RIEMANNIAN MANIFOLDS WITH HARMONIC CURVATURE

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1. INTRODUCTION

The present paper is a survey of results on manifolds with harmonic curvature, i.e., on those Riemannian manifolds for which the divergence of the curvature tensor vanishes identically. The curvatures of such manifolds occur as a special case of Yang-Mills fields. These manifolds also form a natural generalization of Einstein spaces and of conformally flat manifolds with constant scalar curvature.

After describing the known examples of compact manifolds with harmonic curvature, we give, in Sect. 5, a review of theorems concerning such manifolds. Most of their proofs are either omitted or only briefly sketched. For a complete presentation of the results mentioned in this paper (except for Sect. 3, Sect. 7 and 4.4) the reader is referred to the forthcoming book [5], where one of the chapters deals with generalizations of Einstein spaces.

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2. PRELIMINARIES

2.1. Given a Riemannian vector bundle E over a compact Riemannian manifold (M,g), one studies the *Yang-Mills potentials* in E, i.e., those metric connections ∇ in E which are critical points for the *Yang-Mills functional*

$$YM(\nabla) = \tfrac{1}{2} \int_M |R|^2 \ ,$$

where R is the curvature of ∇ and the integration is with respect to the Riemannian measure of (M,g) (see [8]). The obvious operator of *exterior differentiation* d as well as its formal adjoint, the *diver-*

gence d*, can be applied to differential forms on (M,g) valued in Riemannian vector bundles with fixed metric connections (cf. [6]). The Yang-Mills potentials in E now are characterized by $d^*R = 0$. In view of the Bianchi identity $dR = 0$ for the 2-form R, this means that the curvature form of any Yang-Mills potential is *harmonic*.

2.2. A Riemannian manifold (M,g) is said to have *harmonic curvature* if its Levi-Civita connection ∇ in the tangent bundle TM satisfies $d^*R = 0$. If M is compact, this just means that ∇ is a Yang-Mills potential in TM (i.e., a critical point for the Yang-Mills functional in the space of all g-metric connections, where g is *fixed*).

2.3. Let (M,g) be a Riemannian manifold, $\dim M = n \geq 3$. Its curvature tensor R, Ricci tensor Ric, scalar curvature $Scal$ and Weyl conformal tensor W can obviously be viewed as differential forms valued in suitable bundles of exterior forms : $R, W \in \Omega^2(M, \Lambda^2 M)$, $Ric \in \Omega^1(M, \Lambda^1 M)$, $Scal = Scal_g \in \Omega^0(M, \Lambda^0 M)$. The second Bianchi identity $dR = 0$ now easily gives

$$(1) \qquad d^*R = -\, d\,Ric$$

(in local coordinates : $-\nabla^s R_{sijk} = \nabla_k (Ric)_{ij} - \nabla_j (Ric)_{ik}$),

$$(2) \qquad d^*W = -\frac{n-3}{n-2}\, d[Ric - \frac{1}{2(n-1)} Scal \cdot g],$$

$$(3) \qquad d^*Ric = -\tfrac{1}{2}\, d\,Scal,$$

$$(4) \qquad d^*W = -\,Con(dW)\ ,\quad dW = -\,(n-3)^{-1} g \odot d^*W \quad (\text{if } n > 3),$$

where, in (1) - (3), we identify $\Omega^i(M, \Lambda^j M)$ with $\Omega^j(M, \Lambda^i M)$, while, in (4), $\odot$ is a natural bilinear pairing and Con is a suitable contraction.

2.4. Riemannian manifolds satisfying $d^*W = 0$ are said to have *harmonic Weyl tensor*, which is justified by the fact that this condition implies $dW = 0$ (cf. (4)). These manifolds are studied here in order to simplify various arguments involving the equation $d^*R = 0$. In particular, *all examples with* $d^*R = 0$ *discussed below arise from natural classes of manifolds having* $d^*W = 0$ *by requiring that their scalar curvature be constant* (cf. 2.6.ii).

