(y, e^{\mp ar} \xi) = (y, \sigma \xi)$, where in addition to the curve parameter r , two more real variables are used: $\rho = e^{\mp ar}$, and σ related to ρ via (6.4). The chain rule thus yields $d\sigma/dr = \mp a \rho d\sigma/d\rho = \mp Q^{1/2}$, while $\Phi(y, e^{\mp ar} \xi) = x(\sigma)$ for $x(\sigma) = \exp_y \sigma \xi$. Since $\sigma \mapsto x(\sigma)$ is a unit-speed geodesic, (11.2) and (10.1) give $d[x(\sigma)]/d\sigma = \mp Q^{1/2}$, with Q evaluated at $x(\sigma)$, and the sign factor arising from (4.2), as $v = \nabla \tau$. Applying the chain rule again, we obtain $d[x(\sigma)]/dr = v_{x(\sigma)}$ and, consequently, (14.8).

The claim made in (14.7) about $\hat{\mathcal{V}} = \text{Span}(\hat{v}, \hat{u})$ and $\mathcal{V} = \text{Span}(v, u)$ is now obvious from (14.8) and (9.3) along with (8.7).

For the remaining two pairs of summands, (14.7) in the case of $\hat{J}, J$ (or, $\hat{g}, g$) is a direct consequence of (14.6) and (14.1) (or, respectively, of (i) – (ii) in Remark 11.8 along with parts (h2) – (h3) of Theorem 11.6, (14.5) and (8.5)). Note that, by (14.2), Δ leaves $\xi/|\xi|$ unchanged, while $\rho = |\xi|$ in (h2).

Finally, if $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ in (11.9), Remark 11.8(ii) allows us to use (h1) in Theorem 11.6, instead of (h2), obtaining (14.4). $\square$

Corollary 14.3. *Suppose that (M, g, τ) is a compact geodesic-gradient Kähler triple. Then, for $\mathcal{V}$ and $\mathcal{H}^\pm$ appearing in (11.9), with either sign $\pm$, the distribution $\mathcal{V} \oplus \mathcal{H}^\pm$ is integrable and its leaves are totally geodesic in (M', g) .*

Proof. Use Theorem 14.2 and Theorem 8.1(b) (or – for integrability – (11.7)). $\square$

15. Immersions of complex projective spaces

In the next result the inclusions $N_y \subseteq P(\mathbb{C} \times N_y)$ and $PN_y \subseteq P(\mathbb{C} \times N_y)$ come from the standard identification (5.1) for $V = N_y$, where $y \in \Sigma^\pm$. Let us also note that, by (11.7) and Corollary 14.3, the restriction to the normal space $N_y = N_y \Sigma^\pm \subseteq N\Sigma^\pm$ of the biholomorphism $\Phi : N\Sigma^\pm \rightarrow M \setminus \Sigma^\mp$ (see Theorem 14.2) constitutes

$$(15.1) \quad \text{a totally geodesic holomorphic embedding } \Phi : N_y \rightarrow M \setminus \Sigma^\mp.$$

Theorem 15.1. *Given a compact geodesic-gradient Kähler triple (M, g, τ) and a fixed sign $\pm$, let y be a point of the critical manifold $\Sigma^\pm$. Then the following conclusions hold.*

- (a) *The embedding $\Phi : N_y \rightarrow M \setminus \Sigma^\mp$ with (15.1) has an extension to a totally geodesic holomorphic immersion $\Psi : P(\mathbb{C} \times N_y) \rightarrow M$.*
- (b) *The mapping Ψ in (a) restricted to the projective hyperplane $PN_y \subseteq P(\mathbb{C} \times N_y)$ at infinity is a totally geodesic holomorphic immersion $F : PN_y \rightarrow \Sigma^\mp$, and the metric that it induces on PN_y equals $2(\tau_+ - \tau_-)/a$ times the Fubini-Study metric, cf. Remark 5.4, arising from the inner product g_y in N_y , for $a, \tau_\pm$ as in Remark 11.1(i).*
- (c) *the images of the immersion $F : PN_y \rightarrow \Sigma^\mp$ in (b) and of its differential at any point $\mathbb{C}\xi$, where $(y, \xi) \in N\Sigma^\pm$ and $\xi \neq 0$, coincide with the $\pi^\mp$ -image of the leaf of $\text{Ker } d\pi^\pm$ in M' passing through $x = \Phi(y, \xi)$ and, respectively, with the subspace $d\pi_x^\mp(\mathcal{H}_x^\mp) = d\pi_x^\mp(\mathcal{V}_x \oplus \mathcal{H}_x^\mp)$ of $T_y \Sigma^\mp$.*

Proof. As a consequence of Theorems 14.2 and 11.6(c), the composite $\pi^\mp \circ \Phi$ maps $N\Sigma^\pm \setminus \Sigma^\pm$ (the complement of the zero section in $N\Sigma^\pm$) holomorphically into $\Sigma^\mp$. The restriction of $\pi^\mp \circ \Phi$ to $N_y \setminus \{0\} \subseteq N\Sigma^\pm \setminus \Sigma^\pm$, being, by (11.6) and (14.1), constant on each punctured complex line through 0, thus descends to

$$(15.2) \quad \text{a holomorphic immersion } F : PN_y \rightarrow \Sigma^\mp,$$

where the immersion property of F is an immediate consequence of the fact, established below, that both $\pi^\mp : \Phi(N_y \setminus \{0\}) \rightarrow \Sigma^\mp$ and $\pi^\mp \circ \Phi : N_y \setminus \{0\} \rightarrow \Sigma^\mp$ have constant (complex) rank, equal to $\dim_{\mathbb{C}} N_y - 1$. As Φ is a biholomorphism, it suffices to verify this last claim for the former mapping; we do it noting that $\Pi = \Phi(N_y \setminus \{0\})$ coincides with the $\pi^\pm$ -preimage of y (due to (14.3) and Remark 11.3), and hence forms a leaf of $\text{Ker } d\pi^\pm = \mathcal{V} \oplus \mathcal{H}^\mp$ restricted to M' , cf. (11.7). That $\pi^\mp : \Pi \rightarrow \Sigma^\mp$ satisfies the required rank condition is now clear: the kernel of its differential at any point x coincides, by (11.7) and (11.9), with $\mathcal{V}_x$, while $\mathcal{V} = \text{Span}(v, u)$.

The mapping $\Psi : P(\mathbb{C} \times N_y) \rightarrow M$, equal to Φ on N_y and to F on PN_y , is continuous. Namely, if it were not, we could pick a sequence $\xi_j \in N_y$, $j = 1, 2, \dots$, such that $|\xi_j| \rightarrow \infty$ and $\xi_j/|\xi_j| \rightarrow \xi$ as $j \rightarrow \infty$ for some unit vector $\xi \in N_y$, while no subsequence of the image sequence $\Psi(\xi_j)$ tends to $F(\mathbb{C}\xi)$. The resulting limit relation $\sigma_j \rightarrow \delta$, where σ_j corresponds to $\rho_j = |\xi_j|$ as in the line preceding (6.4), combined with (14.1), now gives $\Psi(\xi_j) = \Phi(\xi_j) = \text{Exp}^\perp(y, \sigma_j \xi_j / \rho_j)$ which – due to continuity of $\text{Exp}^\perp$ and (11.6) – converges to $\text{Exp}^\perp(y, \delta\xi) = y_\mp$, for a specific point

$y_{\mp}$. However, (a) – (b) in Lemma 11.4 and the definition of F also give $y_{\mp} = F(\mathbb{C}\xi)$, which contradicts our choice of ξ_j , proving continuity of Ψ .

