Special biconformal changes of Kähler surface metrics

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Abstract The term "special biconformal change" refers, basically, to the situation where a given nontrivial real-holomorphic vector field on a complex manifold is a gradient relative to two Kähler metrics, and, simultaneously, an eigenvector of one of the metrics treated, with the aid of the other, as an endomorphism of the tangent bundle. A special biconformal change is called nontrivial if the two metrics are not each other's constant multiples. For instance, according to a 1995 result of LeBrun, a nontrivial special biconformal change exists for the conformally-Einstein Kähler metric on the two-point blow-up of the complex projective plane, recently discovered by Chen, LeBrun and Weber; the real-holomorphic vector field involved is the gradient of its scalar curvature. The present paper establishes the existence of nontrivial special biconformal changes for some canonical metrics on Del Pezzo surfaces, viz. Kähler-Einstein metrics (when a nontrivial holomorphic vector field exists), non-Einstein Kähler-Ricci solitons, and Kähler metrics admitting nonconstant Killing potentials with geodesic gradients.

Keywords Biconformal change · Ricci soliton · conformally-Einstein Kähler metric · special Kähler-Ricci potential · geodesic gradient

Mathematics Subject Classification (2000) Primary 53C55 · Secondary 53C25

1 Introduction

By a *metric-potential pair* on a complex manifold M with $\dim_{\mathbb{C}} M \geq 2$ we mean any pair (g,τ) formed by a Kähler metric g on M and a nonconstant Killing potential τ for g, that is, a function $\tau : M \to \mathrm{I\!R}$ such that $J(\nabla\tau)$ is a nontrivial Killing field on the Kähler manifold (M,g). Another metric-potential pair $(\hat{g},\hat{\tau})$ on the same complex manifold M is said to arise from (g,τ) by a *special biconformal change* if

$$\text{i)} \quad \hat{g} = fg - \theta(d\tau\otimes d\tau + \xi\otimes\xi), \qquad \text{ii)} \quad \hat{\nabla}\hat{\tau} = \nabla\tau \tag{1.1}$$

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for $\xi = g(J(\nabla\tau), \cdot)$ and some C^∞ functions $f, \theta : M \to \mathbb{R}$. The equality in (1.1.ii) states that the $\hat{g}$-gradient of $\hat{\tau}$ coincides with the g-gradient of τ.

A special biconformal change as above will be called *trivial* if f is a positive constant, $\theta = 0$, and $\hat{\tau}$ equals $f\tau$ plus a constant.

Ganchev and Mihova [10, Section 4] studied biconformal changes of a more general type. In their approach, $\tau : M \to \mathbb{R}$ is not required to be a Killing potential.

The existence of nontrivial special biconformal changes has already been established for some metric-potential pairs (g,τ). LeBrun [14] proved it when g is a Kähler metric on a compact complex surface, conformal to a non-Kähler Einstein metric, and τ is the scalar curvature of g. Both the one-point and two-point blow-ups of $\mathbb{C}\mathrm{P}^2$ are known to admit metrics with the properties just listed (the latter, due to a recent result of Chen, LeBrun and Weber [3]; see also Section 9). On the other hand, Ganchev and Mihova [10] exhibited a nontrivial special biconformal change leading from (g,τ), for any nonflat Kähler metric g of quasi-constant holomorphic sectional curvature, and suitable τ, to a metric-potential pair $(\hat{g}, \hat{\tau})$ in which the Kähler metric $\hat{g}$ is flat.

This paper addresses the existence question for nontrivial special biconformal changes of metric-potential pairs in complex dimension 2. It is not known whether all metric-potential pairs (g,τ) on compact complex surfaces admit such changes. However, nontrivial special biconformal changes of (g,τ) always exist locally, at points where $d\tau \neq 0$ (Remark 2 at the end of Section 5).

Biconformal changes of a more general kind than those defined above are introduced in Section 12, where it is also shown that such a generalized biconformal change exists between any two $\mathrm{U}(2)$-invariant Kähler metrics on $\mathbb{C}\mathrm{P}^2$ or on the one-point blow-up of $\mathbb{C}\mathrm{P}^2$.

Theorems 1 and 3, stated and proved in Sections 6 and 14, provide two general mechanisms allowing one to construct examples of nontrivial special biconformal changes. They are based on criteria for the existence of such changes that are, in addition, required to satisfy a certain functional dependence relation, or to yield a metric in the same Kähler class; in the former case the criterion amounts to a Laplacian condition.

The first main result of the paper, derived from Theorem 1, is the existence of nontrivial special biconformal changes of various canonical metrics on Del Pezzo surfaces. Specifically, they are shown to exist for all metric-potential pairs (g,τ) with suitably chosen τ, on compact complex surfaces M, such that g is

(i) any Kähler-Einstein metric with positive scalar curvature (and M admits a nontrivial holomorphic vector field), or
(ii) any non-Einstein Kähler-Ricci soliton, or
(iii) any Kähler metric admitting a special Kähler-Ricci potential τ.

The second main result is Theorem 2, establishing the existence of nontrivial special biconformal changes of (g,τ) whenever (M,g) is a compact Kähler surface and the integral curves of $\nabla\tau$ are reparametrized geodesics. Being a special Kähler-Ricci potential is sufficient for τ to have this last property, but it is not necessary; more general examples are described in the Appendix.

Two Kähler metrics on a given complex surface cannot be nontrivially conformal. The relation of "general biconformal equivalence" is not of much interest here either, since it holds locally, almost everywhere, for any two Kähler surface metrics (Section 5). On the other hand, on compact complex surfaces, a special biconformal change between two given metric-potential pairs exists sometimes, though not very often, and if it does exist, it amounts to an explicit description of one Kähler metric in terms of the other. For instance, as shown at the end of Section 13, the one-point blow-up of $\mathbb{C}\mathrm{P}^2$ admits a biconformal change of a more general type, introduced in Section 12, leading from the Kähler-Ricci soliton constructed by Koiso [13] and, independently, Cao [2], to one of Calabi's extremal Kähler metrics [1], conformal to the non-Kähler, Einstein metric found by Page [17].

2 Preliminaries

All manifolds and Riemannian metrics are assumed to be of class C^∞. A manifold is by definition connected.

Given a Riemannian manifold (M,g), the *divergence* of a vector field w or a bundle morphism $A : TM \to TM$ is defined as usual, by $\operatorname{div} w = \operatorname{tr}\nabla w$ and $\operatorname{div} A = \xi$, for the 1-form ξ sending any vector field w to the function $\xi(w) = \operatorname{div}(Aw) - \operatorname{tr}(A\nabla w)$. The inner product $\langle\,,\rangle$ of 2-forms is characterized by $2\langle\sigma,\sigma\rangle = -\operatorname{tr}A^2$, where $A : TM \to TM$ is the bundle morphism with $g(Aw,\cdot) = \sigma(w,\cdot)$ for all vector fields w. In coordinates, $\operatorname{div} w = w^j_{\,,j}$, $(\operatorname{div} A)_j = A^k_{j,k}$ and $2\langle\sigma,\sigma\rangle = \sigma_{jk}\sigma^{jk}$. Also, for any 2-form σ and vector fields w,w',

$$\langle\sigma,\alpha\wedge\alpha'\rangle = \sigma(w,w'), \qquad \text{where } \ \alpha = g(w,\cdot), \ \ \alpha' = g(w',\cdot). \tag{2.1}$$

Lemma 1 *Suppose that* $\delta,\varepsilon \in (0,\infty)$ *and* $\tau,\psi : (-\delta,\varepsilon) \to \mathbb{R}$ *are* C^∞ *functions such that, if the dot stands for the derivative with respect to the variable* $t \in (-\varepsilon,\varepsilon)$*,*

a. $\dot\tau(0) = 0 \neq \ddot\tau(0)$ *and* $\dot\tau\ddot\tau \neq 0$ *everywhere in* $(-\delta,0)\cup(0,\varepsilon)$*,*
b. $\tau : (-\delta,0) \to \mathbb{R}$ *and* $\tau : (0,\varepsilon) \to \mathbb{R}$ *both have the same range* $\mathbf{I} \subset \mathbb{R}$*,*
c. $\psi(t) = G(\tau(t))$ *for some function* $G : \mathbf{I} \to \mathbb{R}$ *and all* $t \in (-\delta,0)\cup(0,\varepsilon)$*.*

