New examples of compact Weyl-parallel manifolds

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Abstract

We prove the existence of compact pseudo-Riemannian manifolds with parallel Weyl tensor which are neither conformally flat nor locally symmetric, and represent all indefinite metric signatures in all dimensions $\boldsymbol{n \geq 5}$. Until now such manifolds were only known to exist in dimensions $\boldsymbol{n = 3j + 2}$, where $\boldsymbol{j}$ is any positive integer [1]. As in [1], our examples are diffeomorphic to nontrivial torus bundles over the circle and arise from a quotient-manifold construction applied to suitably chosen discrete isometry groups of diffeomorphically-Euclidean "model" manifolds.

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Introduction

Essentially conformally symmetric (briefly, *ECS*) *manifolds* [2] are those pseudo-Riemannian manifolds of dimensions $n \geq 4$ which have parallel Weyl tensor ($\nabla W = 0$) without being conformally flat ($W = 0$) or locally symmetric ($\nabla R = 0$). Their existence, for every $n \geq 4$, was established by Roter [3, Corollary 3], who also showed that their metrics are all indefinite [4, Theorem 2]. A local description of all ECS metrics is given in [5].

Manifolds with $\nabla W = 0$ are often called *conformally symmetric* [6]. This class represents one of the natural linear conditions imposed on ∇R, cf. [7, Chapter 16], and due to its naturality it attracted the attention of other authors, including Cahen and Kerbrat [8], Hotloś [9], Mantica and Suh [10, Section 3], Schliebner [11], Deszcz et al. [12, Sect. 4], [13, Theorem 6.1]. Results on ECS manifolds, as well as techniques used in obtaining them have been applied to more general classes of manifolds [14, Example 2.2], [15], [16, Theorem 3], [17], [18, Theorem 3.9], [19, Lemma 3], [20], [21, proofs of Theorems 1.1 and 4.5] and, more recently, [22].

Every ECS manifold M carries a distinguished null parallel distribution $\mathcal{D}$ of dimension $d \in \{1, 2\}$, discovered by Olszak [23]. See also [5, p. 119]. We will refer to d as the *rank* of M. Explicitly, the sections of $\mathcal{D}$ are the vector fields corresponding via the metric to 1-forms ξ such that $\xi \wedge [W(v, v', \cdot\,, \cdot)] = 0$ for all vector fields v, v'.

Compact rank-one ECS manifolds which are also geodesically complete and not locally homogeneous are known to exist [1, Theorem 1.1] in all dimensions $n \geq 5$ with $n \equiv 5 \pmod 3$. Quite recently [24] we constructed geodesically incomplete locally-homogeneous compact rank-one ECS manifolds of all *odd* dimensions $n \geq 5$.

It still remains an open question whether a compact ECS manifold may be of rank two, or of dimension four.

Our main result may be viewed as an improvement on [1, Theorem 1.1], since it covers every dimension $n \geq 5$, rather than just those congruent to 5 modulo 3.

Theorem A. *There exist compact rank-one ECS manifolds of all dimensions $n \geq 5$ and all indefinite metric signatures, diffeomorphic to nontrivial torus bundles over the circle, geodesically complete, and not locally homogeneous. Their local-isometry types, in each fixed dimension and metric signature, form – in a natural sense – an infinite-dimensional moduli space.*

The significance of the "model" manifolds (see Section 2) used to produce our compact examples is twofold: in addition to representing all rank-one local isometry types, they constitute the universal coverings of a large class of compact rank-one ECS manifolds [25, Corollary D].

Finally, the bundle structure in Theorem A reflects a general principle: in [26] we show that a non-locally-homogeneous compact rank-one ECS manifold, replaced if necessary by a two-fold isometric covering, *must be a bundle over the circle, having $\mathcal{D}^\perp$ as the vertical distribution.*

Outline of the construction

The paper is structured as follows. After preliminaries comes Section 2, presenting rank-one ECS model manifolds, which exist in all dimensions $n \geq 4$ (even though our construction of their compact quotients requires assuming that $n \geq 5$). The purpose of Section 3 is to show that the particular model manifolds which we focus on are geodesically complete, but not locally homogeneous, so that the same conclusion holds for our compact quotients. In Section 4 we observe (Lemma 4.1) that, given an integer $m \geq 3$, there exists a $\mathrm{GL}(m, \mathbb{Z})$ polynomial with m distinct real positive roots different from 1. The next two sections, crucial for our existence argument, are devoted to proving, in Theorem 6.2, that all the m-element $\mathrm{GL}(m, \mathbb{Z})$-spectra just mentioned – and even a wider class characterized by condition (4.2) – arise via a specific integral formula from periodic curves $\mathbb{R} \ni t \mapsto B = B(t)$ of diagonal $m \times m$ matrices satisfying an ordinary differential equation of the form $\dot{B} + B^2 = f + A$, with a function f and matrix A appearing in a suitable rank-one ECS model manifold $\widehat{M}$ of dimension $n = m + 2$. Finally, Section 7 provides the existence proof: a curve $t \mapsto B(t)$ realizing, for any given $m \geq 3$, one of the $\mathrm{GL}(m, \mathbb{Z})$-spectra of Lemma 4.1, is used to construct a group Γ acting on the corresponding model $\widehat{M}$ freely and properly discontinuously by isometries, with a compact quotient manifold $M = \widehat{M}/\Gamma$.