2.5. By a *Codazzi tensor* on a Riemannian manifold (M,g) we mean any *symmetric* C^∞ tensor field b of type (0,2) on M which, viewed as a Λ^1-valued 1-form, satisfies the relation $db = 0$; this is clearly nothing else than the *Codazzi equation* $\nabla_i b_{jk} = \nabla_j b_{ik}$.

2.6. Given a Riemannian manifold (M,g), $\dim M \geq 4$, it follows from 2.3 that equation $d^*R = 0$ for (M,g) is equivalent to either of the following two conditions :

i) $d\,Ric = 0$, i.e., Ric is a Codazzi tensor on (M,g) (hence, by (3), $Scal$ is constant).

ii) (M,g) has harmonic Weyl tensor and constant scalar curvature.

If $\dim M = 3$, $d^*R = 0$ is still equivalent to i) and it characterizes conformally flat 3-manifolds with constant scalar curvature (cf. 2.7).

2.7. In view of (2), a Riemannian manifold (M,g) with $\dim M = n \geqq 4$ has harmonic Weyl tensor if and only if $Ric - (2n-2)^{-1}Scal\cdot g$ is a Codazzi tensor. If $n = 3$, the latter condition means that (M,g) is conformally flat ([20], p. 306).

2.8. THE SIMPLEST EXAMPLES. By 2.6 and 2.7, the manifolds listed in $1_W, 2_W$ and 3_W (resp., $1_R, 2_R$ and 3_R) of our table of examples actually have harmonic Weyl tensor (resp., harmonic curvature). The problem of finding metrics of type 3_R on compact conformally flat manifolds is obviously related to a special case of Yamabe's conjecture (see [17]).

3. THE MODULI SPACE

3.1. Consider a metric g on a compact manifold M, satisfying the condition $d^*R = 0$. The linearized version of this condition, restricted to a slice through g (see [15]) in a suitable Sobolev space of metrics implies that the corresponding slice vector lies in the kernel of a third order differential operator with injective symbol. Consequently, *the moduli space of all metrics with* $d^*R = 0$ *modulo the group of diffeomorphisms of the underlying compact manifold is locally finite dimensional.*

3.2. The assertion of 3.1 fails to hold for the weaker condition $d^*W = 0$; counterexamples are provided by $3_W, 4_W, 5_W, 6_W$ and 7_W of the table.

4. FURTHER EXAMPLES

4.1. For pointwise conformal metrics g and $\bar{g} = e^{2\sigma}g$ on a manifold M, the tensor d^*W for g and the corresponding quantity $\overline{d^*W}$ for $\bar{g}$ are related by

$$\overline{d^*W} = d^*W - (n-3)\,W(\nabla\sigma,\cdot,\cdot,\cdot)\,, \quad n = \dim M \geqq 3. \tag{5}$$

4.2. For Riemannian products of Einstein manifolds like those described in 4_W and 5_W of the table, with any function σ which is constant along the second factor, both terms on the right-hand side of (5) vanish, and so the conformally related metrics of 4_W and 5_W must also have harmonic Weyl tensor. In order that these metrics have constant scalar curvature (and hence harmonic curvature, cf. 2.6.ii), the function F on the first factor manifold must satisfy a second order differential equation, which admits a non-constant positive solution under the conditions stated in 4_R and 5_R (see [9], [12], [5]). If the (N,h) that we use are not of constant curvature, the examples in 4_R and 5_R obtained from this construction are neither locally isometric to those of 1_R, 2_R and 3_R, nor to each other ([9], [12]).

4.3. The Weyl tensor of any *four-dimensional oriented* Riemannian manifold (M,g) can be decomposed into its $SO(4)$-*irreducible components:* $W = W^+ + W^-$, which corresponds to the decomposition $\Lambda^2 M = \Lambda^2_+ M + \Lambda^2_- M$ of $\Lambda^2 M$ into the eigenspace bundles of the Hodge star operator ([2]). If, moreover, (M,g) is a Kähler manifold endowed with the natural orientation, the condition $d^* W^+ = 0$ holds for the conformally related metric $\mathrm{Scal}^{-2}\cdot g$, defined wherever $\mathrm{Scal} = \mathrm{Scal}_g \neq 0$ ([10]). Letting (M,g) now be the Riemannian product of two orientable surfaces, we see that the example described in 6_W of the table has $d^* W = 0$ (since $g = g_1 \times g_2$ is a Kähler metric for two complex structures, corresponding to different orientations of M).