Then G *has a* C^∞ *extension to the half-open interval* $\mathbf{I}\cup\tau(0)$*.*

Proof One can view τ as a new C^∞ coordinate on both $(-\varepsilon,0)$ and $(0,\varepsilon)$. Thus, $G : \mathbf{I} \to \mathbb{R}$ is of class C^∞, and so are all the derivatives $d^kG/d\tau^k$ treated as functions on $\mathbf{I}$. Let us prove by induction on $k \geq 0$ that $d^kG/d\tau^k$ is a C^∞ function of the variable $t \in (-\varepsilon,\varepsilon)$ (and, in particular, has a limit at the endpoint $\tau(0)$ of $\mathbf{I}$). The induction step: by (b) – (c), $\chi = d^kG/d\tau^k$ treated as a C^∞ function on $(-\delta,\varepsilon)$ has the same range on $(-\delta,0)$ as on $(0,\varepsilon)$, which also remains true when δ,ε are replaced with suitably related smaller positive numbers δ',ε', and such δ',ε' may be chosen arbitrarily close to 0. Hence $\dot\chi(0) = 0$. As $\dot\tau$ is a new C^∞ coordinate on $(-\varepsilon,\varepsilon)$, vanishing at 0, smooth functions on $(-\varepsilon,\varepsilon)$ that vanish at 0 are smoothly divisible by $\dot\tau$. Consequently, $d^{k+1}G/d\tau^{k+1} = d\chi/d\tau = \dot\chi/\dot\tau$ is a smooth function of $t \in (-\varepsilon,\varepsilon)$. □

Remark 1 Let F be a C^∞ function $U\times D \to \mathbb{R}$, where $U \subset \mathbb{R}^k$ is an open set and $D \subset \mathbb{C}$ is a disk centered at 0. If $F(y,0) = 0$ and $F(y,zq) = F(y,z)$ for all $(y,z,q) \in \mathbb{R}^k\times\mathbb{C}^2$ with $|q| = 1$, then $F(y,z) = |z|^2h(y,z)$ for some C^∞ function $h : U\times D \to \mathbb{R}$. If, in addition, the Hessian of F is nonzero everywhere in $U\times\{0\}$, then so is h.

In fact, for $r \in \mathbb{R}$ close to 0, the function $(y,r) \mapsto F(y,r)$ is smooth and vanishes when $r = 0$, so that it is smoothly divisible by r (due to the first-order Taylor formula). The same applies to $(y,r) \mapsto F(y,r)/r$. The last claim holds since, on $U\times\{0\}$, the Hessian of F equals $2h$ times the Euclidean metric of $\mathbb{C}$.

We will use the *connectivity lemma* for Morse-Bott functions τ on compact manifolds M, stating that, if the positive and negative indices of the Hessian of τ at every critical point are both different from 1, then the τ-preimage of every real number is connected. See [16, Lemma 3.46 on p. 124].

3 Kähler manifolds

Let M be a complex manifold. Its complex-structure tensor is always denoted by J. Given a real 1-form μ on M, the symbol μJ stands for the 1-form $J^*\mu$, so that

$$(\mu J)_x = \mu_x \circ J_x : T_x M \to \mathrm{I\!R} \tag{3.1}$$

at any point $x \in M$. If g is a Kähler metric on M, the Kähler form of g is $\omega = g(J\cdot,\cdot)$, while ∇ denotes both the Levi-Civita connection of g and the g-gradient. Real-holomorphic vector fields on M are the sections v of TM such that $\pounds_v J = 0$, which, for any fixed Kähler metric g on M, is equivalent to $[J,\nabla v] = 0$. The commutator $[\,,\,]$ is applied here to J and ∇v treated as vector-bundle morphisms $TM \to TM$, the latter acting on vector fields w by $(\nabla v)w = \nabla_w v$. The fact that

$$2i\partial\overline{\partial}\psi = (\nabla d\psi)(J\cdot,\cdot) - (\nabla d\psi)(\cdot,J\cdot) \tag{3.2}$$

for any Kähler metric g on M, and any C^2 function $\psi : M \to \mathrm{I\!R}$, will be used below and in Section 14. In the following (well-known) lemmas, $\imath_v b = b(v,\cdot,\ldots,\cdot)$ for vector fields v and covariant tensor fields b, while $[d(d_v\psi)]J$ is defined by (3.1) with $\mu = d(d_v\psi)$.

Lemma 2 *In a Kähler manifold (M,g) one has $2\imath_v(i\partial\overline{\partial}\psi) = d(d_{Jv}\psi) - [d(d_v\psi)]J$ for any real-holomorphic vector field v on M and any C^2 function $\psi : M \to \mathrm{I\!R}$.*

Proof Let us set $u = Jv$ and $\mu = 2\imath_v(i\partial\overline{\partial}\psi)$. By (3.2), $\mu J = [\imath_u(\nabla d\psi)]J + \imath_v(\nabla d\psi)$. In coordinates, this reads $(\mu J)_k = \psi_{,pq}v^s J_s^p J_k^q + \psi_{,sk}v^s$. However, $\psi_{,pq}v^s J_s^p = (\psi_{,p}v^s)_{,q}J_s^p - \psi_{,p}v^s_{,q}J_s^p$, while, as v is real-holomorphic, $v^s_{,q}J_s^p = J_q^s v^p_{,s}$. Hence $(\mu J)_k = (\psi_{,p}v^s J_s^p)_{,q}J_k^q + \psi_{,s}v^s_{,k} + \psi_{,sk}v^s = \{[d(d_u\psi)]J + d(d_v\psi)\}_k$. □

Lemma 3 *Let there be given C^∞ functions $\tau,\psi : M \to \mathrm{I\!R}$ on a complex manifold (M,g) and two Kähler metrics $g,\hat{g}$ on M such that the Kähler forms ω of g and $\hat{\omega}$ of $\hat{g}$ are related by $\hat{\omega} = \omega + 2i\partial\overline{\partial}\psi$. If the the g-gradient v of τ is real-holomorphic and $d_{Jv}\psi = 0$, then v is also the $\hat{g}$-gradient of $\hat{\tau} = \tau + d_v\psi$.*

Proof By (3.1), $(\imath_v\omega)J = \omega(v,J\cdot) = g(Jv,J\cdot) = g(v,\cdot) = d\tau$. Consequently, $(\imath_v\hat{\omega})J = \hat{g}(v,\cdot)$. Lemma 2 now yields $\hat{g}(v,\cdot) = (\imath_v\hat{\omega})J = d(\tau + d_v\psi)$. □

Here is another well-known lemma.

Lemma 4 *A differentiable 2-form η on a Kähler surface (M,g) is closed if and only if $d\langle\omega,\eta\rangle = -\operatorname{div} JA$, where J is the complex structure, ω denotes the Kähler form of g, and $A : TM \to TM$ is the bundle morphism with $g(Av,\cdot) = \eta(v,\cdot)$ for all vector fields v.*

Proof The operator $\imath_\omega$ sending every differential 3-form ζ on M to the 1-form $\imath_\omega\zeta$ such that $(\imath_\omega\zeta)(v) = \langle\omega,\zeta(v,\cdot,\cdot)\rangle$ for all vector fields v is, by dimensional reasons, an isomorphism, since $\imath_\omega(\xi\wedge\omega) = \xi$ for any 1-form ξ. The assertion now follows from the local-coordinate formula $2(\imath_\omega d\eta)_j = \omega^{kl}(\eta_{kl,j} + \eta_{lj,k} + \eta_{jk,l})$. □

4 Killing potentials

Let τ be a Killing potential on a compact Kähler manifold (M,g). As usual, this means that τ is a C^∞ function $M \to \mathrm{I\!R}$ and $J(\nabla\tau)$ is a Killing field on (M,g). In other words, $\nabla\tau$ is a real-holomorphic vector field, or, equivalently, the 2-tensor field $\nabla d\tau$ is Hermitian. Using the notation

$$v = \nabla\tau, \qquad u = Jv, \qquad \xi = g(u,\cdot\,), \qquad Q = g(v,v), \qquad Y = \Delta\tau, \tag{4.1}$$

here and and throughout the paper, one then has

$$\text{a)}\;\; 2\,\mathrm{Ric}(v,\cdot\,) = -dY, \qquad \text{b)}\;\; 2\nabla d\tau(v,\cdot\,) = dQ. \tag{4.2}$$

In fact, (4.2.b) holds for any C^∞ function τ on a Riemannian manifold, provided that v and Q are still given by (4.1). The identity (4.2.a) is well known, cf. [1].