The most important ingredient of the above argument is the $\mathrm{GL}(m, \mathbb{Z})$-spectrum property of the curve $t \mapsto B(t)$.

1 Preliminaries

By a *lattice* in a real vector space $\mathcal{L}$ with $\dim \mathcal{L} = m < \infty$ we mean, as usual, an additive subgroup of $\mathcal{L}$ generated by some basis of $\mathcal{L}$. Then

$$\Lambda \text{ is a discrete subset of } \mathcal{L}, \tag{1.1}$$

as one sees identifying Λ and $\mathcal{L}$ with $\mathbb{Z}^m$ and $\mathbb{R}^m$.

Suppose that a group Γ acts on a manifold $\widehat{M}$ freely by diffeomorphisms. The action of Γ on $\widehat{M}$ is called *properly discontinuous* if there exists a locally diffeomorphic surjective mapping $\widehat{M} \to M$ onto some manifold M, the preimages of points of M under which are precisely the orbits of the Γ action.

Remark 1.1. *A free left action of a group Γ on a manifold $\widehat{M}$ is properly discontinuous if and only if for any sequences a_j in Γ and y_j in $\widehat{M}$, with j ranging over positive integers, such that both y_j and $a_j y_j$ converge, the sequence a_j is constant except for finitely many j. See [27, Exercise 12-19 on p. 337].*

Remark 1.2. *Any smooth submersion from a compact manifold into a connected manifold is a (surjective) bundle projection. This is the compact case of Ehresmann's fibration theorem [28, Corollary 8.5.13].*

2 The model manifolds

Let $f, p, n, V, \langle \cdot, \cdot \rangle$ and A denote a nonconstant C^∞ function $f : \mathbb{R} \to \mathbb{R}$, periodic of period $p > 0$, an integer $n \geq 4$, a pseudo-Euclidean inner product $\langle \cdot, \cdot \rangle$ on a real vector space V of dimension $n - 2$, and a nonzero, traceless, $\langle \cdot, \cdot \rangle$-self-adjoint linear endomorphism A of V. Consider the pseudo-Riemannian metric [3]

$$\kappa \, dt^2 + \, dt \, ds + \delta \tag{2.1}$$

on the manifold $\widehat{M} = \mathbb{R}^2 \times V \approx \mathbb{R}^n$. The products of differentials stand here for symmetric products, t, s are the Cartesian coordinates on $\mathbb{R}^2$ treated, with the aid of the projection $\widehat{M} \to \mathbb{R}^2$, as functions $\widehat{M} \to \mathbb{R}$, and δ is the pullback to $\widehat{M}$ of the flat (constant) pseudo-Riemannian metric on V arising from the inner product $\langle \cdot, \cdot \rangle$, while $\kappa : \widehat{M} \to \mathbb{R}$ is the function given by $\kappa(t, s, x) = f(t)\langle x, x \rangle + \langle Ax, x \rangle$.

The metric (2.1) turns $\widehat{M}$ into a rank-one ECS manifold [5, Theorem 4.1].

We now define $\mathcal{E}$ to be the vector space of all C^∞ solutions $u:\mathbb{R}\to V$ to the differential equation $\ddot u(t) = f(t)u(t) + Au(t)$, and set $\mathrm{G} = \mathbb{Z}\times\mathbb{R}\times\mathcal{E}$. Whenever $u,w\in\mathcal{E}$, the function $\Omega(u,w) = \langle\dot u, w\rangle - \langle u,\dot w\rangle : \mathbb{R}\to\mathbb{R}$ is constant, giving rise to a nondegenerate skew-symmetric bilinear form Ω on $\mathcal{E}$. We also have a natural linear isomorphism $T:\mathcal{E}\to\mathcal{E}$ with $(Tu)(t) = u(t-p)$. Next, we turn G into a Lie group by declaring the group operation to be

$$(k,q,u)\cdot(\ell,r,w) = (k+\ell, q+r-\Omega(u,T^\ell w), T^{-\ell}u+w), \tag{2.2}$$

and introduce a left action of the Lie group G on the manifold $\widehat{M}$, with

$$(k,q,u)\cdot(t,s,v) = (t+kp, s+q-\langle\dot u(t), 2v+u(t)\rangle, v+u(t)). \tag{2.3}$$