Holomorphicity of Ψ is now obvious from Remark 3.5 applied to $\Pi = \mathbb{P}(\mathbb{C} \times N_y)$ and its codimension-one complex submanifold $\Lambda = \mathbb{P}N_y$. Furthermore,

$$(15.3) \quad \Psi \text{ is an immersion.}$$

To see this, first note that Ψ has two restrictions, F to $\mathbb{P}N_y$ and Φ to the dense open submanifold N_y , already known to be immersions, the former into $\Sigma^{\mp}$, cf. (15.1) – (15.2). Next, for any unit vector $\xi \in N_y$, if Λ' denotes the projective line in $\mathbb{P}(\mathbb{C} \times N_y)$ joining $\mathbb{C}(1, 0)$ to the point $\mathbb{C}\xi \in \mathbb{P}N_y$ (identified via (5.1) with $\mathbb{C}(0, \xi)$), then the restriction of Ψ to Λ' is an embedding with the image $\Lambda = \Psi(\Lambda')$ forming a complex submanifold of M , biholomorphic to $\mathbb{CP}^1$, and intersecting each of Σ^+ and Σ^- orthogonally at a single point. In fact, Lemma 11.4 yields all the claims just made except the ‘embedding’ property; we obtain the latter from Remark 3.4(b), which we use to conclude that the resulting holomorphic mapping $\Psi: \Lambda' \rightarrow \Lambda$, being injective (since so is Φ), must be a biholomorphism. Now (15.3) follows.

For obvious reasons of continuity, (15.1) implies that the holomorphic immersion $\Psi: \mathbb{P}(\mathbb{C} \times N_y) \rightarrow M$ is totally geodesic, which establishes (a). Finally, Remarks 8.4, 11.1(iii), 1.8 and Theorem 14.2 give rise to (b), completing the proof. $\square$

Remark 15.2. For $m, d_{\pm}, k_{\pm}, q$ as in Remark 12.1, the codimension $\dim_{\mathbb{C}} \Sigma^{\mp} - \dim_{\mathbb{C}} N_y$ of the immersion F in Theorem 15.1(b) equals q . In fact, $\dim_{\mathbb{C}} N_y = m - d_{\pm} - 1$, and so, by (12.4), $\dim_{\mathbb{C}} \Sigma^{\mp} - \dim_{\mathbb{C}} N_y = (m - d_{\pm} - 1) - d_{\mp} = q$.

Remark 15.3. Suppose that the distribution $\mathcal{H}$ in (11.9) is 0-dimensional or, in other words, $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$. Then, for either sign $\pm$, the critical manifold $\Sigma^{\pm}$, with its submanifold metric, must be biholomorphically isometric to a complex projective space carrying the Fubini-Study metric multiplied by $2(\tau_+ - \tau_-)/a$.

In fact, the isometric immersion F of Theorem 15.1(b), having codimension zero (cf. Remark 15.2), is necessarily a biholomorphism (Remark 3.9).

16. Consequences of condition (0.3)

The results stated and proved below use Definition 4.2, the notations of (9.1), (11.4), (11.7), and the notion of projectability introduced in Section 2.

Lemma 16.1. *For a compact geodesic-gradient Kähler triple (M, g, τ) , the following three conditions are mutually equivalent.*

- (i) *The distribution $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ on M' is integrable.*
- (ii) *$\text{Ker } d\pi^- = \mathcal{V} \oplus \mathcal{H}^+$ is π^+ -projectable.*
- (iii) *$\text{Ker } d\pi^+ = \mathcal{V} \oplus \mathcal{H}^-$ is π^- -projectable.*

In (ii) – (iii) one may also replace $\mathcal{V} \oplus \mathcal{H}^{\pm}$ by $\mathcal{H}^{\pm}$ or $\mathcal{Z}$. If (i) – (iii) hold, then:

- (iv) *The immersions of Theorem 15.1(c) are all embeddings.*

- (v) The $\pi^\pm$ -images $\mathcal{Z}^\pm$ of the integrable distribution $\mathcal{Z}$ on M' are integrable holomorphic distributions on $\Sigma^\pm$ and have totally geodesic leaves biholomorphically isometric to complex projective spaces carrying $2(\tau_+ - \tau_-)/a$ times the Fubini-Study metric, cf. Theorem 15.1(b). These leaves coincide with the images of the embeddings in (iv), and form the fibres of holomorphic bundle projections $\text{pr}^\pm : \Sigma^\pm \rightarrow B^\pm$ for some compact complex base manifolds $B^\pm$.
- (vi) The summand $\mathcal{H}$ in (11.9) is $\pi^\pm$ -projectable and its $\pi^\pm$ -image coincides with the orthogonal complement of $\mathcal{Z}^\pm$ in $T\Sigma^\pm$.
- (vii) The leaf space $B = M'/\mathcal{Z}$ admits a unique structure of a compact complex manifold such that the quotient projection $M' \rightarrow M'/\mathcal{Z}$ constitutes a holomorphic fibration while, for either sign $\pm$ and $\text{pr}^\pm : \Sigma^\pm \rightarrow B^\pm$ as in (iv), the mapping $B \rightarrow B^\pm$, sending each leaf of $\mathcal{Z}$ to its image under $\text{pr}^\pm \circ \pi^\pm$, is a biholomorphism.
- (viii) There exists a unique holomorphic bundle projection $\pi : M \rightarrow B$ with $\text{Ker } d\pi = \mathcal{Z}$ on M' such that, for both signs $\pm$, the restriction of π to M' equals $\beta^\pm \circ \text{pr}^\pm \circ \pi^\pm$, where $\beta^\pm$ is the inverse of the biholomorphism $B \rightarrow B^\pm$ in (vii).
- (ix) $R^D(w, w') = -ia(\tau_+ - \tau_-)^{-1}h(Jw, w') : N \rightarrow N$, with the notation of (1.2), for the submanifold metric h of $\Sigma^\pm$, the normal connection D in its normal bundle $N = N\Sigma^\pm$, any vector field w' on $\Sigma^\pm$, and any section w of $\mathcal{Z}^\pm$, cf. (v).

Proof. Since $\mathcal{V} \oplus \mathcal{H}^\pm$ are both integrable by (11.7), the mutual equivalence of (i), (ii), (iii) and the integrability claim in (v) are all immediate from Lemma 2.7 applied to $\mathcal{E}^\pm = \mathcal{V} \oplus \mathcal{H}^\pm$, along with (11.7) and (11.9). The immersions mentioned in Theorem 15.1(c) thus have nonsingular images, namely, the leaves Π of the distribution $\mathcal{Z}^\pm$ in (v), so that (iv) follows from Remark 3.9 applied to PN_y standing for $\mathbb{CP}^l$, with $l = k_\mp$ defined in Remark 12.1, and such a leaf Π . The remaining part of (v) is a direct consequence of Theorem 15.1(b) and Remark 3.3.

At any $y \in \Sigma^\pm$, the image $d\pi_x^\pm(\mathcal{H}_x^\pm)$ is now independent of the choice of $x \in M'$ with $\pi^\pm(x) = y$, and hence so is its orthogonal complement $d\pi_x^\pm(\mathcal{H}_x)$ in $T_y\Sigma^\pm$ (see Remark 11.8(iii)), proving assertion (vi).

The mappings $B \rightarrow B^\pm$ in (vii) are obviously bijective, and lead to an identification $B^+ = B^-$ which is a biholomorphism, as one sees restricting $\pi^\pm$ to “local” complex submanifolds of M' which the composite bundle projections $M' \rightarrow \Sigma^\pm \rightarrow B^\pm$ (with fibres provided by the leaves of $\mathcal{Z}$) send biholomorphically onto open submanifolds of $B^\pm$. This yields (vii). For (viii), it suffices to note that the two composite bundle projections $\text{pr}^\pm \circ \pi^\pm : M \setminus \Sigma^\mp \rightarrow B$ agree, by (vii), on the intersection M' of their domains, cf. Remark 11.1(iv), while the union of their domains is M .

For (ix), Theorem 14.2 allows us to identify $M \setminus \Sigma^\mp$ with N so that (8.5.c) and (8.8) hold under the assumptions following (8.5). Since w lies in the $\pi^\pm$ -image $\mathcal{Z}^\pm$ of $\mathcal{H}^\pm$, cf. (ii), (iii), (v), formula (11.8) gives $2Sw = Qw/(\tau - \tau_\mp)$ for its D -horizontal lift, also denoted by w . Replacing $2Sw$ in (8.8) with $Qw/(\tau - \tau_\mp)$ and multiplying the result by $(\tau - \tau_\mp)Q^{-1}$, we get an expression for $g(w, w')$ which, equated to (8.5.c), yields $\langle R^D(w, Jw')\xi, i\xi \rangle = -a(\tau_+ - \tau_-)^{-1}\langle \xi, \xi \rangle h(w, w')$, since $\rho^2 = \langle \xi, \xi \rangle$ while, obviously, $|\tau - \tau_\pm| = \mp(\tau - \tau_\pm)$. Applying the last equality to Jw instead of

w , and using (b) in Section 7 along with Hermitian symmetry of $\langle R^D(w, w')\xi, i\eta \rangle = -\langle iR^D(w, w')\xi, \eta \rangle$ in ξ, η , we obtain the required relation in (ix). $\square$

Note that the above proof of (ix) in Lemma 16.1 actually uses the assumptions (i) – (iii): without them, the formula $\langle R^D(w, Jw')\xi, i\xi \rangle = -a(\tau_+ - \tau_-)^{-1}\langle \xi, \xi \rangle h(w, w')$, rather than being valid for any given $w \in \mathcal{Z}_y^\pm$, $y \in \Sigma^\pm$, and *all* vectors ξ normal to $\Sigma^\pm$ at y , would hold only when w lies in some subspace of $T_y\Sigma^\pm$ depending on ξ .