Lemma 5 *Let there be given a Kähler surface (M,g) with the Kähler form ω, a Killing potential τ on (M,g), and C^∞ functions $f,\theta : M \to \mathrm{I\!R}$. Then, in the notation of* (4.1)*, the 2-form $\eta = f\omega + \theta\,\xi\wedge d\tau$ is closed if and only if $d(f - Q\theta) + (d_v\theta + \theta Y)\,d\tau + (d_u\theta)\xi = 0$.*

Proof For A corresponding to η as in Lemma 4, $JA = \theta(\xi\otimes u + d\tau\otimes v) - f$, where f stands for f times Id. Also, $\mathrm{div}(\xi\otimes u) = \nabla_u\xi$ and $\mathrm{div}(d\tau\otimes v) = Y d\tau + \nabla_v d\tau$, so that $\mathrm{div}(\xi\otimes u + d\tau\otimes v) = Y d\tau$. (Note that $\nabla_u\xi = g(\nabla_u u,\cdot\,)$, which is the opposite of $\nabla_v d\tau = g(\nabla_v v,\cdot\,)$, as $\nabla_u u = \nabla_u(Jv) = J\nabla_u v = J\nabla_v u = \nabla_v(Ju) = -\nabla_v v$.) Thus, $\mathrm{div}JA = \theta Y d\tau + (d_u\theta)\xi + (d_v\theta)d\tau - df$. As $\langle\omega,\eta\rangle = 2f - Q\theta$ by (2.1), Lemma 4 yields our claim. $\square$

Given a nonconstant Killing potential τ on a compact Kähler manifold (M,g) and a C^∞ function $\psi : M \to \mathrm{I\!R}$, one may refer to ψ as *a C^∞ function of τ* if $\psi = G(\tau)$ for some C^∞ function $G : [\tau_{\min},\tau_{\max}] \to \mathrm{I\!R}$. Note that

$$\psi \text{ is a } C^\infty \text{ function of } \tau \text{ if and only if } d\psi\wedge d\tau = 0. \tag{4.3}$$

In fact, let $M' \subset M$ be the open set on which $d\tau \neq 0$. It is well-known that M' is connected and dense in M, and that Killing potentials are Morse-Bott functions (cf. [8, Remark 2.3(ii) and Example 11.1]). The relation $d\psi\wedge d\tau = 0$ clearly means that ψ restricted to M' is, locally, a C^∞ function of τ. Consequently, the word 'locally' can be dropped, since the connectivity lemma, mentioned at the end of Section 2, now implies connectedness of the τ-preimages of all real numbers. Also, due to the Morse-Bott property of τ, its critical manifolds are compact and isolated from one another, so that their number is finite, and, as τ is constant on each of them, τ has a finite set Γ of critical values. Next, we show that the function $G : [\tau_{\min},\tau_{\max}] \smallsetminus \Gamma \to \mathrm{I\!R}$ with $\psi = G(\tau)$ on $M \smallsetminus \tau^{-1}(\Gamma)$ has a C^∞ extension to $[\tau_{\min},\tau_{\max}]$. To this end, we fix $\tau_* \in \Gamma$ and a point $x \in M$ such that $\tau(x) = \tau_*$ and $d\tau_x = 0$. The nullspace of the Hessian of τ at x coincides with the tangent space at x of the critical manifold of τ containing x (cf. [8, Remark 2.3(iii-d)]). One may thus choose $\delta,\varepsilon \in (0,\infty)$ and a C^∞ curve $(-\varepsilon,\varepsilon) \ni t \mapsto x(t)$ in M with $x(0) = x$, for which the assumptions, and hence the conclusion, of Lemma 1 are satisfied if one lets the symbols τ and ψ stand for the functions $\tau(x(t))$ and $\psi(x(t))$ of the variable t.

5 Special biconformal changes

Two Riemannian metrics $g,\hat g$ on a manifold M are sometimes referred to as *biconformal* [9, 10] if there exist vector subbundles $\mathscr{V}$ and $\mathscr{H}$ of TM with $TM = \mathscr{V} \oplus \mathscr{H}$ such that, for some positive C^∞ functions $f,\chi : M \to \mathrm{I\!R}$,

$$\hat g = fg \ \text{ on } \mathscr{H}, \qquad \hat g = \chi g \ \text{ on } \mathscr{V}, \qquad g(\mathscr{H},\mathscr{V}) = \hat g(\mathscr{H},\mathscr{V}) = \{0\}. \tag{5.1}$$

This kind of biconformality is of little interest in the case of two Kähler metrics on a given complex surface M, since, locally, in a dense open subset of M, (5.1) always holds, due to the existence of eigenspace bundles of $\hat g$ relative to g.

The special biconformal changes defined in the Introduction represent a particular case of the situation described above. Namely, relation (1.1.i), in the open set $M' \subset M$ where $d\tau \neq 0$, amounts to (5.1) with $\chi = f - Q\theta$, $\mathscr{V} = \mathrm{Span}_{\mathrm{I\!R}}(v,u)$ and $\mathscr{H} = \mathscr{V}^\perp$ (notation of (4.1)), so that $TM' = \mathscr{V} \oplus \mathscr{H}$, while

$$f \text{ and } f - Q\theta \text{ are the eigenvalue functions of } \hat g \text{ relative to } g. \tag{5.2}$$

Given a Kähler manifold (M,g) with a nonconstant Killing potential τ and C^∞ functions $f,\theta : M \to \mathrm{I\!R}$, let a twice-covariant symmetric tensor field $\hat g$ on M be Hermitian relative to the underlying complex structure J. Thus, $\hat\omega = \hat g(J\,\cdot\,,\,\cdot\,)$ is a 2-form. Then (1.1.i) holds, for $\xi = g(J(\nabla\tau),\,\cdot\,)$, if and only if

$$\hat\omega \;=\; f\omega \;+\; \theta\,\xi \wedge d\tau. \tag{5.3}$$

A nontrivial special biconformal change (1.1) of a metric-potential pair (g,τ), if it exists, is never unique. Namely, it gives rise to a three-parameter family of such changes, leading to the metric-potential pairs $(p\hat g + qg, p\hat\tau + q\tau + s)$, with any constants p,q,s such that $p\hat g + qg$ is positive definite (for instance, p,q may both be positive). In fact, (1.1) holds if one replaces $\hat g, f, \theta$ and $\hat\tau$ by $g' = p\hat g + qg, pf + q, p\theta$ and $\tau' = p\hat\tau + q\tau + s$. (Specifically, (1.1.ii) for τ' is immediate as $\imath_v g' = p\imath_v\hat g + q\imath_v g$, for $v = \nabla\tau$.) In addition, g' is a Kähler metric, since the 2-form $\omega' = g'(J\,\cdot\,,\,\cdot\,)$ equals $p\hat\omega + q\omega$, and so $d\omega' = 0$.

The existence of a special biconformal change (1.1) for a pair (g,τ), with prescribed f and θ, obviously amounts to requiring $\hat g$ given by (1.1.i) to be a Kähler metric such that $v = \nabla\tau$ is the $\hat g$-gradient of some C^∞ function $\hat\tau$. The following lemma describes a condition equivalent to this in the case of Kähler surfaces; a similar result, valid in all complex dimensions, was obtained by Ganchev and Mihova [10, the text following Definition 4.1].

Lemma 6 *Given a metric-potential pair* (g,τ) *on a compact complex surface* M *and* C^∞ *functions* $f,\theta,\hat\tau : M \to \mathrm{I\!R}$*, one has* (1.1) *for a metric-potential pair of the form* $(\hat g,\hat\tau)$ *on* M *if and only if, in the notation of* (4.1)*,*

i. $\hat\tau = P(\tau)$ *for some* C^∞ *function* $P : [\tau_{\min},\tau_{\max}] \to \mathrm{I\!R}$*,*
ii. $f - Q\theta = H(\tau)$*, with* $H = dP/d\tau$*,*
iii. $d_u\theta = 0$ *and* $d_v\theta + \theta Y = -H'(\tau)$*, where* $H' = dH/d\tau$*,*
iv. $f > \max(Q\theta, 0)$.