With all triples assumed to be elements of G, one then has

$$(k,q,u)\cdot(0,r,w)\cdot(k,q,u)^{-1} = (0, r-2\Omega(u,w), T^k w). \tag{2.4}$$

Our G also acts on the manifold $\mathbb{R}^2\times\mathcal{E}$, diffeomorphic to $\mathbb{R}^{2n-2}$, by

$$(k,q,u)\cdot(t,z,w) \;=\; (t+kp,\, z+q-\Omega(u,w),\, T^k(w+u)), \tag{2.5}$$

and the following mapping is equivariant relative to the G-actions (2.5) and (2.3):

$$\mathbb{R}^2\times\mathcal{E}\ni(t,z,w)\mapsto(t,s,v) = (t,\, z-\langle\dot w(t), w(t)\rangle,\, w(t))\in\widehat{M}. \tag{2.6}$$

All the above facts are established in [1, p. 77], where it is also shown that

$$\text{the group } \mathrm{G} \text{ acts on } \widehat{M} \text{ by isometries of the metric (2.1).} \tag{2.7}$$

Remark 2.1. *If, at the beginning of this section, $f:I\to\mathbb{R}$ is just a nonconstant C^∞ function on an open interval $I\subseteq\mathbb{R}$, rather than being defined on $\mathbb{R}$ and periodic, while the remaining data $n, V, \langle\cdot,\cdot\rangle$ and A are as before, $I\times\mathbb{R}\times V$ with the metric (2.1) will still be a rank-one ECS manifold; conversely, in any n-dimensional rank-one ECS manifold, every point at which the covariant derivative of the Ricci tensor is nonzero has a neighborhood isometric to an open subset of a manifold of this type [5, Theorem 4.1].*

3 Geodesic completeness

Later in this section, showing that local homogeneity implies relation (3.3), we use much weaker assumptions than necessary for the purposes of the present paper. The reason is that we need to cite such a general conclusion when proving a result in another paper, namely, [26, Theorem 7.3].

For the manifold $\widehat{M} = \mathbb{R}^2\times V\approx\mathbb{R}^n$ with the metric (2.1),

$$\widehat{M} \text{ is geodesically complete, but not locally homogeneous.} \tag{3.1}$$

To see this, we let i, j range over $2, \ldots, n-1$, fix linear coordinates x^i on V which, along with $x^1 = t$ and $x^n = s/2$, form a global coordinate system on $\widehat{M}$. The possibly-nonzero components of (2.1), its reciprocal metric, and the Levi-Civita connection ∇ then are those algebraically related to

$$\begin{array}{llll} g_{11} = \kappa, \quad g_{1n} = 1, & g^{1n} = 1, \quad g^{nn} = -\kappa, & \text{(constants) } g_{ij} \text{ and } g^{ij}, \\ \Gamma_{11}^{n} = \partial_1 \kappa/2, & \Gamma_{11}^{i} = -g^{ij}\partial_j \kappa/2, & \Gamma_{1i}^{n} = \partial_i \kappa/2. \end{array} \tag{3.2}$$

See [3, p. 93]. With $(\,)^{\boldsymbol{\cdot}}$ referring to the geodesic parameter, the geodesic equations read $\ddot{x}^1 = 0$ (and so $c = \dot{x}^1$ is constant), $\ddot{x}^i = -c^2 \Gamma_{11}^{i}$ and $\ddot{x}^n = -c^2 \Gamma_{11}^{n} - 2c\Gamma_{1i}^{n}\dot{x}^i$. Note that κ has the form $\kappa = f(t) g_{ij} x^i x^j + a_{ij} x^i x^j$, with constants g_{ij} and a_{ij}. Being linear in the parameter of a maximal geodesic, x^1 is defined on $\mathbb{R}$, and the same follows for all x^i (as they now satisfy a system of linear second-order equations) and for x^n (which then has a prescribed second derivative). Thus, completeness follows. The second claim of (3.1) is a consequence of what we show later in this section: namely, more generally, for any metric on $I \times \mathbb{R} \times V$ as in Remark 2.1, with an open interval $I \subseteq \mathbb{R}$, if $(t, s, x) \in I \times \mathbb{R} \times V$ has a product-type neighborhood U involving a subinterval I' of I with a Killing field v on U such that $d_v t \neq 0$ everywhere in U (or, equivalently, $v^1 \neq 0$ everywhere in U for the above coordinates $x^1, \ldots, x^n$), then

$$(|f|^{-1/2})^{\boldsymbol{\cdot}} \text{ is constant on every subinterval of } I' \text{ on which } |f| > 0, \tag{3.3}$$