4.4. Let (S^2, g_c) be the sphere of constant curvature $c > 0$ and suppose that we are given a compact Riemannian surface (N,h) with non-constant Gaussian curvature K such that $K + c > 0$ on N and

$$(K + c)^3 - 3(K + c)\cdot\Delta K - 6|dK|^2 - \lambda^3 = 0 \tag{6}$$

for some real $\lambda > 0$, where the quantities $\Delta = -h^{ij}\nabla_i\nabla_j$ and $|\ |^2$ refer to the geometry of (N,h). Then the compact Riemannian 4-manifold $(N \times S^2, (K + c)^{-2}\cdot(h \times g_c))$ has harmonic curvature (since it is of type 6_W and has constant scalar curvature by (6)). However, it is not clear whether this construction really gives new compact 4-manifolds with $d^* R = 0$:

a) For any real c, λ with $c > \lambda > 0$, *the torus* T^2 *admits a metric* h *with non-constant curvature* K *satisfying* $K + c > 0$ *and* (6). Namely, we can define h first on $\mathbb{R}^2$ (with coordinates t, y) by $h = dt^2 + (K(t) + c)^2 dy^2$, taking for $K = K(t)$ any non-constant periodic solution to $3(dK/dt)^2 = -2K^3 - 3cK^2 + c^3 - \lambda^3$ with $K + c > 0$ (which is easily seen to exist), and then project it onto $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with an appropriate action of $\mathbb{Z}^2$. Unfortunately, the metric with $d^* R = 0$ on $T^2 \times S^2$, obtained as above with this h, turns out to be a Riemannian product (of S^1 and $S^1 \times S^2$ with suitable metrics) and hence it is *nothing really new*. (However, one easily shows that for compact surfaces (N,h) with non-constant K satisfying $K + c > 0$ and (6) which are not locally isometric to a torus as above, the corresponding metrics with $d^* R = 0$ on $N \times S^2$ are never locally isometric to examples $1_R, 2_R$ or 3_R of the table.)

b) Let $\lambda_o > 0$ be a *simple* eigenvalue of the Laplace operator on a compact surface (N, h_o) of constant curvature -1. To any function x on N and to $\lambda \in \mathbb{R}$ we can assign the function $f(x,\lambda)$, de-

TABLE OF EXAMPLES

Known examples of compact Riemannian manifolds with $d^*W = 0$ or $d^*R = 0$ (up to local isometry)	
with $d^*W = 0$	with $d^*R = 0$
1_W. Einstein spaces	1_R. Einstein spaces
2_W. Riemannian products of manifolds with $d^*R = 0$	2_R. Riemannian products of manifolds with $d^*R = 0$
3_W. Conformally flat manifolds	3_R. Conformally flat manifolds with constant scalar curvature
4_W. Warped products $(S^1 \times N, F\cdot(dt^2 \times h))$, F any positive function on S^1, (N,h) any Einstein space	4_R. As in 4_W, with $\mathrm{Scal}_h > 0$ and for a suitable non-constant F (which always exists on S^1 of appropriate length). Also, *twisted warped products* $(\mathbb{R}\times N)/\mathbb{Z}$, where $\mathbb{R}\times N$ has the pull-back metric and the $\mathbb{Z}$-action involves an isometry of h
5_W. Warped products $(M\times N, F\cdot(g\times h))$, (M,g) of constant curvature K, (N,h) Einstein with $\mathrm{Scal}_h = -\dim N\cdot(\dim N - 1)K$, F any positive function on M	5_R. As in 5_W, with $K < 0$ and $\dim N > \dim M - \lambda_1/K$, λ_1 being the lowest positive eigenvalue of Δ in (M,g), for a suitable non-constant F (which must exist)
6_W. $(M_1\times M_2, (K_1+K_2)^{-2}(g_1\times g_2))$, $\dim M_i = 2$, (M_i, g_i) having Gaussian curvature K_i $(i=1,2)$ with $\|K_1+K_2\| > 0$	6_R ? Not known whether 6_W contains new compact manifolds with $d^*R = 0$ (see 4.4)
7_W. $(M_1\times M_2, \|K_1+K_2\|^{2/(1-n)}\cdot(g_1\times g_2))$, $\dim M_1 = 2 < \dim M_2 = n$, (M_1,g_1) with non-constant curvature K_1, (M_2,g_2) of constant curvature K_2, $\|K_1 + K_2\| > 0$	7_R ? Not known whether 7_W gives new compact manifolds with $d^*R = 0$