Let us now fix a Kähler manifold $(\hat{\Sigma}, \hat{h})$, and consider pairs $N, \langle, \rangle$ formed by a holomorphic complex vector bundle N over $\hat{\Sigma}$ and the real part $\langle, \rangle$ of a Hermitian fibre metric in N , the Chern connection of which – see Section 7 – satisfies the curvature condition $R^D(w, w') = 2i\hat{h}(Jw, w') : N \rightarrow N$ for any vector fields w, w' tangent to $\hat{\Sigma}$, where the notation of (1.2) is used.

Lemma 16.2. *Whenever $\hat{\Sigma}$ is simply connected and such $N, \langle, \rangle$ exist, they are essentially unique, in the sense that, given another pair $N', \langle, \rangle'$ with the same property, some holomorphic vector-bundle isomorphism $N \rightarrow N'$ takes $\langle, \rangle$ to $\langle, \rangle'$.*

Proof. Remark 1.9 implies that the Chern connections D and D' induce a flat metric connection in the bundle $\text{Hom}(N, N')$. The required isomorphism is now provided by a global parallel section of $\text{Hom}(N, N')$ chosen so as to transform $\langle, \rangle$ into $\langle, \rangle'$ at one point, and its holomorphicity follows from (e) in Section 7. $\square$

Theorem 16.3. *For a compact geodesic-gradient Kähler triple (M, g, τ) , the following two conditions are equivalent.*

- (i) (M, g, τ) is isomorphic to a CP triple, defined as in Section 5.
- (ii) $d_+ + d_- = m - 1$, where $m = \dim_{\mathbb{C}} M$ and $d_\pm = \dim_{\mathbb{C}} \Sigma^\pm$. In other words, cf. Remark 12.1, $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$, that is, $\mathcal{H}$ in (11.9) is 0-dimensional.

In this case, the assertion of Theorem 14.2, including (14.4), is satisfied by $(M \setminus \Sigma^\mp, g, \tau)$, with either fixed sign $\pm$ and (Σ, h) biholomorphically isometric to a complex projective space carrying $2(\tau_+ - \tau_-)/a$ times the Fubini-Study metric, N and $\langle, \rangle$ being, up to a holomorphic vector-bundle isomorphism, the normal bundle of the latter treated as a linear variety in $\mathbb{CP}^m$ and its Hermitian fibre metric induced by the Fubini-Study metric of $\mathbb{CP}^m$.

Furthermore, the isomorphism types of CP triples (M, g, τ) having any given values of $d_\pm$ and m in (ii) are in a natural bijective correspondence, obtained by applying Remark 11.1(i), with quadruples $\tau_-, \tau_+, a, \tau \mapsto Q$ that satisfy (6.1).

Proof. First, (i) implies (ii) according to (12.7).

Assuming now (ii), let us use Remark 13.4 to select a CP triple $(\mathbb{CP}^m, g', \tau')$ realizing the same data $d_\pm, \tau_\pm, a$ and $\tau \mapsto Q$, in (ii) above and Remark 11.1(i), as our (M, g, τ) (which also establishes the surjectivity part of the final clause). With either fixed sign $\pm$, denoting $\Sigma^\mp, \Sigma^\pm$ by Σ, Π , and their analogs for $(\mathbb{CP}^m, g', \tau')$ by Σ', Π' , we choose the isomorphisms $N \rightarrow M \setminus \Pi$ and $N' \rightarrow \mathbb{CP}^m \setminus \Pi'$ by applying Theorem 14.2(i) to both triples. As (i) has already been shown to yield (ii), we may now also apply Remark 15.3 to both of them, identifying the critical manifolds

Σ, Σ' (and their submanifold metrics) with a complex projective space $\hat{\Sigma}$ (and, respectively, with the Fubini-Study metric $\hat{h}$ multiplied by $2(\tau_+ - \tau_-)/a$). Next, (ix) in Lemma 16.1 holds for both triples, so that the pairs $N, \langle, \rangle$ and $N', \langle, \rangle'$ associated with them via Theorem 14.2 satisfy, along with $\hat{\Sigma} = \Sigma = \Sigma'$ and $\hat{h}$, the assumptions – as well as the conclusion – of Lemma 16.2. Thus, some holomorphic vector-bundle isomorphism $N \rightarrow N'$ takes $\langle, \rangle$ to $\langle, \rangle'$ and, since the metrics $\hat{g}, \hat{g}'$ on N and N' constructed in Section 8 depend only on $\langle, \rangle, \langle, \rangle'$ (aside from the data fixed above and shared by both triples), this isomorphism is a holomorphic isometry of $(N, \hat{g})$ onto $(N', \hat{g}')$, sending τ to its analog on N' . In view of [8, Lemma 16.1], it can be extended to an isomorphism between the triples (M, g, τ) and $(\mathbb{CP}^m, g', \tau')$. We thus obtain injectivity in the final clause and the fact that (ii) yields (i). $\square$

17. Horizontal extensions of CP triples

Once again, we use the notation of (9.1), (4.2) and (11.3), assuming (M, g, τ) to be a compact geodesic-gradient Kähler triple (Definition 4.2).

Lemma 17.1. *Suppose that conditions (i) – (iii) along with the other assumptions of Lemma 16.1 hold for a triple (M, g, τ) , and π, B are as in Lemma 16.1(viii).*

- (a) *Given a π -projectable nonzero local section w of the distribution $\mathcal{H}$ in (11.9),*
 - (a1) *w commutes with the vector fields $v = \nabla \tau$ and $u = Jv$,*
 - (a2) *w is $\pi^\pm$ -projectable for both signs $\pm$,*
 - (a3) *the local flow of w in M' preserves the distributions $\mathcal{V}, \mathcal{H}^+$ and $\mathcal{H}^-$.*
- (b) *The leaves of the integrable distribution $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ on M' are totally geodesic complex submanifolds of M' and all the local flows mentioned in (a3) act between them via local isometries.*

Proof. Any w in (a) is normal to the totally geodesic leaves of the integrable distributions $\mathcal{V} \oplus \mathcal{H}^\pm$ (see Corollary 14.3), while v, u are both tangent to them, as $\mathcal{V} = \text{Span}(v, u)$. Therefore, $\nabla_v w$ and $\nabla_u w$, being, as a result, also normal to those leaves for both signs $\pm$, are – by (11.9) – sections of $\mathcal{H}$. The same is true of $\nabla_w v, \nabla_w u$ (and hence of $[v, w], [u, w]$) due to S -invariance in (11.9), with $S = \nabla v$ and $\nabla u = A = JS = SJ$, cf. (9.2.a). At the same time, π -projectability of w implies, via Remark 2.1 and Lemma 16.1(viii), that $[v, w]$ and $[u, w]$ are sections of $\mathcal{Z} = \text{Ker } d\pi = \mathcal{H}^\perp$. We thus obtain (a1) along with (a3) for $\mathcal{V}$. Next, (a2) follows: due to π -projectability of w , with $y \in \Sigma^\pm$ fixed, $d\pi_x w_x$ is independent of the choice of $x \in M'$ such that $\pi^\pm(x) = y$, and hence so must be $d\pi_x^\pm w_x$, as the differential at y of the bundle projection $\beta^\pm \circ \text{pr}^\pm : \Sigma^\pm \rightarrow B$ (see (vii) – (viii) in Lemma 16.1) sends $d\pi_x^\pm w_x$ to $d\pi_y w_y$, which determines $d\pi_x^\pm w_x$ uniquely due to its being orthogonal, by Lemma 16.1(v), to $\mathcal{Z}_y^\pm$, for the vertical distribution $\mathcal{Z}^\pm$ of $\beta^\pm \circ \text{pr}^\pm$.