$(\,)^{\boldsymbol{\cdot}}$ this time denoting d/dt, and (3.3) in turn easily yields

$$\text{positivity of } |f| \text{ and constancy of } (|f|^{-1/2})^{\boldsymbol{\cdot}} \text{ on } I', \tag{3.4}$$

since a maximal open subinterval I'' of I' with $|f| > 0$ on I'' must equal I' (or else I'' would have a finite endpoint lying in I', at which $|f|^{-1/2}$, being a linear function, would have a finite limit, contrary to maximality of I''). Local homogeneity of (2.1) on $\widehat{M} = \mathbb{R}^2 \times V$ would, by (3.4) for $I' = \mathbb{R}$, imply that the nonconstant periodic function $|f|^{-1/2}$ is linear. This contradiction proves (3.1).

We now establish (3.3), assuming what is stated in the five lines preceding (3.3), and using the coordinates $x^1, \ldots, x^n$ on $I \times \mathbb{R} \times V$ mentioned there. (Thus – cf. Remark 2.1 – we are looking at an *arbitrary* n*-dimensional rank-one ECS manifold*, rather than just $\widehat{M}$ with the metric (2.1), where f is periodic.) First,

$$\begin{array}{l} \text{the function } t = x^1 \text{ is determined, uniquely up to affine substitu-} \\ \text{tions, by the local geometry of the metric (2.1), while the assign-} \\ \text{ment } t \mapsto f(t), \text{ modulo its replacements by } t \mapsto q^2 f(qt+p), \text{ with} \\ q, p \in \mathbb{R} \text{ and } q \neq 0, \text{ is a local geometric invariant of (2.1) as well.} \end{array} \tag{3.5}$$

In fact, by (3.2), the coordinate vector field ∂_n is parallel. Hence so is the 1-form $dt = dx^1$ corresponding to ∂_n via the metric (2.1). According to [3, p. 93], where the convention about the sign of the curvature tensor is the opposite of ours,

$$\text{the metric (2.1) has the Ricci tensor } \mathrm{Ric} = (2-n) f(t)\, dt \otimes dt, \tag{3.6}$$

and the only possibly-nonzero components of its Weyl tensor W are those algebraically related to W_{1i1j}. Thus, for any vector fields v, v', the 2-form $W(v, v', \cdot\,, \cdot\,)$ is $\wedge$-divisible by $dt = dx^1$ and, consequently, the parallel gradient $\partial_n = \nabla t$ spans the Olszak distribution $\mathcal{D}$ described in the Introduction. This yields the first claim of (3.5), while the second one is then obvious from (3.6). By (3.5), the local flow of our Killing field v on U, with $d_v t \neq 0$, sends t to affine functions of t, and so $d_v t = \pounds_v t = at + b$, where $a, b \in \mathbb{R}$ and $(a, b) \neq (0, 0)$. Thus, $\pounds_v dt = d\pounds_v t = a\,dt$ and $\pounds_v[f(t)] = d_v[f(t)] = \dot f(t)\,d_v t = (at + b)\dot f(t)$. From (3.6) and the Leibniz rule, $0 = \pounds_v[f(t)\,dt \otimes dt] = [(at + b)\dot f(t) + 2af(t)]\,dt \otimes dt$. As $(a, b) \neq (0, 0)$, (3.3) follows.

Remark 3.1. *Let* G *be the group defined in Section 2, acting on* $\widehat{M}$ *via (2.3). Due to (3.5), if a family of metrics arises from (2.1) on quotient manifolds* $M = \widehat{M}/\Gamma$ *of a fixed dimension* $n \geq 5$, *for subgroups* $\Gamma \subseteq \mathrm{G}$ *acting on* $\widehat{M}$ *freely and properly discontinuously, where* f *used in (2.1) ranges over an infinite-dimensional manifold of* C^∞ *functions of a given period* p, *then such a family of metrics forms an infinite-dimensional moduli space of local-isometry types.*