fined to be the left-hand side of (6) formed with the metric $h = e^{2x}\cdot h_0$ and with $c = \lambda + 1$ (so that f is a fourth order non-linear operator in x with parameter λ). The equation $f(x,\lambda) = 0$ has the curve of *trivial* solutions with $x = 0$ and any λ. At $x = 0$ and $\lambda = \lambda_0$ this equation has a bifurcation point, since the hypotheses of Theorem (4.1.12) of [3] (p. 155) are satisfied. Thus, there is a curve $(x(\varepsilon),\lambda(\varepsilon))$ of non-trivial solutions through $x(0) = 0$, $\lambda(0) = \lambda_0$, and for small $\varepsilon \neq 0$ the metric $h_\varepsilon = e^{2x(\varepsilon)}\cdot h_0$ has non-constant curvature K with $K + c > 0$ and (6) for $c = \lambda(\varepsilon) + 1$. By the above construction, this would lead to new compact Riemannian 4-manifolds with $d^*R = 0$; however, the author does not know whether the Laplace operator of any compact surface with constant negative curvature admits a simple positive eigenvalue.

4.5. For g_1, g_2 as in 7_W, one easily sees ([19]) that the metric $g_1 \times g_2$ satisfies $2|W|^2\cdot\nabla W = d(|W|^2) \otimes W$. Together with (5) and with the fact that, for $g_1 \times g_2$, $|W|$ is proportional to $|K_1 + K_2|$, this implies that the conformally related metric described in 7_W really satisfies $d^*W = 0$.

5. PROPERTIES OF MANIFOLDS WITH HARMONIC CURVATURE

5.1. Most of the known results on manifolds with $d^*R = 0$ are valid in more generality. From $d^*R = 0$ it follows that $d^*W = 0$, which in turn implies the existence of a natural Codazzi tensor b on the manifold (M,g) ($b = \mathrm{Ric} - (2n-2)^{-1}\mathrm{Scal}\cdot g$, $n = \dim M$). On the other hand, $d^*R = 0$ means that $b = \mathrm{Ric}$ is a Codazzi tensor with constant trace (see 2.7 , 2.8). The results presented in this section will be stated under appropriate weaker hypotheses, as described above.

5.2. THEOREM (D. DeTurck - H. Goldschmidt, [14]). *Every Riemannian manifold with* $d^*R = 0$ *is analytic in suitable local coordinates.*

5.3. THEOREM ([13]). *Given a Codazzi tensor* b *on* (M,g), $x \in M$ *and eigenspaces* V, V' *of* $b(x)$, *the curvature operator* $R(x) \in \mathrm{End}\,\Lambda^2 T_xM$ *leaves* $V \wedge V' \subset \Lambda^2 T_xM$ *invariant.*

Proof. Assume $\det b \neq 0$ near x (taking $b + tg$ instead of b). Viewing b as a bundle automorphism of TM (near x), we easily conclude from $db = 0$ that $\overline{\nabla} = b^*\nabla$ is the Levi-Civita connection of $\overline{g} = b^*g$ ([16]). Hence $\overline{g}$ has the (0,4) curvature tensor $\overline{R}(X,Y,Z,U) = R(X,Y,bZ,bU)$. Our assertion now follows from the first Bianchi identity for $\overline{R}$.