We obtain the remainder of (a3) by noting that, for either fixed sign $\pm$,

- (17.1) $\begin{aligned} &\text{all vectors in } \mathcal{H}^\pm \text{ are realized by } \pi^\pm\text{-projectable local} \\ &\text{sections } w^\pm \text{ of } \mathcal{H}^\pm \text{ commuting with } w, \text{ the } \pi^\pm\text{-images} \\ &\hat{w}^\pm \text{ of which also commute with the } \pi^\pm\text{-image } \hat{w} \text{ of } w. \end{aligned}$

Namely, since the $\pi^\pm$ -image $\hat{w}$ of w is obviously $\mathcal{Z}^\pm$ -projectable, we may prescribe the $\pi^\pm$ -image $\hat{w}^\pm$ of $w^\pm$ to be a local section of $\mathcal{Z}^\pm$ commuting with $\hat{w}$ (see (v) – (vi) in Lemma 16.1 and Remark 2.2), and then lift $\hat{w}^\pm$ to $\mathcal{H}^\pm$, using Remark 11.8(iii). For the resulting lift $w^\pm$, (1.8) and parts (ix), (vi) of Lemma 16.1 give $[w, w^\pm] = 0$.

We now derive (b) from Remark 2.5. According to Remark 2.6, it suffices to establish (i) in Remark 2.5 for local sections of $\mathcal{Z}$ having the form $w' = w^0 + w^+ + w^-$ with $w^\pm$ satisfying (17.1) and w^0 equal to a constant-coefficient combination of v and u . Orthogonality in (11.9) combined with (9.3) shows that $g(w', w')$ equals a constant multiple of Q plus the sum of the terms $g(w^\pm, w^\pm)$. Verifying part (i) of Remark 2.5 thus amounts to showing that $d_w Q = d_w[g(w^\pm, w^\pm)] = 0$. In terms of the $\pi^\pm$ -image $\hat{w}^\pm$ of $w^\pm$, Remark 11.8(iv) gives $(\tau_+ - \tau_-)g(w^\pm, w^\pm) = (\tau - \tau_\mp)h(\hat{w}^\pm, \hat{w}^\pm)$, where h is the submanifold metric of $\Sigma^\pm$. Since $\tau_\pm$ are constants, our claim is thus reduced to two separate parts, $d_w Q = d_w \tau = 0$ and $d_w[h(\hat{w}^\pm, \hat{w}^\pm)] = 0$. The former part is immediate: Q is a function of τ , cf. Remark 11.1(i), while w and $v = \nabla \tau$ are sections of the mutually orthogonal summands $\mathcal{H}$ and $\mathcal{V} = \text{Span}(v, u)$ in (11.9). For the latter part, (a2) allows us to replace w by its $\pi^\pm$ -image $\hat{w}$, noting that $h(\hat{w}^\pm, \hat{w}^\pm)$ is the $\pi^\pm$ -pullback of a function defined (locally) in $\Sigma^\pm$. Now $d_w[h(\hat{w}^\pm, \hat{w}^\pm)] = 0$ due to the fact that (ii) implies (i) in Remark 2.5, and $\hat{w}$, or $\hat{w}^\pm$, is normal or, respectively, tangent to the totally geodesic leaves of the integrable distribution $\mathcal{Z}^\pm$, while $\hat{w}$, besides being – as noted above – projectable along $\mathcal{Z}^\pm$, also commutes with $\hat{w}^\pm$ (see (v) – (vi) in Lemma 16.1 and (17.1)). $\square$

We say that a (locally-trivial) holomorphic fibre bundle carries a specific local-type *fibre geometry* if such a geometric structure is selected in each of its fibres and suitable local C^∞ trivializations make the structures appear the same in all nearby fibres. For instance, holomorphic complex vectors bundle endowed with Hermitian fibre metrics may be referred to as

- (i) holomorphic bundles of Hermitian vector spaces.

The fact that (i) leads to the presence of the distinguished Chern connection (Section 7) has obvious generalizations to two situations (ii) – (iii) discussed below.

By a *horizontal distribution* for a holomorphic bundle projection $\pi : M \rightarrow B$ between complex manifolds, also called a *connection in the holomorphic bundle* M over B , we mean any C^∞ real vector subbundle $\mathcal{H}$ of TM , complementary to the vertical distribution $\text{Ker } d\pi$, so that TM is the direct sum of $\text{Ker } d\pi$ and $\mathcal{H}$. *Horizontal lifts* of vectors tangent to B , and of piecewise C^1 curves in B , as well as *parallel transports* along such curves, are then defined in the usual fashion, although the maximal domain of a lift of a curve (or, of a parallel transport) may in general be a proper subinterval of the original domain interval. This last possibility does not, however, occur in bundles with compact fibres, or in vector bundles with linear connections, where horizontal lifts of curves and parallel transports are all *global*.

We proceed to describe the *Chern connection* $\mathcal{H}$ in the cases of

- (ii) holomorphic bundles of Fubini-Study complex projective spaces, and
- (iii) holomorphic bundles of CP triples, over any complex manifold B .

Their fibre geometries consist of Fubini-Study metrics (Remark 5.4) and, respectively, the structures of a CP triple (Section 5).

For (ii), $\mathcal{H}$ arises since local C^∞ trivializations mentioned earlier may be chosen so as to share their domains with local holomorphic trivializations; the former make the fibre geometry appear constant, and the latter turn the bundle, locally, into the projectivization (5.2) of a holomorphic vector bundle E endowed with a Hermitian fibre metric $(\cdot, \cdot)$ that induces the Fubini-Study metrics of the original fibres. Since $(\cdot, \cdot)$ is unique up to multiplications by positive functions (Remark 5.4), Lemma 7.1(iv) easily implies that its choice does not affect the resulting parallel transports between the projectivized fibres, thus giving rise to $\mathcal{H}$.

The Chern connection $\mathcal{H}$ now also arises in case (iii) since, according to Remark 13.5, (iii) is a subcase of (ii). The situation is, however, more special: the critical manifolds – analogs of (4.2) – in the fibres now constitute two holomorphic bundles $\Sigma^\pm$ of Fubini-Study complex projective spaces over B (with fibre dimensions that need not be both positive; see Remark 15.3), contained as subbundles in the original bundle, and invariant under all $\mathcal{H}$ -parallel transports. Also, the fibre-geometry gradients and their J -images (analogous to what we normally denote by $v = \nabla \tau$ and u) together form two holomorphic vertical vector fields v and $u = Jv$ on the total space. This is immediate from the preceding paragraph, with the two subbundles $\Sigma^\pm$ corresponding to a $(\cdot, \cdot)$ -orthogonal holomorphic decomposition $E = E^+ \oplus E^-$ of the locally-defined vector bundle E , cf. (5.3.ii) and (5.5.c) – (5.5.d), while the flow of u , described in the lines following (5.3), acts in both $E^\pm$ via multiplications by two (unrelated) constant unit complex scalars. In case (ii), or (iii),

$$(17.2) \quad \begin{array}{l} \text{the } \mathcal{H}\text{-parallel transports are holomorphic isometries or,} \\ \text{respectively, CP-triple isomorphisms between the fibres,} \end{array}$$

which holds for (ii) since it does for (i), cf. Section 7 and, consequently, also extends to the case of (iii) via the canonical modifications in Remarks 13.3 and 13.5.

The following assumptions and notations will now be used to construct compact geodesic-gradient Kähler triples, each of which we call a *horizontal extension* of the CP triple provided by any fibre $(\pi^{-1}(z), g^z, \tau^z)$.