Sufficiency of (i) – (iv) *remains true without the compactness hypothesis.*

Proof Necessity: first, (1.1) implies (i). In fact, $d\hat\tau = \imath_v\hat g = (f - Q\theta)d\tau$ in view of (1.1), so that (4.3) gives (i) and (ii). Next, by Lemma 5, (5.3) and (ii), $[d_v\theta + \theta Y + H'(\tau)]d\tau + (d_u\theta)\xi = 0$, and (iii) follows since ξ is orthogonal to $d\tau$. Finally, (iv) amounts to positive definiteness of $\hat g$, cf. (5.2).

Sufficiency: conditions (ii) – (iv) combined with Lemma 5 and (5.3) show that $\hat{g}$ defined by (1.1.i) is a Kähler metric. Also, in view of (1.1.i) and (4.1), $\imath_v\hat{g} = (f - Q\theta)d\tau$, which, according to (i) and (ii), equals $d\hat{\tau}$. This proves (1.1.ii). □

Remark 2 For any metric-potential pair (g,τ) on a Kähler surface, nontrivial special biconformal changes of (g,τ) exist locally, at points where $d\tau \neq 0$, and the C^∞ function P of the variable τ, such that $\hat{\tau} = P(\tau)$ for the resulting pair $(\hat{g},\hat{\tau})$, may be prescribed arbitrarily, as long as $dP/d\tau > 0$. (Cf. Lemma 6(i)-(ii) and (5.2).) This is clear from the final clause of Lemma 6, since conditions (iii) and (iv) in Lemma 6 can be realized by solving an ordinary differential equation, with suitably chosen initial data, along each integral curve of v.

6 One general construction

The following theorem provides a method of constructing examples of nontrivial special biconformal changes in complex dimension 2. In the next four sections this method will be applied to four specific classes of Kähler surface metrics.

Theorem 1 *Given a nonconstant Killing potential* τ *on a compact Kähler surface* (M,g), *the following two conditions are equivalent*:

a. (g,τ) *admits a nontrivial special biconformal change as in* (1.1)*, with* θ *which is a* C^∞ *function of* τ,
b. $\Delta[S(\tau)] = -H'(\tau)$ *for some nonconstant* C^∞ *functions* $S,H : [\tau_{\min},\tau_{\max}] \to \mathrm{I\!R}$ *and* $H' = dH/d\tau$.

Then, up to additive constants, H *in* (b) *coincides with* H *appearing in Lemma* 6*, while* θ *in* (a) *and* S *in* (b) *are related by* $\theta = dS/d\tau$.

Proof Assuming (a) and using Lemma 6(iii), one obtains (b) for any S with $dS/d\tau = \theta$. Conversely, (b) easily implies condition (iii) in Lemma 6(iii) for $\theta = dS/d\tau$. Adding a suitable constant to H, one also gets (iv) in Lemma 6 for P, f and $\hat{\tau}$ chosen so as to satisfy (ii) and (i) in Lemma 6. □

7 Kähler-Einstein surfaces

On any compact Kähler-Einstein manifold (M,g) such that the constant λ with $\mathrm{Ric} = \lambda g$ is positive and M admits a nontrivial holomorphic vector field, there exists a nonconstant Killing potential. In fact, by Matsushima's theorem [15], $\mathfrak{h} = \mathfrak{g} \oplus J\mathfrak{g}$ for the spaces $\mathfrak{h}$ and $\mathfrak{g}$ of all real-holomorphic vector fields and, respectively, all real-holomorphic gradients, where $J\mathfrak{g}$ consists of all Killing fields on (M,g).

Using Theorem 1 one sees that *a nontrivial special biconformal change of* (g,τ) *exists whenever* g *is a Kähler-Einstein metric with positive Einstein constant* λ *on a compact complex surface* M *and* τ *is a nonconstant Killing potential on* (M,g). Namely, by (4.2.a), $\Delta\tau = a - 2\lambda\tau$ for some $a \in \mathrm{I\!R}$. Thus, condition (b) in Theorem 1 holds for $S(\tau) = \tau$ and $H(\tau) = \lambda\tau^2 - a\tau$.

8 Kähler-Ricci solitons

A *Ricci soliton* [11] is a Riemannian manifold (M,g) with the property that, for some constant λ, the tensor field $\lambda g - \mathrm{Ric}$ is the Lie derivative of g in the direction of some vector field. Perelman [18, Remark 3.2] proved that, if M is compact, such a vector field must be the sum of a Killing field and a gradient, or, equivalently, there exists a C^∞ function $\tau : M \to \mathrm{I\!R}$ with

$$\nabla d\tau + \mathrm{Ric} = \lambda g \quad \text{for a constant } \lambda\,. \tag{8.1}$$

If a C^∞ function $\tau : M \to \mathrm{I\!R}$ satisfies (8.1), then [4, p. 201]

$$c = \Delta\tau - g(\nabla\tau,\nabla\tau) + 2\lambda\tau \tag{8.2}$$

is a constant. In fact, adopting the notation of (4.1) except for the formulae involving J, and applying to both sides of (8.1) either $-2\,\mathrm{div}$, or tr_g followed by d, or, finally, $2\iota_v$, one obtains $-2dY - 2\,\mathrm{Ric}(v,\cdot\,) - d\mathrm{s} = 0$, $dY + d\mathrm{s} = 0$, and, by (4.2.b), $dQ + 2\,\mathrm{Ric}(v,\cdot\,) = 2\lambda\,d\tau$. (Here s denotes the scalar curvature, while $\mathrm{div}\nabla d\tau = dY + \mathrm{Ric}(v,\cdot\,)$ by the Bochner identity, which has the coordinate form $v^k{}_{,jk} = v^k{}_{,kj} + R_{jk}v^k$, and $\mathrm{div}\,\mathrm{Ric} = d\mathrm{s}/2$ in view of the Bianchi identity for the Ricci tensor.) Adding these three equalities produces the relation $d[\Delta\tau - g(\nabla\tau,\nabla\tau) + 2\lambda\tau] = 0$.

By a *Kähler-Ricci soliton* one means a Ricci soliton which is at the same time a Kähler manifold [19,20]. A function τ with (8.1) then is a Killing potential. (Since g and Ric are Hermitian, so must be $\nabla d\tau$ as well.) Also, (8.2) can be rewritten as $\Delta e^{-\tau} = (2\lambda\tau - c)e^{-\tau}$. Thus, by Theorem 1, *for every non-Einstein compact Kähler-Ricci soliton* (M,g) *of complex dimension* 2*, the pair* (g,τ)*, where* τ *is a function satisfying* (8.1)*, admits a nontrivial special biconformal change*. Specifically, condition (b) in Theorem 1 then holds for $S(\tau) = e^{-\tau}$ and $H(\tau) = [2\lambda(\tau+1) - c]e^{-\tau}$.

9 Conformally-Einstein Kähler surfaces

Let (M,g) be a conformally-Einstein, non-Einstein compact Kähler surface. The scalar curvature s of g then is a nonconstant Killing potential, and so g is an extremal Kähler metric [1], while $\mathrm{s} > 0$ everywhere and $\mathrm{s}^3 + 6\mathrm{s}Y - 12Q = 12c$ for some constant $c > 0$ (notation of (4.1)). See [5, Prop. 4 on p. 419 and Theorem 2 on p. 428], [14, Lemma 3 on p. 169].

Conformally-Einstein, non-Einstein Kähler metrics are known to exist on both the one-point and two-point blow-ups of $\mathbb{C}\mathrm{P}^2$. The former, found by Calabi [1], is conformal to the Page metric [17], for reasons given in [5, the top of p. 430]; the existence of the latter is a result of Chen, LeBrun and Weber [3].