5.4. BOURGUIGNON'S COMMUTATION THEOREM ([6]). *Any Codazzi tensor* b *on* (M,g) *commutes with* Ric, *while the endomorphism* $g \owedge b$ *of* $\Lambda^2 M$

commutes with R *and* W *(here* $\textcircled{\scriptsize\wedge}$ *is the bilinear pairing of symmetric 2-tensor fields, giving rise to algebraic curvature tensors).*

Proof. At $x \in M$, decompose $\Lambda^2 T_x M$ using a $b(x)$-eigenspace decomposition of $T_x M$ and apply 5.3.

5.5. THEOREM ([13]). *Let* b *be a Codazzi tensor on* (M,g), $\dim M = n$. *If* b *has* n *distinct eigenvalues almost everywhere in* M, *then all Pontryagin forms of* (M,g) *vanish identically.*

5.6. THEOREM (cf. [6], [13]). *If* (M,g) *satisfies* $d^*W = 0$ *and* $x \in M$, *then the inner product* $g(x)$ *and the algebraic curvature tensor* $R(x)$ *in* $T_x M$ *cannot be completely arbitrary; in other words, condition* $d^*W = 0$ *always imposes algebraic restrictions on the curvature. For instance, we have*

i) $\mathrm{Ric}(x)$ *has a multiple eigenvalue,*

or

ii) $T_x M$ *admits an orthonormal basis* $e_1,\ldots,e_n$ *which diagonalizes* $R(x)$ *in the sense that all* $e_i \wedge e_j$ *$(i<j)$ are eigenvectors of* $R(x) \in \mathrm{End}\,\Lambda^2 T_x M$. *This in turn implies that all Pontryagin forms of* (M,g) *vanish at* x.

Proof of 5.5 and 5.6. If $b(x)$ (resp., $\mathrm{Ric}(x)$) has $n = \dim M$ distinct eigenvalues, 5.3 gives rise to 1-dimensional invariant subspaces and hence to eigenvectors of $R(x)$. The assertion concerning the Pontryagin forms now follows immediately.

5.7. COROLLARY. *Let* (M,g) *satisfy* $d^*R = 0$. *If, for some* $x \in M$, *all eigenvalues of* $\mathrm{Ric}(x)$ *are simple, then all real Pontryagin classes of* M *are zero.*

Proof. Immediate from 5.6 and 5.2.

5.8. THEOREM (cf. [6], [13]). *For a Codazzi tensor* b *on* (M,g) *with* $\dim M = 4$, *we have* $P_1 \otimes (b - \frac{1}{4}\mathrm{tr}\, b \cdot g) = 0$, P_1 *being the (first) Pontryagin form of* (M,g). *More precisely, at any point* x *where* b *is not a multiple of* g, *the endomorphisms* $W^+(x)$ *of* $\Lambda^2_+ T_x M$ *and* $W^-(x)$ *of* $\Lambda^2_- T_x M$ *have equal spectra.*

Idea of proof (see [5] for details). We may assume that the number of distinct eigenvalues of b is locally constant at x. The assertion can now be obtained from 5.3 using the algebraic properties of W, except for the case of two double eigenvalues for $b(x)$, where an argument involving differentiation is needed ([13], [5]).

5.9. COROLLARY (J. P. Bourguignon, [6]). *The signature* τ *and the Euler characteristic* χ *of any compact oriented Riemannian four-ma-*

nifold (M,g) *with harmonic curvature satisfy*

$$(2\chi - 3|\tau|)\cdot|\tau| \geqq 0 \ .$$

More precisely, if $\tau \neq 0$ *,* (M,g) *must be Einstein and so* $2\chi \geqq 3|\tau|$ *by Thorpe's inequality* ([22]).

Proof. Immediate from 5.8 and 5.2 .

5.10. THEOREM (D. DeTurck - H. Goldschmidt, [14], cf. [5]). *Let* (M,g) *satisfy* $d^*W = 0$. *If, for some point* x *,* $W(x) = 0$ *and all eigenvalues of* Ric(x) *are simple, then* $W = 0$ *identically on* M .