4 $\mathrm{GL}(m, \mathbb{Z})$ polynomials

By a $\mathrm{GL}(m, \mathbb{Z})$ *polynomial* we mean here any degree m polynomial with integer coefficients, leading coefficient $(-1)^m$, and constant term 1 or -1. It is well known, cf. [1, p. 75], that these are precisely the characteristic polynomials of matrices in $\mathrm{GL}(m, \mathbb{Z})$, the group of invertible elements in the ring of $m \times m$ matrices with integer entries. Equivalently, the $\mathrm{GL}(m, \mathbb{Z})$ polynomials are

$$\text{the characteristic polynomials of endomorphisms of an } m\text{-dimensional real vector space } V \text{ sending some lattice } \Lambda \text{ in } V \text{ onto itself.} \tag{4.1}$$

On $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ one may impose the following condition:

$$\begin{array}{l}\{\lambda_1, \ldots, \lambda_m\} \text{ is a subset of } (0, \infty) \smallsetminus \{1\}, \text{ not} \\ \text{of the form } \{\lambda\} \text{ or } \{\lambda, \lambda^{-1}\} \text{ with any } \lambda > 0.\end{array} \tag{4.2}$$

This amounts to requiring that $\lambda_1, \ldots, \lambda_m \in (0, \infty)$ and the absolute values $|\log \lambda_1|, \ldots, |\log \lambda_m|$ be all positive, but not all equal. Clearly,

$$\text{if } \{\lambda_1, \ldots, \lambda_m\} \subseteq (0, \infty) \smallsetminus \{1\} \text{ has more than two elements, (4.2) follows.} \tag{4.3}$$

Lemma 4.1. *For every integer* $m \geq 3$ *there exists a* $\mathrm{GL}(m, \mathbb{Z})$ *polynomial, the roots* $\lambda_1, \ldots, \lambda_m$ *of which are all real, positive, distinct, and different from* 1.

Proof. For $m = 3$, according to [1, Lemma 2.1], whenever $k, \ell \in \mathbb{Z}$ and $2 \leq k < \ell \leq k^2/4$, the polynomial $\lambda \mapsto -\lambda^3 + k\lambda^2 - \ell\lambda + 1$ has three distinct real roots λ, μ, ν with $1/\ell < \lambda < 1 < \mu < k/2 < \nu < k$, as required. Since the quadratic polynomial $\lambda \mapsto \lambda^2 + k\lambda + 1$ with any integer $k < -2$ has one root in $(0, 1)$ and another in $(0, \infty)$, both of them depending on k via an injective function, products of such quadratic polynomials with different values of k realize our claim for all even m, while the case of odd m is settled by the same products, further multiplied by the above cubic polynomial. $\square$

The quartic polynomials obtained in the above proof have very special sets of roots, of the form $\{\lambda, \lambda^{-1}, \mu, \mu^{-1}\}$. To obtain more diverse spectra, consider $\lambda \mapsto P(\lambda) = \lambda^4 - m\lambda^3 + \ell\lambda^2 - k\lambda + 1$, with $k, \ell, m \in \mathbb{Z}$. The inequalities

$$P(2) \leq 16P(1/2) < 0 < P(1)$$

sufficient for our conclusion will follow once $k \geq 7$, as we may then choose m with $k \leq m \leq 2k - 7$ (and so $16P(1/2) - P(2) = 6(m-k) \geq 0$). This also gives $2(k+m) - 2 < 4k + m - 8$, and hence $2(k+m) - 4 < 2\ell < 4k + m - 8$ for some ℓ, which amounts to $P(1) > 0 > 16P(1/2)$.

5 Smoothness-preserving retractions

The following fact will be needed in Section 6. A *retraction* from a set onto a subset is, as usual, a mapping equal to the identity on the subset, and by the *function component* of a pair (f, A) we mean the C^k function f.

Lemma 5.1. *Let there be given an integer $k \geq 0$, a compact smooth manifold Q, finite-dimensional real vector spaces $\mathcal{X}$ and $\mathcal{Z}$, an open set $\mathcal{U}' \subseteq C^k(Q, \mathbb{R}) \times \mathcal{X}$, a smooth mapping $\Phi : \mathcal{U}' \to \mathcal{Z}$, and a point $y \in \mathcal{U}'$. If the differential $d\Phi_y$ is surjective, and $z = \Phi(y)$, then there exist a neighborhood $\mathcal{U}$ of y in $\mathcal{U}'$ and a smooth retraction $\pi : \mathcal{U} \to \mathcal{U} \cap \Phi^{-1}(z)$ such that, for every $x \in \mathcal{U}$ having a smooth function component, the function component of $\pi(x)$ is smooth as well.*