- (a) $\pi : M \rightarrow B$ and $\mathcal{H}$ are the bundle projection and the Chern connection of a holomorphic bundle of CP triples with a compact base B and the CP-triple fibres $(\pi^{-1}(z), g^z, \tau^z)$, $z \in B$, while $\Sigma^\pm$ stand for the above subbundles of Fubini-Study complex projective spaces, invariant under $\mathcal{H}$ -parallel transports.
- (b) We let $\tau_\pm, a$ be the data associated with some/any fibre $(\pi^{-1}(z), g^z, \tau^z)$ as in Remark 11.1(i), and $\tau : M \rightarrow \mathbb{R}$ (or, $\pi^\pm : M \setminus \Sigma^\mp \rightarrow \Sigma^\pm$) be the C^∞ function (or, holomorphic bundle projection) which, restricted to each $\pi^{-1}(z)$, equals τ^z or, respectively, the version of (11.4) corresponding to $(\pi^{-1}(z), g^z, \tau^z)$. We also set $M' = M \setminus (\Sigma^+ \cup \Sigma^-)$.
- (c) One is given two Kähler metrics $h^\pm$ on the total spaces $\Sigma^\pm$ of our holomorphic bundles of Fubini-Study complex projective spaces such that either $h^\pm$ makes

- the fibres $\Sigma_z^\pm$, $z \in B$, orthogonal to $\mathcal{H}$ along $\Sigma^\pm$ and, restricted to each fibre, $h^\pm$ equals $2(\tau_+ - \tau_-)/a$ times the Fubini-Study metric of $\Sigma_z^\pm$.
- (d) We define a Riemannian metric g on M' by requiring that $\mathcal{H}$ be g -orthogonal to the vertical distribution $\text{Ker } d\pi$, that g agree on the fibres $\pi^{-1}(z)$ with the metrics g^z , and that $(\tau_+ - \tau_-)g = (\tau - \tau_-)h^+ + (\tau_+ - \tau)h^-$ on $\mathcal{H}$, the symbols $h^\pm$ being also used for the $\pi^\pm$ -pullbacks of $h^\pm$, cf. (b) – (c).
- (e) Our final assumption is that the Riemannian metric g on the dense open submanifold M' has an extension to a Kähler metric on M (still denoted by g).

Remark 17.2. Under the hypotheses (a) – (e), the resulting horizontal extension (M, g, τ) is actually a geodesic-gradient Kähler triple. Namely, being a part of the geometry of the fibres $(\pi^{-1}(z), g^z, \tau^z)$, the functions τ^z are preserved by $\mathcal{H}$ -parallel parallel transports, that is, τ is constant along $\mathcal{H}$, and so its (vertical) g -gradient must, by Remark 1.2. coincide with the holomorphic vertical vector field v described in the lines preceding (17.2). On the other hand, the function $Q = g(v, v)$, equal - consequently - to its fibre version, is a specific function of τ . Thus, by Lemma 4.1, τ has a holomorphic geodesic g -gradient.

Remark 17.3. Whenever a compact geodesic-gradient Kähler triple (M, g, τ) is a horizontal extension arising as in Remark 17.2, the distribution $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ on M' coming from the decomposition (11.9) for (M, g, τ) coincides, on M' , with the vertical distribution $\text{Ker } d\pi$ of the bundle projection $\pi : M \rightarrow B$ (see (a) above) and, consequently, $\mathcal{Z}$ is integrable.

In fact, applying Remark 2.5 to $\mathcal{H}$ -horizontal lifts w of local vector fields on B , we see that, by (17.2), $\text{Ker } d\pi$ has totally geodesic leaves. Using (11.5) for both (M, g, τ) and the fibres $(\pi^{-1}(z), g^z, \tau^z)$, we now conclude that the projections $\pi^\pm : M \setminus \Sigma^\mp \rightarrow \Sigma^\pm$ defined in (b) are the same as those in (11.4). (Note that, due to the orthogonality requirement in (c), the minimizing geodesic segment in $\pi^{-1}(z)$, $z \in B$, joining a point $x \in \pi^{-1}(z)$ to $\Sigma_z^\pm$ a normal to $\Sigma_z^\pm$, serves as the segment with the same properties for M rather than $\pi^{-1}(z)$.) Now (11.7) implies that the distribution $\mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ is contained in $\text{Ker } d\pi$ and, restricted to every fibre, equals the analog of $\mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ for the fibre, that is, its tangent bundle (see Theorem 16.3(ii)). Thus, $\mathcal{Z}$ coincides with the full vertical distribution $\text{Ker } d\pi$.

Theorem 17.4. *A geodesic-gradient Kähler triple (M, g, τ) , with compact M , satisfies one/all of the mutually-equivalent conditions (i) – (iii) of Lemma 16.1, if and only if it is isomorphic to a horizontal extension of a CP triple, defined as above using (a) – (e).*

Proof. Remark 17.3 clearly yields the ‘if’ part of our claim.

Conversely, let (M, g, τ) satisfy (i) – (iii) in Lemma 16.1. Lemma 16.1(viii) states that $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ coincides, on M' , with the vertical distribution $\text{Ker } d\pi$ of the holomorphic bundle projection $\pi : M \rightarrow B$. Also, in view of Remark 4.3, the leaves of $\mathcal{Z}$ form geodesic-gradient Kähler triples, due to their being complex submanifolds of M tangent to $v = \nabla \tau$ (since $\mathcal{V} = \text{Span}(v, u)$) and, as they are also totally geodesic (see Lemma 17.1(b)), (11.8) and the S -invariance in (11.9), with

$S = \nabla v$, imply via Theorem 16.3 that they are all isomorphic to CP triples. The local isometries of Lemma 17.1(b) can obviously be made global due to compactness (see the lines preceding (ii) above) which, consequently, turns M into a holomorphic bundle of CP triples over B , in the sense of (iii).

On the other hand, the g -orthogonal complement of $\mathcal{Z} = \text{Ker } d\pi$ is equal, on M' , to the summand $\mathcal{H}$ in (11.9). Thus, $\mathcal{H}$ constitutes a connection in the bundle M over B , as defined in the lines following (i), and – being the intersection of the horizontal distribution of the Chern connections $\mathcal{V} \oplus \mathcal{H}^\pm$ in the normal bundles $N = N\Sigma^\pm$, cf. Theorem 14.2(ii) – $\mathcal{H}$ itself is, according to (a) in Section 7, the Chern connection of the holomorphic bundle M of CP triples over B .

This provides parts (a) – (b) of the data (a) – (e) required above, with $\Sigma^\pm$ and $\tau_\pm, a$ given by (4.2) and, respectively, Remark 11.1(i). The submanifold metrics $h^\pm$ of $\Sigma^\pm$ have, by (v) – (vi) in Lemma 16.1 and the final clause of Theorem 11.6(b), all the properties needed for (c).

To show that g satisfies (d), consider two π -projectable nonzero local sections w, w' of the distribution $\mathcal{H} = \mathcal{Z}^\perp$, cf. (11.9). According to Lemma 17.1(a) and the last line of Remark 2.4, w and w' are projectable along $\mathcal{V}$ and $\mathcal{H}^\pm$, as well as $\pi^\pm$ -projectable, for either sign $\pm$. Their restrictions to any fixed normal geodesic segment Γ emanating from $\Sigma^\pm$ thus lie in the space $\mathcal{W}$ (cf. (11.2) and (i) – (ii) in Theorem 10.1) and, by Theorem 11.6(g), $g(w, w')$ restricted to Γ is a (possibly nonhomogeneous) linear function of τ . The same linearity condition obviously holds for $g(w, w')$ when g is defined as in (d), rather than being the metric of our triple (M, g, τ) . The two definitions of $g(w, w')$ must now agree, as the two linear functions have – in view of Remark 11.8(iii) and the final clause of Theorem 11.6(b) – the same values $h^\pm(w, w')$ at either endpoint $\tau_\pm$ of the interval $[\tau_-, \tau_+]$. $\square$

Remark 17.5. All compact SKRP triples of Class 1 (cf. Section 8) must be

(17.3) isomorphic to horizontal extensions of CP triples of complex dimension 1,

while those of Class 2 are themselves CP triples of a special type. The former claim is easily verified using [8, Theorem 16.3]; for the latter, see Lemma 8.2.

The classification result of [6, Theorem 6.1] may be rephrased as the conclusion (17.3) about all compact geodesic-gradient Kähler triples (M, g, τ) with $\dim_{\mathbb{C}} M = 2$ other than Class 2 SKRP triples are. Similarly, (17.3) is the case – by their very construction – for the gradient Kähler-Ricci solitons of Koiso [16] and Cao [4], mentioned in the Introduction.