Theorem 1 implies that *for every conformally-Einstein, non-Einstein compact Kähler surface* (M,g)*, the pair* (g,τ)*, with* $\tau = \mathrm{s}$*, admits a nontrivial special biconformal change*. In fact, the equality $\mathrm{s}^3 + 6\mathrm{s}Y - 12Q = 12c$ yields condition (b) in Theorem 1 for $\tau = \mathrm{s}$ and $S(\tau) = -\tau^{-1}$, with $H(\tau) = c\,\tau^{-2} + \tau/6$. The existence of such a biconformal change in this case was first discovered by LeBrun [14, p. 171, the end of the proof of Prop. 2], who proved that, with ρ and ω standing for the Ricci and Kähler forms of g,

$$\rho + 2i\partial\overline{\partial}\log\mathrm{s} = [(Q+c)\mathrm{s}^{-2} + \mathrm{s}/6]\,\omega + \mathrm{s}^{-2}\xi\wedge d\mathrm{s} \tag{9.1}$$

(notation of (4.1). Equality (9.1) easily implies (1.1.i) with a new Kähler metric $\hat{g}$. Namely, the right-hand side of (9.1) coincides with $\hat{\omega}$ in (5.3), for suitable f and θ, while the Hermitian 2-tensor field $\hat{g}$ characterized by $\hat{\omega} = \hat{g}(J\cdot,\cdot\,)$ is positive definite, cf. Lemma 6(iv); at the same time, the left-hand side of (9.1) is a closed 2-form.

10 Special Kähler-Ricci potentials

A *special Kähler-Ricci potential* [7] on a compact Kähler surface (M,g) is any nonconstant Killing potential $\tau : M \to \mathbb{R}$ such that both $Q = g(\nabla\tau,\nabla\tau)$ and $Y = \Delta\tau$ are C^∞ functions of τ. This definition, although different from the one given in [7, § 7] for all complex dimensions $m \geq 2$, is equivalent to it when $m = 2$, as a consequence of (4.2) and (4.3). Special Kähler-Ricci potentials on compact Kähler manifolds (M,g) of any given complex dimension $m \geq 2$ were classified in [8]. They turn out to be biholomorphic to $\mathbb{C}\mathrm{P}^m$ or to holomorphic $\mathbb{C}\mathrm{P}^1$ bundles over complex manifolds admitting Kähler-Einstein metrics. If $m \geq 3$, there are other natural conditions which imply the existence of a Kähler-Ricci potential [7, Corollary 9.3], [12, Theorem 6.4].

If (M,g) is a compact Kähler surface with a special Kähler-Ricci potential τ, *any* nonconstant C^∞ function $S : [\tau_{\min},\tau_{\max}] \to \mathbb{R}$ satisfies condition (b) in Theorem 1. Consequently, by Theorem 1, *the pair* (g,τ) *then admits a nontrivial special biconformal change* (1.1), *in which* θ *may be any prescribed* C^∞ *function of* τ *other than the zero function.*

11 Geodesic gradients

We say that a nonconstant Killing potential τ on a compact Kähler manifold (M,g) has a *geodesic gradient* if all the integral curves of $\nabla\tau$ are reparametrized geodesics. By (4.3), this amounts to requiring that $Q = g(\nabla\tau,\nabla\tau)$ be a C^∞ function of τ, since (4.2.b) gives $2\nabla_v v = \nabla Q$ (notation of (4.1)).

Thus, every special Kähler-Ricci potential (Section 10) has a geodesic gradient. Further examples, which are not special Kähler-Ricci potentials, are described in the Appendix.

Theorem 2 *For every nonconstant Killing potential* τ *with a geodesic gradient on a compact Kähler surface* (M,g)*, other than a special Kähler-Ricci potential, there exists a nontrivial special biconformal change* (1.1) *of the pair* (g,τ)*, for which H, defined in Lemma* 6*, can be, up to an additive constant, any prescribed nonconstant* C^∞ *function of the variable* $\tau \in [\tau_{\min},\tau_{\max}]$ *such that* $H' = dH/d\tau$ *is* L^2*-orthogonal to linear functions of* τ*.*

For a proof, see the final paragraph of the Appendix.

12 Biconformal changes defined on an open submanifold

The last five sections described examples of nontrivial special biconformal changes that naturally arise in certain classes of compact Kähler surfaces. As we will see below and in Section 14, there are also circumstances in which, for a given metric-potential pair (g,τ) on a compact complex surface M, one naturally obtains a nontrivial special biconformal change (1.1) of (g,τ) restricted to the dense open submanifold M' characterized by the condition $d\tau \neq 0$, while the functions $f, Q\theta, \hat{\tau}$ in (1.1), cf. (4.1), and the metric $\hat{g}$, all have C^∞ extensions to M. The only difference between this case and the standard one (defined in the Introduction) is that θ, unlike $Q\theta$, may now fail to have a C^∞ extension to M.

To provide an example of such a situation, we let M stand either for $\mathbb{C}\mathrm{P}^2$ or for the one-point blow-up of $\mathbb{C}\mathrm{P}^2$, so that M is a simply connected compact complex surface with an effective action of $\mathrm{U}(2)$ by biholomorphisms. For any $\mathrm{U}(2)$-invariant Kähler metric g on M, a fixed vector field u generating the action of the center $\mathrm{U}(1) \subset \mathrm{U}(2)$ is a $\mathrm{U}(2)$-invariant g-Killing field. Thus, $u = J(\nabla\tau)$ for some nonconstant Killing potential τ on (M,g). (Cf.

[7, Lemma 5.3].) One may say that such τ is *associated with* g. As the principal orbits of the $\mathrm{U}(2)$ action are three-dimensional, every $\mathrm{U}(2)$-invariant C^∞ function $M \to \mathbb{R}$ is, by (4.3), a C^∞ function of τ. Applied to the functions $Q = g(\nabla\tau, \nabla\tau)$ and $Y = \Delta\tau$, this shows that τ then is a special Kähler-Ricci potential on (M, g), cf. Section 10.

Lemma 7 *For M and $M' \subset M$ as above, any $\mathrm{U}(2)$-invariant Kähler metrics $g, \hat{g}$ on M, and nonconstant Killing potentials $\tau, \hat{\tau}$ associated with them, the pair $(\hat{g}, \hat{\tau})$ restricted to M' arises from (g, τ) by a biconformal change* (1.1). *If, in addition, M is the one-point blow-up of $\mathbb{C}\mathrm{P}^2$, then the functions f and $Q\theta$ in* (1.1)*, as well as the corresponding eigenspace bundles $\mathscr{V}$ and $\mathscr{H}$, introduced in Section* 5*, all have C^∞ extensions to M such that $f > \max(Q\theta, 0)$ on M, cf.* (4.1).

Proof Let us denote by $\mathscr{V}$ the complex-line subbundle of TM', spanned by $v = \nabla\tau$ (that is, by $u = Jv$). The g-orthogonal complement $\mathscr{H} = \mathscr{V}^\perp$ of $\mathscr{V}$ in TM', relative to any $\mathrm{U}(2)$-invariant Kähler metric g, does not depend on the choice of such a metric. In fact, being g-orthogonal to $v = -Ju$, the subbundle $\mathscr{H}$ is contained in the real three-dimensional subbundle tangent to the orbits of the $\mathrm{U}(2)$ action. (As v is the g-gradient of the $\mathrm{U}(2)$-invariant function τ, it is orthogonal to the orbits.) Independence of $\mathscr{H}$ from g now follows since a real three-dimensional subspace of a complex two-dimensional vector space contains only one complex subspace of complex dimension 1.

For any two $\mathrm{U}(2)$-invariant Kähler metrics $g, \hat{g}$, one clearly has (5.1) with some positive C^∞ functions $f, \chi : M' \to \mathbb{R}$. Defining θ, on M', by $\chi = f - Q\theta$, we now obtain (1.1.i) on M', while (1.1.ii) is obvious as $\hat{\nabla}\hat{\tau} = -Ju = \nabla\tau$. Finally, if M is the one-point blow-up of $\mathbb{C}\mathrm{P}^2$, then $\mathscr{V}$ (and hence $\mathscr{H}$ as well) has a C^∞ extension to a subbundle of TM, due to the fact that M is a holomorphic $\mathbb{C}\mathrm{P}^1$ bundle over $\mathbb{C}\mathrm{P}^1$ and $\mathscr{V}$ is tangent to the fibres. □

13 More on $\mathrm{U}(2)$-invariant Kähler metrics

Let us consider the one-point blow-up M of $\mathbb{C}\mathrm{P}^2$, with the effective action of $\mathrm{U}(2)$ by biholomorphisms. By *central automorphisms* of M we mean transformations that belong the holomorphic action of $\mathbb{C}^*$ on M, generated by the action of the center $\mathrm{U}(1) \subset \mathrm{U}(2)$. They all commute with the action of $\mathrm{U}(2)$. Thus, the pullback of any $\mathrm{U}(2)$-invariant Kähler metric on M under any central automorphism of M is again a $\mathrm{U}(2)$-invariant Kähler metric on M.