Proof. Let $x_1, \ldots, x_m \in C^k(Q, \mathbb{R}) \times \mathcal{X}$ be representatives of a basis of the quotient space $[C^k(Q, \mathbb{R}) \times \mathcal{X}]/\mathrm{Ker}\, d\Phi_y \approx \mathcal{Z}$, having smooth function components. The smooth mapping $F : C^k(Q, \mathbb{R}) \times \mathcal{X} \times \mathbb{R}^m \to \mathcal{Z}$ sending $(x, a^1, \ldots, a^m)$ to $\Phi(x + a^i x_i)$, with $x \in C^k(Q, \mathbb{R}) \times \mathcal{X}$ and summation over $i = 1, \ldots, m$, has the value z at $(y, 0, \ldots, 0)$, while the differential at $(y, 0, \ldots, 0)$ of the restriction of F to $\{y\} \times \mathbb{R}^m$, given by $(a^1, \ldots, a^m) \mapsto d\Phi_y(a^i x_i)$, is an isomorphism due to our choice of $x_1, \ldots, x_m$ as a quotient basis. The implicit mapping theorem [29, p. 18] thus provides neighborhoods $\mathcal{U}$ of y in $\mathcal{U}'$ and $\mathcal{U}''$ of 0 in $\mathbb{R}^m$ such that every $x \in \mathcal{U}$ has $F(x, a^1(x), \ldots, a^m(x)) = z$ with a unique $(a^1(x), \ldots, a^m(x)) \in \mathcal{U}''$, which also depends smoothly on x. We may now set $\pi(x) = x + a^i(x) x_i$. $\square$

6 Nearly-arbitrary positive spectra

Given $p \in (0, \infty)$ and an integer $m \geq 3$, we denote by Δ_m the space of all diagonal $m \times m$ matrices with real entries. Let us consider the set $\mathcal{X}_m$ of all ordered $(m+1)$-tuples $(f, \lambda_1, \ldots, \lambda_m)$ formed by a nonconstant periodic C^∞ function $f : \mathbb{R} \to \mathbb{R}$ of period p and positive real numbers $\lambda_1, \ldots, \lambda_m$ such that, for some nonzero traceless matrix $A \in \Delta_m$ and some C^∞ curve $\mathbb{R} \ni t \mapsto B = B(t) \in \Delta_m$, periodic of period p, one has

$$\dot{B} + B^2 = f + A \quad \text{and} \quad \operatorname{diag}(\log \lambda_1, \dots, \log \lambda_m) = -\int_0^p B(t)\, dt, \tag{6.1}$$

where $(\ )\dot{} = d/dt$ and f stands for f times Id or, equivalently, for $\operatorname{diag}(f, \dots, f)$.

In the remainder of this section, we fix an integer $k \geq 1$ and treat real or matrix-valued functions of period p as mappings with the domain S^1.

Remark 6.1. *If $c \in \mathbb{R} \smallsetminus \{0\}$, the linear operator $C^k(S^1, \mathbb{R}) \to C^{k-1}(S^1, \mathbb{R})$ sending y to $\dot{y} + cy$ is an isomorphism: its kernel consists of multiples of $t \mapsto e^{-ct}$, while solving the equation $\dot{y} + cy = u$ with $u \in C^{k-1}(S^1, \mathbb{R})$ for $y : \mathbb{R} \to \mathbb{R}$ gives us $y(t+p) = y(t) + ae^{-ct}$, where $a \in \mathbb{R}$. Now $t \mapsto y(t) + (1 - e^{-pc})^{-1} ae^{-ct}$ is the unique periodic solution to $\dot{y} + cy = u$.*

Theorem 6.2. *If $\lambda_1, \dots, \lambda_m$ satisfy (4.2), then $(f, \lambda_1, \dots, \lambda_m) \in \mathcal{X}_m$ for all f from some infinite-dimensional manifold of C^∞ functions.*

Proof. At any nonsingular $C = \operatorname{diag}(c_1, \dots, c_m) \in \Delta_m$ such that $|c_1|, \dots, |c_m|$ are not all equal, viewed as a constant mapping $C : S^1 \to \Delta_m$, the C^∞ mapping $S : C^k(S^1, \Delta_m) \to C^{k-1}(S^1, \Delta_m)$ with $S(B) = \dot{B} + B^2$ has the differential given by $dS_C Y = \dot{Y} + 2CY$, which is an isomorphism (Remark 6.1). Let $C^2 = h + A$, with h a (constant) multiple of Id and (nonzero) traceless A. Due to the inverse mapping theorem [29, p. 13], S has a local C^∞ inverse from a C^{k-1}-neighborhood of $h + A$ onto a C^k-neighborhood of C. If $f \in C^{k-1}(S^1, \mathbb{R})$ is C^{k-1}-close to the constant h, and $E \in \Delta_m$ constant, traceless as well as close to A, applying to $f + E$ this local inverse followed by the mapping $B \mapsto (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ characterized by the second part of (6.1), we get the composite $f + E \mapsto \varPhi(f + E) = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ of three mappings: first, the above local inverse of S (restricted to the set $\mathcal{U}'$ of $f + E$ with f near h and constant traceless E near A), then the linear operator $B \mapsto -\int_0^p B(t)\, dt \in \Delta_m$ and, finally, the entrywise exponentiation of diagonal $m \times m$ matrices. The differential $d\varPhi_{h+A}$ is thus the composite