18. Constant-rank multiplications

In this section all vector spaces are finite-dimensional and complex. Bilinear mappings of the type discussed here arise in any compact geodesic-gradient Kähler triple (see Theorem 18.4), which leads to the dichotomy conclusion of Theorem 19.1.

A *constant-rank multiplication* is any bilinear mapping $\mu : \mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$, where $\mathcal{N}, \mathcal{T}, \mathcal{Y}$ are vector spaces, such that the function $\mathcal{N} \setminus \{0\} \ni \xi \mapsto \text{rank } \mu(\xi, \cdot)$ is

constant or, equivalently, $\dim \operatorname{Ker} \mu(\xi, \cdot)$ is the same for all nonzero $\xi \in \mathcal{N}$. When $\dim \operatorname{Ker} \mu(\xi, \cdot) = k$ for all $\xi \in \mathcal{N} \setminus \{0\}$, we also say that $\mu : \mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$ has the constant rank $\dim \mathcal{T} - k$. With the notations of Section 5, such μ leads to a mapping

$$(18.1) \quad \varepsilon : \mathcal{PN} \rightarrow \operatorname{Gr}_k \mathcal{T} \quad \text{given by} \quad \varepsilon(\mathbb{C}\xi) = \operatorname{Ker} \mu(\xi, \cdot) \quad \text{for } \xi \in \mathcal{N} \setminus \{0\}.$$

Lemma 18.1. *For μ and ε as above, $\mathcal{N} \setminus \{0\} \ni \xi \mapsto \operatorname{Ker} \mu(\xi, \cdot) \in \operatorname{Gr}_k \mathcal{T}$ and ε are both holomorphic. In terms of the identification (5.6), the differential of the former mapping at any $\xi \in \mathcal{N} \setminus \{0\}$ sends $\eta \in \mathcal{N}$ to the unique $H \in \operatorname{Hom}(\mathcal{W}, \mathcal{T}/\mathcal{W})$ with $\mu(\eta, w) = \mu(\xi, -\tilde{H}w)$ for all $w \in \mathcal{W} = \varepsilon(\mathbb{C}\xi)$, where $\tilde{H} : \mathcal{W} \rightarrow \mathcal{T}$ is any linear lift of H .*

Proof. This is obvious if one sets $F(\xi) = \mu(\xi, \cdot)$ in Remark 5.7. $\square$

Example 18.2. Any given constant-rank multiplication $\mu : \mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$ leads to further such multiplications, $\mu' : \mathcal{N} \times \mathcal{T}' \rightarrow \mathcal{Y}'$ and $\mu^* : \mathcal{N} \times \mathcal{Y}^* \rightarrow \mathcal{T}^*$, obtained by setting $\mu'(\xi, \cdot) = \gamma[\mu(\xi, \alpha \cdot)]$ and $\mu^*(\xi, \cdot) = [\mu(\xi, \cdot)]^*$. Here $\mathcal{T}', \mathcal{Y}'$ are vector spaces, $\alpha : \mathcal{T}' \rightarrow \mathcal{T}$ (or, $\gamma : \mathcal{Y} \rightarrow \mathcal{Y}'$) is surjective (or, injective) and linear, while $[\]^*$ stands for the dual of a vector space or a linear operator.

Lemma 18.3. *If $\mu : \mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$ has the constant rank $\dim \mathcal{T} - k$ and ε with (18.1) is nonconstant, then ε is a holomorphic embedding.*

Whether ε is constant, or not, the same is the case for all multiplications $\mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$ of the constant rank $\dim \mathcal{T} - k$, sufficiently close to μ .

Proof. Let $\mathcal{W} \in \operatorname{Gr}_k \mathcal{T}$. The subset of $\mathcal{N}$ consisting of 0 and all $\xi \in \mathcal{N} \setminus \{0\}$ with $\varepsilon(\mathbb{C}\xi) = \mathcal{W}$ is a vector subspace. In fact, if $\xi, \eta \in \mathcal{N} \setminus \{0\}$ and $\mathcal{W} = \operatorname{Ker} \mu(\xi, \cdot) = \operatorname{Ker} \mu(\eta, \cdot)$, then $\mathcal{W} \subseteq \operatorname{Ker} \mu(\zeta, \cdot)$ for any $\zeta \in \operatorname{Span}(\xi, \eta)$ and, unless $\zeta = 0$, this inclusion is actually an equality due to the constant-rank property of μ .

Therefore, ε -preimages of points of $\operatorname{Gr}_k \mathcal{T}$ are linear subvarieties in $\mathcal{PN}$. If ε is nonconstant, all these subvarieties are zero-dimensional, that is, ε has to be injective. Namely, by Lemma 3.2, for the Kähler form ω of any Kähler metric on $\operatorname{Gr}_k \mathcal{T}$, the integral of $\varepsilon^* \omega$ over any projective line $\mathcal{L}$ in $\mathcal{PN}$ is nonzero, and so $\mathcal{L}$ cannot lie in the ε -preimage of a point. Also, Lemma 18.1 guarantees holomorphicity of ε .

Let ε now be nonconstant. Then ε must be an embedding, that is, $d\varepsilon_{\mathbb{C}\xi}$ is injective at any $\mathbb{C}\xi \in \mathcal{PN}$ or, equivalently, the differential of $\xi \mapsto \operatorname{Ker} \mu(\xi, \cdot)$ at any $\xi \in \mathcal{N} \setminus \{0\}$ has the kernel $\mathbb{C}\xi$. Namely, in Lemma 18.1 we may set $\tilde{H} = 0$ when $H = 0$, and so η lies in the kernel if and only if the inclusion $\mathcal{W} \subseteq \operatorname{Ker} \mu(\eta, \cdot)$ holds for $\mathcal{W} = \varepsilon(\mathbb{C}\xi)$. Unless $\eta = 0$, this inclusion is, as before, an equality, and injectivity of ε then yields $\eta \in \mathbb{C}\xi$, which completes the proof, the final clause being an immediate consequence of that in Lemma 3.2. $\square$

Given a compact geodesic-gradient Kähler triple (M, g, τ) , we use the notation of (9.1) and (4.2) to set, for $\xi, \eta \in N_y \Sigma^\pm$ and $w \in T_y \Sigma^\pm$, with either fixed sign $\pm$,

$$(18.2) \quad Z_y^\pm(\xi, \eta)w = ag_y(\xi, \eta)w + (\tau_+ - \tau_-)R_y(\xi, J_y \eta)J_y w.$$

Thus, $Z_y^\pm(\xi, \eta)w \in T_y \Sigma^\pm$, as ξ, η are tangent, and w normal, to the totally geodesic leaf through y of the J -invariant integrable distribution $\operatorname{Ker} d\pi^\pm = \mathcal{V} \oplus \mathcal{H}^\mp$, cf.

(11.7), Theorem 11.6(c), Corollary 14.3, and the first line of Remark 11.3. Also, denoting by $Z_y^\pm(\xi, \eta)$ the endomorphism $w \mapsto Z_y^\pm(\xi, \eta)w$ of $T_y\Sigma^\pm$, one has

$$(18.3) \quad Z_y^\pm(\xi, \eta) = Z_y^\pm(\eta, \xi) = Z_y^\pm(J_y\xi, J_y\eta)w, \quad J_y[Z_y^\pm(\xi, \eta)] = [Z_y^\pm(\xi, \eta)]J_y,$$

as an obvious consequence of (3.3) and (3.4). Next, we define a complex-bilinear mapping $\mu_y^\pm : N_y\Sigma^\pm \times T_y\Sigma^\pm \rightarrow \overline{\text{Hom}}_{\mathbb{C}}(N_y\Sigma^\pm, T_y\Sigma^\pm)$ by

$$(18.4) \quad \mu_y^\pm(\xi, w) = Z_y^\pm(J_y\xi, \cdot)w + Z_y^\pm(\xi, \cdot)J_yw.$$

By $\overline{\text{Hom}}_{\mathbb{C}}$ we mean here ‘the space of antilinear operators’ and $\overline{\text{Hom}}_{\mathbb{C}}(N_y\Sigma^\pm, T_y\Sigma^\pm)$ is treated as a complex vector space in which the multiplication by i acts via composition with J_y from the left. (The product thus equals the given operator $N_y\Sigma^\pm \rightarrow T_y\Sigma^\pm$ followed by J_y .) Antilinearity of $\mu_y^\pm(\xi, w)$ and complex-bilinearity of $\mu_y^\pm$ are both obvious from (18.3).

