SOME REMARKS ON THE LOCAL STRUCTURE OF CODAZZI TENSORS

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Given a Codazzi tensor A on a Riemannian manifold M, let M_A denote the set of all points x of M such that A in a neighborhood of x has differentiable *eigenvalue functions* of constant multiplicities. Thus, M_A is a dense open subset of M. The tangent bundle of each connected component U of M_A splits (differentiably) as the orthogonal Whitney sum of the *eigenspace bundles* of A.

In the sequel we denote by V_λ the eigenspace bundle corresponding to an eigenvalue function λ and by $\langle\ ,\ \rangle_M$ (or simply $\langle\ ,\ \rangle$) the metric of the Riemannian manifold M.

§1. SOME GENERAL PROPERTIES OF CODAZZI TENSORS.

(1.1). PROPOSITION. Let A be a Codazzi tensor on a Riemannian manifold M. In each connected component of M_A we have

(i) For an eigenvalue function λ of A, any two local sections u, v of V_λ (i.e., local vector fields such that $Au = \lambda u$, $Av = \lambda v$) satisfy the relation

$$A\nabla_v u = \lambda\nabla_v u + (v\lambda)u - \langle u,v\rangle\nabla\lambda. \tag{1}$$

(ii) If u, v, w are mutually orthogonal local sections of the eigenspace bundles of A corresponding to the (not necessarily distinct) eigenvalue functions λ_u, λ_v, λ_w, then

$$(\nabla_w A)(u,v) = (\lambda_u - \lambda_v)\ \langle\nabla_w u,v\rangle \tag{2}$$

and

$$(\lambda_u - \lambda_v)\ \langle\nabla_w u,v\rangle = (\lambda_u - \lambda_w)\ \langle\nabla_v u,w\rangle = (\lambda_v - \lambda_w)\ \langle\nabla_u v,w\rangle\ . \tag{3}$$

(iii) Given distinct eigenvalue functions λ, μ of A and local sections u of V_λ, v of V_μ with $|u| = 1$, we have

$$v\lambda = (\lambda - \mu)\ \langle\nabla_u u,v\rangle\ . \tag{4}$$

PROOF. For any vector field w, the Leibniz rule implies $\langle A\nabla_v u,w\rangle = \langle\nabla_v(\lambda u) - (\nabla_v A)u,w\rangle = \langle(v\lambda)u + \lambda\nabla_v u,w\rangle - (\nabla_w A)(u,v)$ and $(\nabla_w A)(u,v) = w\langle Au,v\rangle - \langle\nabla_w u,Av\rangle - \langle Au,\nabla_w v\rangle = (w\lambda)\langle u,v\rangle + \lambda\{w\langle u,v\rangle - \langle\nabla_w u,v\rangle - \langle u,\nabla_w v\rangle\} = \langle\langle u,v\rangle\nabla\lambda,w\rangle$, which proves (i). Now, if u, v are local sections of eigenspace bundles, corresponding to the eigenvalue functions λ_u, λ_v, and $\langle u,v\rangle = 0$, then, for each vector field w, $(\lambda_u-\lambda_v)\langle\nabla_w u,v\rangle = -\lambda_u\langle u,\nabla_w v\rangle - \lambda_v\langle\nabla_w u,v\rangle = -A(u,\nabla_w v) - A(\nabla_w u,v) = (\nabla_w A)(u,v)$, so that (ii) is immediate from the Codazzi equation. To obtain (iii), it is sufficient to set $\lambda_u = \lambda$,

$\lambda_v = \mu$ and $w = u$, which gives $(\lambda - \mu)\langle\nabla_u u,v\rangle = (\nabla_u A)(u,v) = (\nabla_v A)(u,u) = \nabla_v(A(u,u))$ $= v\lambda$, as required.

(1.2). REMARK. From (1) one easily obtains the well-known fact that, for a Codazzi tensor with *constant* eigenvalues, the eigenspace bundles are integrable and their leaves are totally geodesic. For arbitrary Codazzi tensors we have the following weaker result.

(1.3). THEOREM. Let A be a Codazzi tensor on a Riemannian manifold M. Then, in each connected component of M_A,

(i) The eigenspace bundles of A are integrable and their leaves are totally umbilic in M.

(ii) Every eigenvalue function λ of multiplicity greater than one is constant along the leaves of V_λ.

