COMPACT RIEMANNIAN MANIFOLDS WITH HARMONIC CURVATURE AND NON-PARALLEL RICCI TENSOR

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For any (pseudo) Riemannian manifold, the divergence δR of its curvature tensor R satisfies the well-known identity

(1) $$\delta R = dS ,$$

where S is the Ricci tensor (viewed as a vector-valued 1-form) and dS denotes its Riemannian exterior derivative (so that, using the metric, one may consider dS as a 1-form with values in exterior 2-forms). The local coordinate expression for (1) is

(2) $$\nabla^i R_{hijk} = \nabla_k S_{hj} - \nabla_j S_{hk} .$$

While every manifold with parallel Ricci tensor has harmonic curvature (i.e. satisfies $\delta R = 0$), there are examples ([3], Theorem 5.2) of open Riemannian manifolds with $\delta R = 0$ and $\nabla S \neq 0$. In [1] J.P. Bourguignon has asked the question whether the Ricci tensor of a *compact* Riemannian manifold with harmonic curvature must be parallel.

The aim of this note is to describe an easy example answering this question in the negative. More precisely, metrics with $\delta R = 0$ and $\nabla S \neq 0$ are exhibited on $S^1 \times N^3$, N^3 being e.g. the 3-sphere or a lens space. By taking products of these manifolds with themselves or with arbitrary compact Einstein manifolds, one gets similar examples in all dimensions greater than three.

THEOREM. *Let* (N^3,h) *be a three-dimensional Riemannian manifold with constant positive sectional curvature* K. *Define the Riemannian manifold* (M^4,g) *by* $M^4 = S^1 \times N^3$,

$$g_{((\cos t,\, \sin t),x)}(u+X, v+Y) = \langle u,v\rangle + F(t)h_x(X,Y) ,$$

for $u,v \in T_{(\cos t,\, \sin t)}S^1$ *and* $X,Y \in T_xN^3$, *where* $\langle\ ,\ \rangle$ *is the standard metric of* $S^1 = R/2\pi Z$ *and*

$$F(t) = 2Km^{-2} + A\cos mt + B\sin mt , \tag{3}$$

the positive integer m *and real numbers* A,B *being chosen so that*

$$0 < A^2 + B^2 < 4Km^{-4} ,$$

which implies that the function F *is non-constant, positive and periodic.*
Then (M^4,g) *has harmonic curvature tensor, but its Ricci tensor field is not parallel.*

Proof. It seems convenient to use local coordinates. In a product chart $x^o = t$, x^1, x^2, x^3 for $S^1 \times N^3$ we have, setting $q(t) = \log F(t)$,
$g_{oo} = 1$, $g_{oi} = 0$, $g_{ij} = e^q h_{ij}$, $\Gamma^o_{oo} = \Gamma^o_{oi} = \Gamma^i_{oo} = 0$, $\Gamma^o_{ij} = -\frac{1}{2}q'e^q h_{ij}$,
$\Gamma^i_{oj} = \frac{1}{2}q'\delta^i_j$, $\Gamma^i_{jk} = H^i_{jk}$, $S_{oo} = -\frac{3}{4}(2q'' + (q')^2)$, $S_{oi} = 0$,
$S_{ij} = (2K - \frac{1}{2}e^q q'' - \frac{3}{4}e^q(q')^2)h_{ij}$, $\nabla_o S_{oo} = -\frac{3}{2}(q''' + q'q'') = -3\frac{d}{dt}(F^{-\frac{1}{2}}\frac{d^2}{dt^2}(F^{\frac{1}{2}}))$,
$\nabla_o S_{io} = \nabla_i S_{oo} = 0$, $\nabla_o S_{ij} = -(2Kq' + \frac{1}{2}e^q q''' + \frac{3}{2}e^q q'q'')h_{ij}$,
$\nabla_i S_{oj} = -(Kq' + \frac{1}{2}e^q q'q'')h_{ij}$, $\nabla_k S_{ij} = 0$, where i,j,k run through $\{1,2,3\}$ and the Γ's (resp. H's) are the Christoffel symbols of g (resp. of h with respect to the chart x^1, x^2, x^3 of N^3). From (3) we obtain $q'' + (q')^2 - 2Ke^{-q} + m^2 = 0$, whence $q''' + 2q'q'' + 2Kq'e^{-q} = 0$, i.e. $\nabla_o S_{ij} = \nabla_i S_{oj}$, and $\delta R = 0$ is now immediate from (2). On the other hand, $\nabla S \neq 0$, since $\nabla_o S_{oo} = 0$ would mean that the non-constant positive periodic function $F^{\frac{1}{2}}$ is an eigenfunction of d^2/dt^2. This completes the proof.

REMARK 1. It is easy to verify that the manifold (M^4,g) defined above is *conformally flat*, that is, its Weyl tensor $W = 0$. (Conformal flatness together with constancy of the scalar curvature is well-known to imply harmonicity of the curvature tensor. On the other hand, the scalar curvature is constant whenever $\delta R = 0$.)

REMARK 2. One can prove ([2], Theorem 3) that every four-dimensional compact analytic Riemannian manifold with harmonic curvature, whose Ricci tensor is not parallel and has less than three distinct eigenvalues at any point, is covered isometrically by $S^1 \times S^3$ with a metric closely related to the one described above.

REFERENCES

[1] J.P. Bourguignon, On harmonic forms of curvature type (preprint).

[2] A. Derdziński, Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor, to appear in Mathematische Zeitschrift.

[3] A. Gray, Einstein-like manifolds which are not Einstein, Geometriae dedicata, 7(1978), 259-280.