Proof. Choose an orthonormal C^∞ frame field diagonalizing Ric (and R, W) near x (cf. 5.6.ii). Equations $d^*W = 0$, $dW = 0$ (see (4)) mean that the components of W in this frame field satisfy a first order system of *linear* differential equations, solved for the directional derivatives. As $W(x) = 0$, $W = 0$ near x . Since W is a solution to the elliptic system $(dd^* + d^*d)W = 0$, $W = 0$ everywhere in view of Aronszajn's unique continuation theorem (cf. [1]).

5.11. THEOREM (M. Berger, cf.[4] and [5]). *Let* b *be a Codazzi tensor with constant trace on a compact Riemannian manifold with sectional curvature* $K > 0$. *Then* b *is a constant multiple of the metric. Thus, a compact manifold* (M,g) *with* $d^*R = 0$ *and* $K > 0$ *must be Einstein.*

Proof. See [4] or [5].

5.12. THEOREM (Y. Matsushima [18], S. Tanno [21]). *Let* b *be a Hermitian Codazzi tensor on a Kähler manifold. Then* b *is parallel. In particular, a Kähler manifold with* $d^*W = 0$ *must have* $\nabla \mathrm{Ric} = 0$.

Proof. The expression $(\nabla_X b)(Y,JZ)$, where J is the complex structure tensor, is symmetric in X,Y and skew-symmetric in Y,Z , so that it vanishes.

5.13. THEOREM ([9], cf. [5]). *Let* (M,g) *be compact and satisfy* $d^*R = 0$. *If its Ricci tensor is not parallel and has, at each point, less than 3 distinct eigenvalues, then* (M,g) *admits a finite Riemannian covering by a manifold* $S^1 \times N$ *endowed with a twisted warped product metric as described in* 4_R *of the table. Conversely, all examples of* 4_R *satisfy the above hypotheses.*

Proof. See [9] or [5].

5.14. THEOREM. *Suppose we are given an oriented Riemannian four-manifold* (M,g) *with* $d^*W = 0$ *, and a point* x *of* M *at which* $W \neq 0$ *and* $4\,\mathrm{Ric} \neq \mathrm{Scal}\cdot g$. *If the endomorphism* W^+ *of* $\Lambda^2_+ M$ *has less than three distinct eigenvalues at every point, then, in a neighborhood of*

X , g *is obtained by a conformal deformation of a product of surface metrics as in* 6_W *of our table.*

Proof. See [5].

6 . SOME OPEN QUESTIONS

6.1. Does there exist a compact simply connected Riemannian manifold with $d^*R = 0$, the Ricci tensor of which is not parallel ? (cf. [7]).

6.2. Are there compact Riemannian 4-manifolds satisfying $d^*R = 0$ and not locally isometric to examples $1_R, 2_R, 3_R$ of our table ([6]; cf. 4.4).

6.3. Must a locally homogeneous Riemannian manifold with $d^*R = 0$ have parallel Ricci tensor ?

6.4. Do the classes 6_W , 7_W of our table contain new compact Riemannian manifolds with harmonic curvature ? (cf. 4.4).

7 . THE CLASSIFICATION PROBLEM IN DIMENSION FOUR

7.1. In this section we present some steps towards a classification of compact Riemannian 4-manifolds with harmonic curvature. By a *classification* we mean a description of all Riemannian universal covering spaces of such manifolds which are different from the "classical" examples (Einstein, conformally flat, products). The results we discuss below consist in excluding certain a priori possible cases. Their proofs, too long to be reproduced here, are available from the unpublished manuscript [11].

7.2. Let us recall the *known examples of compact Riemannian four-manifolds with* $d^*R = 0$. First, we have the types 1_R, 2_R and 3_R of our table (which obviously include many compact manifolds). Explicitly, these are

I. Einstein spaces

II. Conformally flat manifolds with constant scalar curvature

III. Manifolds locally isometric to Riemannian products $\mathbb{R} \times N$, where N is a conformally flat 3-manifold with constant scalar curvature

IV. Manifolds locally isometric to Riemannian products of surfaces with constant curvatures

Examples 4_R and 5_R of the table yield nothing new in dimension four. Finally, it seems convenient to list here also the examples of 6_R :

V. Manifolds locally isometric to $(N \times S^2, (K + c)^{-2}(h \times g_c))$ as described in 4.4.

Although we do not know if new examples with *compact* surfaces N really occur in V., this is clearly true for non-compact N (just take N to be a surface of revolution, which reduces (6) to an ordinary differential equation).