$$f + E \mapsto Y \mapsto Z = -\int_0^p Y(t)\, dt \mapsto e^{-pC} Z \tag{6.2}$$

of the differentials of our three mappings, at the points $h+A$, C and $-\int_0^p C\, dt = -pC$. Note that the first arrow in (6.2) sends $f + E$ to Y with $\dot{Y} + 2CY = f + E$, while the entrywise exponentiation has at $-pC$ the differential $Z \mapsto e^{-pC} Z$. Integrating $\dot{Y} + 2CY = f + E$ from 0 to p, we obtain $2C\int_0^p Y(t)\, dt = \int_0^p f(t)\, dt + pE$ due to periodicity of Y and constancy of both C and E, so that the second arrow in (6.2) takes Y to $Z = -(2C)^{-1}[\int_0^p f(t)\, dt + pE]$. Applying to this Z the last arrow of (6.2), we see that $d\varPhi_{h+A}(f+E) = -e^{-pC}(2C)^{-1}[\int_0^p f(t)\, dt + pE]$, and so $d\varPhi_{h+A}$ is manifestly surjective onto Δ_m. The preimage $\varPhi^{-1}(e^{-pC})$ is thus an infinite-dimensional submanifold of the manifold formed by our $f + E$, with the tangent space at $h + A$ equal to $\operatorname{Ker} d\Phi_{h+A}$, and hence consisting of all $f+E$ with $E=0$ and $\int_0^p f(t)\,dt=0$. See [29, p. 30].

The hypotheses of Lemma 5.1 are now satisfied by the circle $Q=S^1$, the space $\mathcal{Z}$ of all diagonal $m\times m$ matrices, its subspace $\mathcal{X}$ consisting of traceless ones, the points $y=h+A$ and $z=e^{-pC}$, and our Φ along with its domain $\mathcal{U}'$, treated as a subset of $C^k(S^1,\mathbb{R})\times\mathcal{X}$ via the identification of each $f+E$ with the pair (f,E). For the smooth retraction π arising from Lemma 5.1, ε near 0 in $\mathbb{R}$, and any given $f\in C^\infty(S^1,\mathbb{R})$ with $\int_0^p f(t)\,dt=0$, the curve $\varepsilon\mapsto\pi(h+\varepsilon f,A)$ lies in the preimage $\Phi^{-1}(e^{-pC})$, consists of function-matrix pairs having a smooth function component, and its velocity vector at $\varepsilon=0$ is f (as one sees applying $d/d\varepsilon$ and noting that the differential of π at $y=h+A$ equals the identity when restricted to the tangent space of $\Phi^{-1}(e^{-pC})$). Since f was just any smooth function $S^1\to\mathbb{R}$ with $\int_0^p f(t)\,dt=0$, such curves realize the infinite-dimensional manifold of C^∞ functions named in our assertion. In addition, the curve $\varepsilon\mapsto S^{-1}(\pi(h+\varepsilon f,A))$ consists of smooth matrix-valued functions B due to regularity of solutions for the differential equation $\dot B+B^2=f+E$. Since $(\lambda_1,\dots,\lambda_m)=e^{-pC}$ was an arbitrary m-tuple with (4.2), this completes the proof. $\square$

While, as we just showed, condition (4.2) is sufficient for $(\lambda_1,\dots,\lambda_m)$ to lie in the image of $\mathcal{X}_m$ under the mapping $(f,\lambda_1,\dots,\lambda_m)\mapsto(\lambda_1,\dots,\lambda_m)$, a weaker version of (4.2) is also *necessary* for it: this version, allowing some λ_i to equal 1, states that $\{\lambda_1,\dots,\lambda_m\}\subseteq(0,\infty)$ does not have the form $\{\lambda\}$ or $\{\lambda,\lambda^{-1}\}$ with any $\lambda>0$. To see its necessity, write $B=(b_1,\dots,b_m)$ and $A=(a_1,\dots,a_m)$, so that the first equality in (6.1) amounts to $\dot b_i+b_i^2=f+a_i$ for $i=1,\dots,m$. Next,

$$\text{if } \lambda_i \text{ equals } \lambda_j \text{ or } \lambda_j^{-1} \text{ for some distinct } i,j, \text{ then } a_i=a_j. \tag{6.3}$$