Suppose, in addition, that $g, \hat{g}$ are $\mathrm{U}(2)$-invariant Kähler metrics on M, and let $\tau, \hat{\tau}$ denote the nonconstant Killing potentials associated with them (Section 12). According to the final clause Lemma 7 and (5.2), one has (5.1) with positive C^∞ functions $f, \chi : M \to \mathbb{R}$, where $\chi = f - Q\theta$. Both f and χ are constant on either of the two exceptional orbits $\Sigma^\pm$ of the $\mathrm{U}(2)$ action, biholomorphic to $\mathbb{C}\mathrm{P}^1$. (In fact, f and χ are $\mathrm{U}(2)$-invariant, since so are both metrics.) Let the constants $f^\pm$ and $\chi^\pm$ be the values of f and χ on $\Sigma^\pm$. The positive real number

$$\mathrm{d}(g, \hat{g}) = \chi^+\chi^-/(f^+f^-) \tag{13.1}$$

is an invariant which remains unchanged when one of the metrics $g, \hat{g}$ is replaced with its pullback under any central automorphism of M (since the pair (χ^+, χ^-) then is replaced by $(r\chi^+, r^{-1}\chi^-)$ for some $r \in (0, \infty)$). On the other hand, $\mathrm{d}(g, \hat{g}) = 1$ when $(\hat{g}, \hat{\tau})$ arises from (g, τ) by a special biconformal change: in fact, $f^\pm = \chi^\pm$, as $\chi = f - Q\theta$ and $Q = 0$ on $\Sigma^\pm$.

This shows that (b) implies (a) in the following proposition.

Proposition 1 *For any $\mathrm{U}(2)$-invariant Kähler metrics $g, \hat{g}$ on the one-point blow-up of $\mathbb{C}\mathrm{P}^2$, and nonconstant Killing potentials $\tau, \hat{\tau}$ associated with them in the sense of Section* 12*, the following two conditions are equivalent*:

a. $\mathrm{d}(g,\hat{g}) = 1$,
b. $(\hat{g},\hat{\tau})$ *arises by a special biconformal change from the pullback of* (g,τ) *under some central automorphism of M.*

Proof It suffices to verify that (a) leads to (b). Because of how $\chi^{\pm}$ change under the action of a central automorphism (see above), one may use a pullback as in (b) to replace χ^{+} with the value f^{+}. As $\mathrm{d}(g,\hat{g}) = 1$, (13.1) now gives $\chi^{-} = f^{-}$ as well. Since $\chi = f - Q\theta$, it follows that $Q\theta = 0$ on $\Sigma^{\pm}$.

In view of the final clause of Lemma 7, the assertion will follow if one shows that θ (and not just $Q\theta$) has a C^{∞} extension to M. To this end, fix a point $x \in \Sigma^{\pm}$ and identify a neighborhood of x in M diffeomorphically with $U \times D$, so that the flow of u consists of the rotations $(y,z) \mapsto (y,zq)$, where $q \in \mathbb{C}$ and $|q| = 1$ (notation of Remark 1, for $k = 2$). According to Remark 1, both Q and $Q\theta$ is smoothly divisible by $|z|^2$, while $Q/|z|^2$ is positive on $U \times \{0\}$, and so $\theta = [(Q\theta)/|z|^2]/(Q/|z|^2)$ is smooth everywhere in $U \times D$. That the Hessian of Q is nonzero everywhere in $\Sigma^{\pm}$ follows since the same is true of the Hessian of τ, cf. [7, Remark 5.4]. Namely, differentiating (4.2.b) one sees that the former Hessian equals twice the square of the latter, if both are identified with morphisms $TM \to TM$ as in the lines preceding (2.1). ☐

Let g and $\hat{g}$ denote the two distinguished U(2)-invariant Kähler metrics on the one-point blow-up of $\mathbb{C}\mathrm{P}^2$, mentioned at the end of the Introduction. According to Lemma 7, the corresponding pairs (g,τ) and $(\hat{g},\hat{\tau})$ arise from each other by the weaker version of a biconformal change, described at the beginning of Section 12. The value of $\mathrm{d}(g,\hat{g})$ in this case is not known; if that value turns out to be 1, a stronger conclusion will be immediate from Proposition 1.

14 Another construction

In contrast with Theorem 1, the following result may lead to biconformal changes of a more general kind, introduced at the beginning of Section 12.

Theorem 3 *Suppose that* τ *is a nonconstant Killing potential on a compact Kähler surface* (M,g) *and a Kähler metric* $\hat{g}$ *on* M *represents the same Kähler cohomology class as* g. *Using the notation of* (4.1)*, let us fix a* C^{∞} *function* $\psi : M \to \mathbb{R}$ *such that the Kähler forms* ω *of* g *and* $\hat{\omega}$ *of* $\hat{g}$ *are related by* $\hat{\omega} = \omega + 2i\partial\overline{\partial}\psi$*, and denote by* M' *the open subset of* M *on which* $d\tau \neq 0$.

If there exists a special biconformal change of (g,τ) *leading to a pair* $(\hat{g},\hat{\tau})$*, for some nonconstant Killing potential* $\hat{\tau}$ *on* $(M,\hat{g})$*, then*

$$d_v\psi \text{ is a } C^{\infty} \text{ function of } \tau \text{ and } d_u\psi = 0. \tag{14.1}$$

Conversely, if (14.1) *holds, then, for some nonconstant Killing potential* $\hat{\tau}$ *on* $(M,\hat{g})$*, the pair* $(\hat{g},\hat{\tau})$*, restricted to* M'*, arises from* (g,τ) *by a special biconformal change on* M'*, and, on* M'*, one has* (1.1) *with*

$$f = \Delta\psi + 1 - d(d_v\psi)/d\tau, \quad \theta = [\Delta\psi - 2d(d_v\psi)/d\tau]/Q, \quad \hat{\tau} = \tau + d_v\psi. \tag{14.2}$$

Proof Let some special biconformal change, applied to (g,τ), produce $(\hat{g},\hat{\tau})$. Since u is a Killing field for both g and $\hat{g}$ (Section 2), the Lie derivatives $\pounds_u\omega$ and $\pounds_u\hat{\omega}$ both vanish, while $\pounds_u$ commutes with $\partial\overline{\partial}$ as u is holomorphic. Thus, $d_u\psi$ lies in the kernel of $\partial\overline{\partial}$ and vanishes at points where $d\tau = 0$, which is only possible if $d_u\psi = 0$ identically. Now, by (1.1.ii) and Lemma 3, $d\hat{\tau} = d\tau + d(d_v\psi)$. Consequently, Lemma 6(i) yields (14.1).

Conversely, let us assume (14.1) and define $f, \theta, \hat{\tau}$ by (14.2). Lemma 3 gives $\hat{g}(v,\cdot) - g(v,\cdot) = d\hat{\tau} - d\tau = d(d_v\psi)$, so that $\hat{g}(v,\cdot)$ equals the function $1 + d(d_v\psi)/d\tau = f - Q\theta$ times $d\tau = g(v,\cdot)$. Hence v is, at every point of M', an eigenvector, for the eigenvalue $f - Q\theta$, of $\hat{g}$ treated, with the aid of g, as a bundle morphism $TM' \to TM'$. The Hermitian 2-tensor field $\pi = (d\tau \otimes d\tau + \xi \otimes \xi)/Q$ (notation of (4.1)) is, obviously, the orthogonal projection onto the complex-line subbundle $\mathscr{V}$ of TM', spanned by v, provided that one identifies π, as in the lines preceding (2.1), with a morphism $A : TM' \to TM'$. Similarly, $g - \pi$ is the orthogonal projection onto $\mathscr{H} = \mathscr{V}^\perp$. The other eigenvalue of $\hat{g}$, corresponding to eigenvectors in $\mathscr{H}$, is f. (In fact, the sum of the two eigenvalues is $\mathrm{tr}_g\hat{g}/2$, which equals $2 + \Delta\psi$, as one sees noting that, by (3.2), the relation $\hat{\omega} = \omega + 2i\partial\overline{\partial}\psi$ amounts to $\hat{g} = g + \nabla d\psi + (\nabla d\psi)(J\cdot, J\cdot)$.) The spectral decomposition $\hat{g} = f(g - \pi) + (f - Q\theta)\pi$ now implies (1.1.i), on M', while Lemma 3 yields (1.1.ii), completing the proof. □

15 The integral obstruction

One can ask whether a nontrivial special biconformal change exists for every metric-potential pair (g,τ) on a compact complex surface M. Here are two comments related to this existence question.