7.3. Let (M,g) now be an oriented Riemannian four-manifold with harmonic curvature. Denote by $r_0 \in \{1,2,3,4\}$ (resp., by $w_0 \in \{1,2,3\}$) the maximal number of distinct eigenvalues of Ric (resp., of W^+ acting on $\Lambda^2_+ M$), both attained in an open dense subset of M (cf. 5.2). The following cases are possible:

Case A: $r_0 = 1$, i.e., (M,g) is Einstein, as in 7.2.I.

Case B: $r_0 > 1$, $w_0 = 1$. Then (M,g) must be as in 7.2.II. In fact, W^+ is always traceless in $\Lambda^2_+ M$, which now implies $W^+ = 0$ and, by 5.8, also $W^- = 0$.

Case C: $r_0 > 1$, $w_0 = 2$. Then (M,g) is as in 7.2.IV., or it is of type 7.2.V. (where N may be non-compact and c, λ need not be positive). However, in the latter case $c, \lambda > 0$ and N can be chosen compact, if so is M. See [11] for details.

Case D: $r_0 > 1$, $w_0 = 3$. In an open dense subset M_0 of M we can choose an orthonormal C^∞ frame field $e_1, \ldots, e_4$ diagonalizing R in the sense of 5.6.ii. For $l \in \{1,2,3,4\}$ we say that $A_l \neq 0$ if there exist i,j,k with $\{i,j,k,l\} = \{1,2,3,4\}$ and $g(\nabla_{e_i} e_j, e_k) \neq 0$ somewhere in the given connected component of M_0. Define $m_0 \in \{0,1,2,3,4\}$ to be the number of l with $A_l \neq 0$. It turns out that $m_0 \leq 2$ ([11]) and so three subcases can only occur:

Case D.0 : $m_0 = 0$. See 7.5 below.

Case D.1 : $m_0 = 1$. Then (M,g) is as in 7.2.III. (see [11]).

Case D.2 : $m_0 = 2$. If, moreover, (M,g) is complete, then $|d|\mathrm{Ric}|^2|$ is unbounded on M. Thus, *if* M *is compact,* Case D.2 *cannot occur.*

7.4. According to 7.3, our classification problem for compact Riemannian 4-manifolds with $d^*R = 0$ (cf. 7.1) has been reduced to cases C and D.0, since cases A,B and D.1 imply a "classical" situation (I.-IV. of 7.2), and Case D.2 is impossible. In Case C this problem amounts to the question of existence and classification for compact surfaces (N,h) having the properties stated in 4.4 (cf. 4.4.a,b). As for Case D.0

(which may occur for certain *non-compact* manifolds of type 7.2.III.), we do not know whether compact manifolds of this kind exist.

7.5. Some local properties of Riemannian 4-manifolds (M,g) with $d^*R = 0$ in Case D.0. Let $e_1,\dots,e_4$ be the orthonormal C^∞ frame field, defined at the "generic" points of M and diagonalizing R (and Ric). Since $m_0 = 0$, we have $\nabla_{e_i} e_j = F_{ji} e_i$ and $[e_i, e_j] = F_{ji} e_i - F_{ij} e_j$ for $i \neq j$, with certain functions F_{ji} (in particular, the generic subset of M admits local coordinates with mutually orthogonal coordinate lines). Set $b = Ric - \frac{1}{4} Scal\cdot g$, $\lambda_i = b(e_i, e_i)$, $\sigma_{ij} = W(e_i, e_j, e_i, e_j)$ $(i \neq j)$, so that $\Sigma_i \lambda_i = 0$ and $\sigma_{ij} = \sigma_{ji} = \sigma_{kl}$ if $\{i,j,k,l\} = \{1,2,3,4\}$. Conditions $db = 0$, $d^*W = 0$ and the fact that $e_1,\dots,e_4$ diagonalizes R (and W) now imply $D_k \lambda_i = (\lambda_k - \lambda_i) F_{ki}$, $D_k \sigma_{ji} = (\sigma_{ki} - \sigma_{ji}) F_{kj} + (\sigma_{kj} - \sigma_{ji}) F_{ki}$ and $D_k F_{ji} = F_{ki}(F_{jk} - F_{ji})$ for $i \neq j \neq k \neq i$, D_k being the directional derivative along e_k. The integrability conditions for these systems impose certain algebraic relations on the components λ_i, σ_{ij} and F_{ij}.