Theorem 18.4. *For a compact geodesic-gradient Kähler triple (M, g, τ) , a fixed sign $\pm$, and any point $y \in \Sigma^\pm$, the mapping $\mu_y^\pm$ with (18.4) is a constant-rank multiplication, cf. Section 18. Furthermore, if $\varepsilon = \varepsilon_y^\pm$ corresponds to $\mu = \mu_y^\pm$ as in (18.1) and ξ is any nonzero vector normal to $\Sigma^\pm$ at y , then*

$$(18.5) \quad \begin{aligned} \text{i)} \quad & \varepsilon_y^\pm(\mathbb{C}\xi) = d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm), \quad \text{where } x = \Phi(y, \xi), \\ \text{ii)} \quad & \varepsilon_y^\pm(\mathbb{C}\xi) = \text{Ker } Z_y^\pm(\xi, \xi), \quad \text{for } Z_y^\pm(\xi, \xi) \text{ as in (18.3).} \end{aligned}$$

Proof. Whenever $x = \Phi(y, \xi)$ and $\xi \in N_y\Sigma^\pm \setminus \{0\}$, we have

$$(18.6) \quad \text{Ker } Z_y^\pm(\xi, \xi) = d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm).$$

In fact, let $x = x(t) \in \Gamma$ as in Theorem 11.6, with some fixed $t \in (t_-, t_+)$. According to (11.8) and parts (iii), (iv), (vi) of Theorem 10.1, the vectors forming $\mathcal{H}_x^\pm$ are precisely the values $w(t)$ for all w as in Theorem 11.6(e) which also have the property that $2(\tau - \tau_\mp)Q^{-1}g(Sw, w') = g(w, w')$ whenever w' satisfies the hypotheses of Theorem 11.6(e). Since the values $w'_\pm$ in Theorem 11.6(h2) fill $T_y\Sigma^\pm$ (cf. assertions (d) – (f) of Theorem 11.6), replacing $g(w, w')$ and $g(Sw, w')$ in the last equality with the expressions provided by Theorem 11.6(h2) and Remark 11.7, we easily verify, using (3.3) and Remark 11.8(i), that $w(t) \in \mathcal{H}_x^\pm$ if and only if $Z_y^\pm(\xi, \eta)w_\pm = 0$. Now the final clause of Theorem 11.6(b) (or, Remark 11.9) yields the first (or, second) equality in (18.6).

To simplify notations, let us write g, Z, J rather than $g_y, Z_y^\pm, J_y$. Since $x = \Phi(y, \xi)$ in (18.6) and Φ is holomorphic (Theorem 14.2), (18.6) and Remark 11.9 clearly imply that, for a suitable integer $k = k_\pm$, the resulting mapping

$$(18.7) \quad N_y\Sigma^\pm \setminus \{0\} \ni \xi \mapsto \text{Ker } Z(\xi, \xi) \in \text{Gr}_k(T_y\Sigma^\pm) \quad \text{is holomorphic.}$$

The C^∞ version of the assumptions listed in Remark 5.7 is thus satisfied if one chooses $U, \mathcal{T}, \mathcal{Y}$ to be $N_y\Sigma^\pm \setminus \{0\}, T_y\Sigma^\pm, T_y\Sigma^\pm$ and sets $F(\xi) = Z(\xi, \xi)$. By (5.8), the differential of (18.7) at any nonzero $\xi \in N_y\Sigma^\pm$ sends any $\eta \in N_y\Sigma^\pm$ to the unique

$H : W \rightarrow \mathcal{T}/W$, where $W = \text{Ker } Z(\xi, \xi)$, with a linear lift $\tilde{H} : W \rightarrow \mathcal{T} = T_y \Sigma^\pm$ such that $Z(\xi, \xi) \circ \tilde{H}$ equals the restriction of $-2Z(\xi, \eta)$ to W . (We have $dF_\xi = 2Z(\xi, \cdot)$ since $Z(\xi, \eta)$ is real-bilinear and symmetric in ξ, η , cf. (18.3).) Consequently,

$$(18.8) \quad 2Z(\xi, \eta)w = -Z(\xi, \xi)\tilde{H}w \quad \text{for all } w \in \text{Ker } Z(\xi, \xi).$$

Complex-linearity of the differential, due to (18.7), means that (18.8) will still hold if we replace η with $J\eta$ and $\tilde{H}$ with $J\tilde{H}$. Then, from (18.3) and (18.8), $2Z(J\xi, \eta)w = -2Z(\xi, J\eta)w = Z(\xi, \xi)J\tilde{H}w = J[Z(\xi, \xi)\tilde{H}w] = -2J[Z(\xi, \eta)w] = -2Z(\xi, \eta)Jw$. In other words, $Z(J\xi, \eta)w + Z(\xi, \eta)Jw = 0$ whenever $w \in \text{Ker } Z(\xi, \xi)$ and $\eta \in N_y \Sigma^\pm$. Thus, by (18.4), $\text{Ker } Z(\xi, \xi) \subseteq \varepsilon_y^\pm(\mathbb{C}\xi) = \text{Ker } \mu_y^\pm(\xi, \cdot)$, while the opposite inclusion is obvious since (18.3) gives $Z(\xi, J\xi) = 0$, and so the expression $Z(J\xi, \eta)w + Z(\xi, \eta)Jw = 0$ for $\eta = J\xi$ equals $Z(\xi, \xi)w$.

The equality $\text{Ker } Z(\xi, \xi) = \varepsilon_y^\pm(\mathbb{C}\xi)$ and (18.6) – (18.7) complete the proof. $\square$

The description of $\dot{x}_\pm$ in the lines preceding (18.7) also gives

$$(18.9) \quad g_y(Z_y^\pm(\xi, \xi)w, w) \geq 0 \quad \text{for all } \xi \in N_y \Sigma^\pm \text{ and } w \in T_y \Sigma^\pm,$$

which one sees taking the limit of the equality in Theorem 11.6(h2) with $w' = w$ as $t \in (t_-, t_+)$ approaches the other endpoint $t_\mp$ (and so $\tau \rightarrow \tau_\mp$).

19. The dichotomy theorem

This section uses the notations listed at the beginning of Section 11 and the symbols $k_\pm$ of Remark 12.1. Any $y \in \Sigma^\pm$ leads to the assignment

$$(19.1) \quad N_y \Sigma^\pm \setminus \{0\} \ni \xi \mapsto d\pi_x^\pm(\mathcal{H}_x^\pm) \in \text{Gr}_k(T_y \Sigma^\pm), \text{ where } x = \Phi(y, \xi) \text{ and } k = k_\pm,$$

$\Phi = \Phi^\pm$ being defined by (14.1). (Due to (11.7) and (11.9), $d\pi_x^\pm$ is injective on $\mathcal{H}_x^\pm$.)

Theorem 19.1. *Given any compact geodesic-gradient Kähler triple (M, g, τ) , one and only one of the following two cases occurs.*

- (a) *Either the mappings (19.1) are all constant, for both signs $\pm$, or*
- (b) *each of (19.1), for both signs $\pm$, descends to a nonconstant holomorphic embedding $\text{PN}_y \rightarrow \text{Gr}_k(T_y \Sigma^\pm)$, where PN_y is the projective space of $N_y = N_y \Sigma^\pm$.*

Condition (a) holds if and only if (M, g, τ) satisfies (i) – (iii) in Lemma 16.1.

Proof. In view of Theorem 18.4, we may use Lemma 18.3 for $\varepsilon = \varepsilon_y^\pm$ corresponding to $\mu = \mu_y^\pm$ as in (18.1), concluding (from an obvious continuity argument) that, with either fixed sign $\pm$, all the mappings (19.1) descend to holomorphic embeddings of PN_y unless they are all constant. Their constancy for one sign implies, however, the same for the other, since it amounts to (ii) or (iii) in Lemma 16.1, while (ii) and (iii) are equivalent. This completes the proof. $\square$

Remark 19.2. Case (a) of Theorem 19.1 is equivalent to (0.3), as one sees combining Lemma 16.1(i) with (11.7). According to (iv) – (vi) in Lemma 16.1, the immersions

of Theorem 15.1(c) are then embeddings and their images form the leaves of foliations on $\Sigma^\mp$, both of which have the same leaf space B .