PROOF. Given an eigenvalue function λ with $\dim V_\lambda \geqq 2$ and a fixed local unit section v of V_λ, choose a local unit section u of V_λ with $\langle u,v\rangle = 0$. By (1), $v\lambda =$ $= \langle\lambda\nabla_v u + (v\lambda)u - \langle u,v\rangle\nabla\lambda, u\rangle = A(\nabla_v u,u) = \langle\nabla_v u,\lambda u\rangle = 0$. This implies (ii) if we know that each V_λ is integrable. To prove this, we may assume $\dim V_\lambda \geqq 2$ and consider local sections u,v of V_λ. Thus, $u\lambda = v\lambda = 0$ and (1) yields $A[v,u] = A(\nabla_v u - \nabla_u v) =$ $=\lambda[v,u]$, i.e., $[v,u]$ lies in V_λ, as required. To show that the leaves of V_λ are totally umbilic, consider a vector field v normal to V_λ. The second fundamental form of the leaves of V_λ with respect to v is given by $b^v(u,u) = -\langle\nabla_u u,v\rangle$, u being a local unit section of V_λ. We may choose v to be a section of some V_μ, $\mu \neq \lambda$, so that (4) gives

$$b^v(u,u) = (\mu - \lambda)^{-1} v\lambda \ , \tag{5}$$

i.e., $b^v_x(u,u)$ is the same for all unit vectors u tangent to the leaf at x, which completes the proof.

As a consequence we obtain

(1.4). LEMMA. Let A be a Codazzi tensor on a Riemannian manifold M, $\dim M \geqq 3$. Suppose U is a connected open subset of M_A such that trace A is constant in U and $\nabla A \neq 0$ somewhere in U. If A has exactly two eigenvalue functions λ,μ in U, and $\dim V_\lambda \geqq \dim V_\mu$, then $\dim V_\mu = 1$, the integral curves of V_μ are geodesics and each leaf of V_λ has constant mean curvature.

PROOF. Given a local section u of V_λ, (1.3.ii) yields $u\lambda = 0$ and

$$u\mu = (\dim V_\mu)^{-1} u\{\text{trace } A - (\dim V_\lambda)\lambda\} = 0. \tag{6}$$

If we had $\dim V_\mu \geqq 2$, then, by (6) and (1.3.ii), μ would be constant and so would be λ. By (1.2), the leaves of both V_λ, V_μ would be totally geodesic, which easily implies that V_λ, V_μ would be invariant under parallel displacements, and the local de Rham theorem would give $\nabla A = 0$. This shows that $\dim V_\mu = 1$. Now, fix a local unit section v of V_μ. Clearly, $\langle\nabla_v v,v\rangle = 0$ and, by (4) and (6), $\langle\nabla_v v,u\rangle = 0$ for any section u of V_λ. Hence $\nabla_v v = 0$, i.e., V_μ is geodesic. In view of (5) the mean curvature of the leaves of V_λ is given by $H = (\mu-\lambda)^{-1} v\lambda$. For any section u of V_λ, (6) implies $uH = (\mu-\lambda)^{-1} uv\lambda$, while $uv\lambda = [u,v]\lambda$ and $\langle [u,v], v\rangle = -\langle \nabla_v u, v\rangle = \langle u, \nabla_v v\rangle = 0$. Thus, $[u,v]$ lies in V_λ, which yields $[u,v]\lambda = 0$ by Theorem (1.3.ii). Hence $uH = 0$, which completes the proof.

§2. A SPECIAL CASE.

We can now give a complete description (at generic points) of non-parallel Codazzi tensors, which have constant trace and less than three distinct eigenvalues at any point.

(2.1). REMARK. Consider a *warped product* manifold $M = I \times_F N$ ([4], [1]) of an interval I of $\mathbb{R}$ with a Riemannian manifold N, dim N = dim M - 1, which is nothing but the smooth manifold I x N endowed with the metric g, where $g_{(t,y)}(\xi + X, \eta + Y) = \langle \xi, \eta\rangle_I + F(t)\langle X, Y\rangle_N$ for $\xi,\eta \in T_t I, X, Y \in T_y N$, F beeing a positive function on I. In a suitable product chart $t = x_o, x_1, \ldots, x_{n-1}$ (n = dim M) for I x N, the components of g and its Christoffel symbols are given by $g_{oo} = 1$, $g_{oi} = 0$, $g_{ij} = e^q h_{ij}$ and $\Gamma^o_{oo} = \Gamma^i_{oo} = 0$, $\Gamma^o_{ij} = -\frac{1}{2} e^q q' h_{ij}$, $\Gamma^i_{oj} = \frac{1}{2} q' \delta^i_j$, $\Gamma^i_{jk} = H^i_{jk}$, where q = log F and h_{ij}, H^i_{jk} are components of the metric of N and its Christoffel symbols in the chart $x_1, \ldots, x_{n-1}$ (i,j,k being always assumed to run through 1, ..., n-1). Given a symmetric (0,2) tensor A on M whose local components are of the form