REFERENCES

1. Aronszajn, N.: A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. Math. Pures Appl. 36, 235--249 (1957) (Zbl. 84.304)
2. Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self-duality in four-dimensional Riemannian geometry. Proc. Roy. Soc. London A362, 425-461 (1978) (Zbl. 389.53011)
3. Berger, M.S.: Nonlinearity and Functional Analysis, Academic Press, 1977 (Zbl.368.47001)
4. Berger, M., Ebin, D.: Some decompositions of the space of symmetric tensors on a Riemannian manifold. J. Differential Geometry 3, 379-392 (1969) (Zbl. 194,531)
5. Besse, A.L.: Einstein Manifolds (to appear)
6. Bourguignon, J.P.: Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein. Invent. math. 63, 263-286 (1981) (Zbl. 456.53033)
7. Bourguignon, J.P.: Metrics with harmonic curvature. Global Riemannian Geometry (edited by T.J. Willmore and N.J. Hitchin), 18-26. Ellis Horwood, 1984
8. Bourguignon, J.P., Lawson, H.B.,Jr.: Yang-Mills theory: Its physical origins and differential geometric aspects. Seminar on Differential Geometry (edited by S.T. Yau), Ann. of Math. Studies No. 102, 395-421 (1982) (Zbl. 482.58007)
9. Derdziński, A.: On compact Riemannian manifolds with harmonic curvature. Math. Ann. 259, 145-152 (1982) (Zbl. 489.53042)
10. Derdziński, A.: Self-dual Kähler manifolds and Einstein manifolds of dimension four. Compos. Math. 49, 405-433 (1983) (Zbl. 527.53030)
11. Derdziński, A.: Preliminary notes on compact four-dimensional Riemannian manifolds with harmonic curvature, 1983 (unpublished)
12. Derdziński, A.: An easy construction of new compact Riemannian manifolds with harmonic curvature (preliminary report). SFB/MPI 83-21, Bonn (1983)
13. Derdziński, A., Shen, C.L.: Codazzi tensor fields, curvature and Pontryagin forms. Proc. London Math. Soc. 47, 15-26 (1983) (Zbl. 519.53015)
14. DeTurck, D.: private communication
15. Ebin, D.G.: The manifold of Riemannian metrics. Proc. of Symposia in Pure Math. 15, 11-40 (1970) (Zbl. 205,537)
16. Hicks, N.: Linear perturbations of connexions. Michigan Math. J. 12, 389-397 (1965) (Zbl. 135,225)
17. Lafontaine, J.: Remarques sur les variétés conformément plates. Math. Ann. 259, 313-319 (1982) (Zbl. 469.53036)

18. Matsushima, Y.: Remarks on Kähler-Einstein manifolds. Nagoya Math. J. 46, 161-173 (1972) (Zbl. 249.53050)
19. Roter, W.: private communication
20. Schouten, J.A.: Ricci Calculus. Springer-Verlag, 1954
21. Tanno, S.: Curvature tensors and covariant derivatives. Ann. Mat. Pura Appl. 96, 233-241 (1973) (Zbl. 277.53013)
22. Thorpe, J.: Some remarks on the Gauss-Bonnet integral. J. of Math. Mech. 18, 779--786 (1969) (Zbl. 183,505)