Remark 19.3. When (b) holds in Theorem 19.1, images of the totally geodesic holomorphic immersions of Theorem 15.1(c) pass through every point $y \in \Sigma^\pm$, realizing an uncountable family of tangent spaces: the image of the embedding (19.1).

20. More on Grassmannian triples

We continue using the assumptions and notation of Section 17.

Lemma 20.1. *The leaf space $M'/\mathcal{V}$ of the integrable distribution $\mathcal{V} = \text{Span}(v, u)$ on $M' = M \setminus (\Sigma^+ \cup \Sigma^-)$, cf. Lemma 9.1(a), carries a natural structure of a compact complex manifold of complex dimension $m - 1$, with $m = \dim_{\mathbb{C}} M$, such that the quotient-space projection $M' \rightarrow M'/\mathcal{V}$ forms a holomorphic fibration and, for either sign $\pm$, the projectivization PN of the normal bundle $N = N\Sigma^\pm$, defined as in (5.2), is biholomorphic to $M'/\mathcal{V}$ via the biholomorphisms sending each complex line $\mathcal{L}$ through 0 in the normal space of $\Sigma^\pm$ at any point to the $\text{Exp}^\perp$ -image of the punctured radius δ disk in $\mathcal{L}$, the latter image being a leaf of $\mathcal{V}$ according to Lemma 11.4(a).*

The mappings (11.4), restricted to M' , descend to holomorphic bundle projections

$$(20.1) \quad \pi^\pm : M'/\mathcal{V} \rightarrow \Sigma^\pm,$$

also denoted by $\pi^\pm$, which, under the biholomorphic identifications $M'/\mathcal{V} = \text{P}(N\Sigma^\pm)$ of the preceding paragraph, coincide with the bundle projections $\text{P}(N\Sigma^\pm) \rightarrow \Sigma^\pm$.

Proof. The restrictions $\Phi^\pm = \Phi : N\Sigma^\pm \setminus \Sigma^\pm \rightarrow M'$ given by (14.1) with the two possible signs $\pm$ are biholomorphisms (Theorem 14.2), and hence so is the composite of one of them followed by the inverse of the other. At the same time, by Theorem 14.2(iii), either of them descends to a bijection $\text{P}(N\Sigma^\pm) \rightarrow M'/\mathcal{V}$, and the composite just mentioned yields a biholomorphism between $\text{P}(N\Sigma^\pm)$ and $\text{P}(N\Sigma^\mp)$. This turns $M'/\mathcal{V}$ into a compact complex manifold in a manner independent of the bijection used. Our assertion is now immediate from (14.3). $\square$

Remark 20.2. The direct sum of the two vertical distributions $\text{Ker } d\pi^\pm$ of the projections (20.1) is a distribution on $M'/\mathcal{V}$, since, at every point $A \in M'/\mathcal{V}$, they intersect trivially: $\text{Ker } d\pi_A^+ \cap \text{Ker } d\pi_A^- = \{0\}$. In fact, as a consequence of (11.9), the original vertical distributions on M' , given by (11.7), intersect along $\mathcal{V}$.

For a Grassmannian triple (M, g, τ) obtained as in Section 5 from some data (5.3.i), the descriptions of $\Sigma^\pm$ provided by (5.5.a), and

$$(20.2) \quad \begin{aligned} M'/\mathcal{V} &= \{(W, W') \in \text{Gr}_k \mathcal{V} \times \text{Gr}_{k-1} \mathcal{V} : W' \subseteq W\}, \text{ under which} \\ \pi^\pm \text{ in (20.1) correspond to } (W, W') &\mapsto W \text{ and } (W, W') \mapsto W'. \end{aligned}$$

the equality meaning a natural biholomorphic identification. If (M, g, τ) is in turn a CP triple, arising from (5.3.ii), $\Sigma^\pm$ must be as in (5.5.b), and (20.2) is replaced by $M'/\mathcal{V} = \Sigma^+ \times \Sigma^-$, while $\pi^\pm$ in (20.1) then become the factor projections.

All these claims are immediate consequences of Remark 11.5(d).

Lemma 20.3. *For a finite-dimensional complex vector space V , any $k \in \{1, \dots, \dim V\}$, and M'/V given by (20.2), let $(W_0, W'_0), (W, W') \in M'/V$. Then there exist an integer $p \geq 1$ and $(W_j, W'_j) \in M'/V$, $j = 0, 1, \dots, p$, with $(W_p, W'_p) = (W, W')$ and $(W_{j-1}, W'_{j-1}) \sim (W_j, W'_j)$ whenever $j = 1, \dots, p$, the notation $(\tilde{W}, \tilde{W}') \sim (W, W')$ meaning that $W = \tilde{W}$ or $W' = \tilde{W}'$.*

Proof. If $W_0 = W$, our claim is obvious as $(W_0, W'_0) \sim (W, W')$. Otherwise we may first choose $W_1 = W_0$ and W'_1 such that $W_0 \cap W \subseteq W'_1 \subseteq W_0$, and then select $W'_2 = W'_1$ along with W_2 spanned by W'_1 and a vector in $W \setminus W_0$. Now $(W_0, W'_0) \sim (W_1, W'_1) \sim (W_2, W'_2)$ and $\dim(W_2 \cap W) > \dim(W_0 \cap W)$. This step may be repeated for (W_2, W'_2) instead of (W_0, W'_0) , as long as $W_2 \neq W$. $\square$

Corollary 20.4. *Let (M, g, τ) be any Grassmannian triple, arising from some data (5.3.i) as in Section 5. Then the direct sum $V \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ appearing in Lemma 16.1(i) is a strongly bracket-generating distribution on M' , in the sense that any two points of M' can be joined by a piecewise C^∞ curve tangent to $V \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$.*

Proof. According to (20.2), whenever $(\tilde{W}, \tilde{W}') \sim (W, W')$ in Lemma 20.3, both $(\tilde{W}, \tilde{W}')$ and (W, W') must lie in the same fibre of one of the bundle projections (20.1). As the fibres of either projection (20.1), being complex projective spaces (see the last line in Lemma 20.1), are connected, the strong bracket-generating property thus follows for the direct-sum distribution of Remark 20.2. Our claim is now immediate since $V \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ projects onto that latter distribution under the quotient-space projection $M' \rightarrow M'/V$, which also has connected fibres (biholomorphic to twice-punctured complex projective lines, cf. Lemma 11.4(b)). $\square$

Remark 20.5. A compact geodesic-gradient Kähler triple need not, in general, satisfy conditions (i) – (iii) of Lemma 16.1, that is, (0.3). Examples are provided by all Grassmannian triples (M, g, τ) arising via Lemma 4.4 from data (5.3.i) such that $2 \leq k \leq n - 2$, where $n = \dim_{\mathbb{C}} V$.

Namely, in (12.4), $q = (k - 1)(n - 1 - k)$ as $m = (n - k)k$ (see Remark 5.5) and, similarly, $\{d_+, d_-\} = \{(n - k)(k - 1), (n - 1 - k)k\}$ from (5.5.a) – (5.5.b), where $\dim_{\mathbb{C}} L = 1$ by (5.3.i). Thus, $q > 0$ and $V \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ in (11.9) is a proper subbundle of TM' . Consequently, due to Corollary 20.4, it cannot be integrable.

Remark 20.6. For any compact geodesic-gradient Kähler triple (M, g, τ) , the leaf space M'/V carries what might be called a *holomorphic 2-web of complex projective spaces*, formed by the two holomorphic fibrations (20.1) with fibres biholomorphic to (positive-dimensional) complex projective spaces, having the trivial-intersection property of Remark 20.2. There is also a natural holomorphic complex line bundle over M'/V , the restriction of which to every fibre of π^+ (or, π^-), with (20.1), is biholomorphically isomorphic to the tautological (or, respectively, dual tautological) bundle of the fibre. Specifically, the complex line attached to a leaf $A \subseteq M'$ of V is $\{0\} \cup \Phi^{-1}(A) \subseteq N_y \Sigma^\pm$, cf. Theorem 14.2(iii); that changing the sign $\pm$ to $\mp$ leads to its dual complex line follows from [8, Remark 4.1] and (8.5.a) – (8.5.b).

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