$$A_{oo} = nb + (1-n)G(t), \tag{7}$$
$$A_{io} = 0, \quad A_{ij} = G(t) g_{ij}$$

for some constant b and a function G on I, the only non-trivial components of ∇A are given by

(8) $\nabla_i A_{oj} = \frac{n}{2} e^q q'(b - G) h_{ij}$, $\nabla_o A_{ij} = e^q G' h_{ij}$, $\nabla_o A_{oo} = (1-n)G'$. Therefore A is a Codazzi tensor iff $G = b + ce^{-nq/2} = b + cF^{-n/2}$ for some real c. Moreover, if F $\neq$ constant, then A is not parallel unless c = 0.

Consequently, we have

(2.2). EXAMPLE. In an n-dimensional warped product $I \times_F N$ (I an interval, F non-constant), define the symmetric tensor A by

$$A_{(t,y)}(\xi + X, \eta + Y) =$$

$$(9) \qquad = [b + (1-n)cF^{-n/2}(t)] \langle \xi, \eta \rangle_I + $$
$$+ [bF(t) + cF^{1-n/2}(t)] \langle X, Y \rangle_N ,$$

$c \neq 0$ and b being real numbers. Then A is a Codazzi tensor with constant trace nb and exactly two distinct eigenvalues at each point of M.

The Codazzi tensors of the above type can be characterized as follows.

(2.3). THEOREM. Let A be a non-parallel Codazzi tensor with constant trace on a Riemannian manifold M, $\dim M = n \geq 3$. If x is a point of M such that, in a neighborhood of x, A has precisely two distinct eigenvalues, then x has a neighborhood isometric to a warped product $I \times_F N$, I an interval of $\mathbb{R}$, $F \neq$ constant, in such a way that A is given by (9) with some real numbers b and $c \neq 0$.

PROOF. By (1.3) and (1.4), the tangent bundle of a neighborhood of x splits as the orthogonal direct sum of the eigenspace bundles of A, i.e., of a geodesic line field V_μ and a codimension one foliation V_λ with totally umbilic leaves, each of constant mean curvature. In a suitable local chart $t = x_o, x_1, \ldots, x_{n-1}$ with $\partial/\partial x_o \in V_\mu$, $\partial/\partial x_i \in V_\lambda$ (i,j range over 1, ..., n-1) we have $g_{oi} = A_{oi} = 0$ and, by (1.3.ii), $A_{ij} = Gg_{ij}$ for some function G which depends only on $t = x_o$. Since V_μ is geodesic, $\Gamma^i_{oo} = 0$, i.e., $\partial_i g_{oo} = 0$, and using a coordinate transformation which involves x_o only, we may assume $g_{oo} = 1$. Setting $b = \frac{1}{n}$ trace A we thus obtain formulae (7) for the components of A. As V_λ is totally umbilic, we have $-\Gamma^o_{ij} = Hg_{ij}$, H being the mean curvature of V_λ. Constancy of H along V_λ says now that $\partial_o g_{ij}(t,x_1,\ldots,x_{n-1}) = f(t) g_{ij}(t,x_1,\ldots,x_{n-1})$ for some f. For a function $q(t)$ with $q' = f$, we have now $\partial_o(e^{-q}g_{ij}) = 0$, i.e., $g_{ij}(t,x_1,\ldots,x_{n-1}) = e^{q(t)}h_{ij}(x_1,\ldots,x_{n-1})$ for some h_{ij}. Thus, x has a neighborhood isometric to $I \times_F N$ for some N, where I is an interval and $F = e^q$. Were F constant, so (8) together with the Codazzi equation would give $\nabla A = 0$, contradicting our hypothesis. Our assertion is now immediate from (2.1).

(2.4). REMARK. The above results are slight extensions of some arguments of [2] . The results of [2], concerning compact Riemannian four-manifolds whose Ricci tensor satisfies the Codazzi equation, have been generalized in [3] to the case of arbitrary dimension $n \geq 3$.

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