First, due to compactness of M, for such a special biconformal change (1.1), the functions f and θ are uniquely determined by $H : [\tau_{\min}, \tau_{\max}] \to \mathrm{I\!R}$ appearing in Lemma 6. Namely, the zero function is the only C^∞ solution $\phi : M \to \mathrm{I\!R}$ to the homogeneous equation $d_v\phi + \phi Y = 0$, associated with the equation imposed on θ in Lemma 6(iii). In fact, the Killing potential τ has a finite number of critical manifolds (see the lines following (4.3)). At the same time, $Y = \Delta\tau$ is negative at points where $\tau = \tau_{\max}$, since the Hessian $\nabla d\tau$ is negative semidefinite and nonzero at such points (cf. [7, Remark 5.4]). Thus, there exist $\delta, \varepsilon \in (0,\infty)$ with the property that $d\tau \neq 0$ and $Y \leq -\delta$ everywhere in the open set U on which $0 < \tau_{\max} - \tau < \varepsilon$. Any integral curve $[0,\infty) \ni t \mapsto x(t)$ of $v = \nabla\tau$ with $\tau(x(0)) \in U$ lies entirely in U. A solution ϕ to $d_v\phi + \phi Y = 0$, if not identically zero along the integral curve, may be assumed positive everywhere on the curve, and then $d[\log\phi(x(t))]/dt = -Y(x(t)) \geq \delta > 0$, so that $\phi(x(t)) \to \infty$ as $t \to \infty$, contrary to compactness of M. Thus, H determines θ, and hence f, cf. Lemma 6(ii).

Secondly, let us fix a metric-potential pair (g,τ) on a compact complex surface M. In an attempt to find a nontrivial special biconformal change of (g,τ), one might begin by selecting a nonconstant C^∞ function $H : [\tau_{\min}, \tau_{\max}] \to \mathrm{I\!R}$, which would then become the function H corresponding to such a biconformal change as in Lemma 6(ii). (It is nonconstant as we want the change to be nontrivial, cf. the preceding paragraph.) Using the notation of (4.1), we consider an arbitrary maximal integral curve $\mathrm{I\!R} \ni t \mapsto x(t)$ of $v = \nabla\tau$, and set $(\,)^\cdot = d/dt$ (which is applied to functions restricted to the curve). Our initial task is to find conditions on $W = -dH/d\tau$ and $Y = \Delta\tau$, restricted to the curve, necessary and sufficient for the linear ordinary differential equation

$$\dot{\theta} + \theta Y = W \tag{15.1}$$

to have a solution $\theta : \mathrm{I\!R} \to \mathrm{I\!R}$ with finite limits $\theta(\pm\infty)$. Such θ, if it exists, must be unique. This is obvious from the preceding paragraph (which actually shows more: namely, there is at most one solution θ with a finite limit at ∞, and at most one with a finite limit at $-\infty$).

Writing $\int_a^b[h]$ instead of $\int_a^b h(t)\,dt$ whenever $h:\mathrm{I\!R}\to\mathrm{I\!R}$ is a continuous function and $a,b\in\mathrm{I\!R}\cup\{\infty,-\infty\}$, we see that the condition

$$\int_{-\infty}^{\infty}[\zeta W]=0,\ \text{ where }\ \zeta=e^Z\ \text{ and }\ Z:\mathrm{I\!R}\to\mathrm{I\!R}\ \text{ is an antiderivative of } Y, \tag{15.2}$$

is necessary for the existence of a bounded solution θ to (15.1), as well as sufficient for (15.1) to have a solution θ with finite limits $\theta(\pm\infty)$. (Under our assumptions, ζ is always integrable, and hence so is ζW, while the equality in (15.2) remains unchanged if one replaces Z by another antiderivative of Y.) In fact, necessity of (15.2) follows since, multiplying both sides of (15.1) by ζ, we can rewrite (15.1) as $(\zeta\theta)^{\cdot}=\zeta W$, while boundedness of θ implies that $\zeta\theta$ has limits equal to 0 at both ∞ and $-\infty$, so that integrating the last equality we get (15.2). For sufficiency, note that, in view of l'Hospital's rule, $\theta(\pm\infty)=W(\pm\infty)/Y(\pm\infty)$ if one defines the solution θ to (15.1) by $\theta(t)=[\zeta(t)]^{-1}\int_{-\infty}^{t}[\zeta W]$.

The requirement that (15.2) hold along every maximal integral curve of v, as a restriction on the choice of a nonconstant C^∞ function $H:[\tau_{\min},\tau_{\max}]\to\mathrm{I\!R}$ that would become the function H of Lemma 6(ii) for a nontrivial special biconformal change of (g,τ), is therefore necessary for such a biconformal change to exist. How restrictive this requirement is depends on (g,τ). For instance, if τ is a special Kähler-Ricci potential on (M,g), (15.2) states that W, as a function of $\tau\in[\tau_{\min},\tau_{\max}]$, should be L^2-orthogonal to just one specific function of τ. In general, however, the dependence of τ on t varies with the integral curve, so that (15.2) amounts to a much stronger L^2-orthogonality condition.

16 Remarks on the Ricci form and scalar curvature

Let g be a Kähler metric on a complex manifold M of complex dimension $m\ge 2$. The Ricci form of g then is given by $\rho=\mathrm{Ric}(J\,\cdot\,,\,\cdot\,)$. The Ricci forms of two Kähler metrics $g,\hat g$ on M are related by $\hat\rho=\rho-i\partial\overline{\partial}\log\gamma$, where $\hat\omega^{\wedge m}=\gamma\omega^{\wedge m}$, that is, $\gamma:M\to(0,\infty)$ is the ratio of the volume elements. If (g,τ) and $(\hat g,\hat\tau)$ are metric-potential pairs on M, with a special biconformal change (1.1), this yields

$$\hat\rho\;=\;\rho\;-\;(m-1)i\partial\overline{\partial}\log f\;-\;i\partial\overline{\partial}\log(f-Q\theta), \tag{16.1}$$

since $\gamma=(f-Q\theta)f^{m-1}$, cf. (5.2).

When $m=2$, we have $\gamma=(f-Q\theta)f$, as well as $4\rho\wedge\omega=\mathrm{s}\,\omega\wedge\omega$, while $4(i\partial\overline{\partial}\psi)\wedge\omega=(\Delta\psi)\,\omega\wedge\omega$ for any function ψ, and $4\rho\wedge\xi\wedge d\tau=-(Q\mathrm{s}+d_vY)\,\omega\wedge\omega$. The last three relations are direct consequences of the easily-verified formula

$$4\zeta\wedge\alpha\wedge\xi\;=\;[(\mathrm{tr}_{\mathrm{I\!R}}A)g(v,v)-2g(Av,v)]\,\omega\wedge\omega \tag{16.2}$$

valid whenever (M,g) is a Kähler surface, ω stands for its Kähler form, v is a tangent vector field, $A:TM\to TM$ is a bundle morphism commuting with the complex structure tensor J and self-adjoint at every point, while $\zeta=g(JA\,\cdot\,,\,\cdot\,)$, $\alpha=g(v,\,\cdot\,)$ and $\xi=g(Jv,\,\cdot\,)$. Note that (16.2) gives $4\zeta\wedge\omega=(\mathrm{tr}_{\mathrm{I\!R}}A)\,\omega\wedge\omega$. Hence, by (16.1) with $m=2$ and (5.3), the scalar curvatures s of g and $\hat{\mathrm{s}}$ of $\hat g$ satisfy the relation

$$\gamma\hat{\mathrm{s}}\;=\;(f\,-\,Q\theta)(\mathrm{s}-\Delta\log\gamma)\,+\,\theta\,d_v(d_v\log\gamma-Y). \tag{16.3}$$

The equalities $\gamma=(f-Q\theta)f$ and $f-Q\theta=H(\tau)$ (see Lemma 6(ii)) make it possible to rewrite (16.3) in a number of ways.

Appendix: Killing potentials with geodesic gradients

The following construction generalizes that of [8, §5] (in the case $m = 2$), and gives rise to compact Kähler surfaces (M,g) with nonconstant Killing potentials τ, which have geodesic gradients, but need *not* be special Kähler-Ricci potentials.

One begins by fixing a nontrivial closed interval $\mathbf{I} = [\tau_{\min}, \tau_{\max}]$, a constant $a \in (0,\infty)$, a compact Kähler manifold (N,h) of complex dimension 1 (or, equivalently, a closed oriented real surface N endowed with a Riemannian metric h), C^∞ mappings $\mathbf{I} \ni \tau \mapsto Q \in \mathrm{I\!R}$ and $c : N \to \mathrm{I\!RP}^1 \smallsetminus \mathbf{I}$ such that $Q = 0$ at the endpoints $\tau_{\min}, \tau_{\max}$ and $Q > 0$ on the open interval $\mathbf{I}^\circ = (\tau_{\min}, \tau_{\max})$, while $Q' = 2a$ at $\tau_{\min}$ and $Q' = -2a$ at $\tau_{\max}$. The use of the symbol c conforms to the notations of [8, §5], where $c \in \mathrm{I\!R} \smallsetminus \mathbf{I} \subset \mathrm{I\!RP}^1 \smallsetminus \mathbf{I}$ was a real constant. Here and below, $(\;)' = d/d\tau$ and $\mathrm{I\!R}$ is treated as a subset of $\mathrm{I\!RP}^1$ via the usual embedding $\tau \mapsto [\tau, 1]$ (the brackets denoting, this time, the homogeneous coordinates in $\mathrm{I\!RP}^1$). For algebraic operations involving $\infty = [1,0] \in \mathrm{I\!RP}^1$ and elements of $\mathrm{I\!R} \subset \mathrm{I\!RP}^1$, the standard conventions apply; thus, $\infty^{-1} = 0$. Since we need a canonically selected point τ_* in $\mathbf{I}$, we choose τ_* to be the midpoint of $\mathbf{I}$.

In addition, let us fix a C^∞ complex line bundle $\mathscr{L}$ over N along with an Hermitian fibre metric $\langle\,,\rangle$ in $\mathscr{L}$, and a connection in $\mathscr{L}$ making $\langle\,,\rangle$ parallel and having the curvature form $\Omega = -a(\tau_* - c)^{-1}\omega^{(h)}$, where $\omega^{(h)}$ is the Kähler form of (N,h). (Thus, $\Omega = 0$ at points at which $c = \infty$.) The symbol $\mathscr{L}$ also denotes the total space of the bundle, while $\mathscr{V}$ and $\mathscr{H}$ stand for the vertical distribution $\operatorname{Ker} d\pi$ and the horizontal distribution of our connection, π being the projection $\mathscr{L} \to N$. Treating the norm function $r : \mathscr{L} \to [0,\infty)$ of $\langle\,,\rangle$, simultaneously, as an independent variable ranging over $[0,\infty)$, we finally select a C^∞ diffeomorphism $\mathbf{I}^\circ \ni \tau \mapsto r \in (0,\infty)$ such that $dr/d\tau = ar/Q$.

The above data allow us to define a Riemannian metric g on $M' = \mathscr{L} \smallsetminus N$, where N is identified with the zero section, by $g = (\tau_* - c\circ\pi)^{-1}(\tau - c\circ\pi)\pi^*h$ or $g = \pi^*h$ on $\mathscr{H}$, $g = (ar)^{-2}Q\,\mathrm{Re}\,\langle\,,\rangle$ on $\mathscr{V}$, and $g(\mathscr{H},\mathscr{V}) = \{0\}$. On $\mathscr{H}$, the first formula is to be used in the π-preimage of the set in N on which $c \neq \infty$, and the second one on its complement. Note that, due to the fixed diffeomorphic correspondence between the variables τ and r, we may view τ (and hence Q) as a function $M' \to \mathrm{I\!R}$, while C^∞-differentiability of the algebraic operations in $\mathrm{I\!RP}^1$, wherever they are permitted, implies that g is of class C^∞.

The vertical vector field v on $\mathscr{L}$, the restriction of which to each fibre of $\mathscr{L}$ equals a times the radial (identity) vector field on the fibre, is easily seen to have the property that $d_v = Q\,d/d\tau$, with both sides viewed as operators acting on C^∞ functions of τ. Hence $v = \nabla\tau$, that is, v is the g-gradient of τ.

Clearly, (M',g) becomes an almost Hermitian manifold when equipped with the unique almost complex structure J such that the subbundles $\mathscr{V}$ and $\mathscr{H}$ of TM' are J-invariant and J_x restricted to $\mathscr{V}_x$, or $\mathscr{H}_x$, for any $x \in M'$, coincides with the complex structure of the fibre $\mathscr{L}_{\pi(x)}$ or, respectively, with the $d\pi_x$-pullback of the complex structure of N.

If M now denotes the $\mathbb{C}\mathrm{P}^1$ bundle over N obtained as the projective compactification of $\mathscr{L}$, then g, τ and J have C^∞ extensions to a metric, function and almost complex structure on M, still denoted by g, τ and J. In addition, g is a Kähler metric, that is, $\nabla J = 0$, while τ is a Killing potential with a geodesic gradient on the compact Kähler surface (M,g), but, unless the function $c : N \to \mathrm{I\!RP}^1 \smallsetminus \mathbf{I}$ is constant, τ is not a special Kähler-Ricci potential. For details, see [6].

Proof of Theorem 2 The following classification theorem was established in [6]:

Let τ be a nonconstant Killing potential with a geodesic gradient on a compact Kähler surface (M,g). If τ is not a special Kähler-Ricci potential on (M,g), then, up to a bihol-

omorphic isometry, the triple (M,g,τ) *arises from the above construction applied to some data* $\mathbf{I},a,N,h,\mathscr{L},\langle\,,\rangle,\mathscr{H},c$ *and* $\tau\mapsto Q$ *with the required properties, such that the function* $c:N\to\mathrm{I\!R}\mathrm{P}^1\smallsetminus\mathbf{I}$ *is nonconstant.*

We may thus assume that (M,g) and τ are the objects constructed above. For H as in the statement of Theorem 2 and any fixed $y\in N$, let $F_y:[\tau_{\min},\tau_{\max}]\to\mathrm{I\!R}$ be the antiderivative, vanishing at $\tau_{\min}$, of the function $-[\tau-c(y)]H'(\tau)$ of the variable τ. Thus, $F_y=0$ at both endpoints $\tau_{\min},\tau_{\max}$. Due to the boundary conditions imposed on Q and the first-order Taylor formula, F_y is is smoothly divisible by Q on the whole closed interval $[\tau_{\min},\tau_{\max}]$, that is, $QE_y(\tau)=F_y(\tau)$ for some C^∞ function E_y, and we may define a C^∞ function $\theta:M\to\mathrm{I\!R}$ by $\theta(x)=E_y(\tau)/[\tau-c(y)]$. (Here τ stands for $\tau(x)$, and $y=\pi(x)$, with $\pi:M\to N$ denoting the bundle projection.)

Next, $Y=\Delta\tau$ is given by $Y=(\tau-c\circ\pi)^{-1}Q+dQ/d\tau$ (see [6]). Since, as we noted above, $d_v=Qd/d\tau$, condition (iii) of Lemma 6 follows. Adding a constant to H, we also obtain (iv) in Lemma 6, if P,f and $\hat{\tau}$ are chosen so as to satisfy (ii) and (i) in Lemma 6. This completes the proof. □

Acknowledgements The author thanks Gideon Maschler for helpful comments and suggestions, including the italicized observation at the end of Section 10.

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