Namely, were this not the case, so that $a_i\neq a_j$, while the integrals from 0 to p of b_i and b_j are equal (or, opposite), cf. (6.1), the difference b_i-b_j (or, the sum b_i+b_j) would be the derivative $\dot\chi$ of some periodic function χ and the equality $(b_i-b_j)^\cdot+(b_i+b_j)(b_i-b_j)=a_i-a_j$ would give $\ddot\chi=a_i-a_j$ wherever $\dot\chi=0$ or, respectively, $[(b_i-b_j)e^\chi]^\cdot=(a_i-a_j)e^\chi$. As a_i-a_j is now a nonzero constant, in the former case the critical points of χ would all be strict local maxima, or strict local minima, and in the latter $(b_i-b_j)e^\chi$ would be strictly monotone, both of which contradict periodicity, thus proving (6.3). Combining (6.3) with second equality in (6.1), we now see that $|\log\lambda_1|,\dots,|\log\lambda_m|$ cannot be all equal, as that would give $a_1=\ldots=a_m$, whereas A in (6.1) is nonzero and traceless.

7 Proof of Theorem A: existence

The argument presented in this section proves a special case of an assertion established in [1, Section 9]. For the reader's convenience we chose to proceed as below, rather than cite [1], since this simplifies the exposition.

Existence in Theorem A will follow once we show that, for suitable $f,p,n,$ $V,\langle\cdot,\cdot\rangle,A$ with the properties listed at the beginning of Section 2, where $n\geq 5$ and the metric signature of $\langle\cdot,\cdot\rangle$ are *arbitrary*, and for $\mathrm{G}, \widehat{M}$ appearing in (2.7),

$$\text{some subgroup } \Gamma \subseteq \mathrm{G} \text{ acts on } \widehat{M} \text{ freely and properly dis-} \\ \text{continuously with a compact quotient manifold } M = \widehat{M}/\Gamma. \tag{7.1}$$

To choose $f, p, n, V, \langle\cdot,\cdot\rangle, A$ satisfying the conditions named in Section 2, along with additional objects $\theta, B, \mathcal{L}, \Lambda, \Sigma, \Gamma$ needed for our argument, we let $n \geq 5$ and $p, \theta \in (0,\infty)$ be completely arbitrary, and denote by $\langle\cdot,\cdot\rangle$ a pseudo-Euclidean inner product of any signature in $V = \mathbb{R}^{n-2}$, making the standard basis orthonormal. Lemma 4.1 allows us to fix a $\mathrm{GL}(m,\mathbb{Z})$ polynomial P, where $m = n-2$, the complex roots $\lambda_1, \ldots, \lambda_m$ of which are all real, distinct, and satisfy (4.2). Theorem 6.2, for these $\lambda_1, \ldots, \lambda_m$, yields f, A and a curve $t \mapsto B(t) \in \Delta_m \subseteq \mathrm{End}(V)$, with

$$\text{infinite-dimensional freedom of choosing } f. \tag{7.2}$$

Next, we let $\mathcal{L}$ be the $(n-2)$-dimensional vector space of all solutions $u : \mathbb{R} \to V$ to the differential equation $\dot{u}(t) = B(t)u(t)$, with the *translation operator* $T : \mathcal{L} \to \mathcal{L}$ given by $(Tu)(t) = u(t-p)$. Note that $\mathcal{L} \subseteq \mathcal{E}$ for $\mathcal{E}$ which was defined in Section 2 along with a linear isomorphism $T : \mathcal{E} \to \mathcal{E}$, and our T is the restriction of that one to $\mathcal{L}$. According to [1, Remark 4.2] and (6.1), our $T : \mathcal{L} \to \mathcal{L}$ has the spectrum $\lambda_1, \ldots, \lambda_m$, so that P is its characteristic polynomial and, by (4.1), $T(\Lambda) = \Lambda$ for some lattice Λ in $\mathcal{L}$. (As $\lambda_1, \ldots, \lambda_m$ are all distinct, they uniquely determine the algebraic equivalence type of T.) For later reference, let us also note that

$$T^k \neq \mathrm{Id} \quad \text{for all} \quad k \in \mathbb{Z} \smallsetminus \{0\}, \tag{7.3}$$

for $T : \mathcal{L} \to \mathcal{L}$, as its spectrum $\{\lambda_1, \ldots, \lambda_m\}$ is contained in $(0,\infty) \smallsetminus \{1\}$. Now $\Sigma = \{0\} \times \mathbb{Z}\theta \times \Lambda$ is both a subset of $\mathrm{G} = \mathbb{Z} \times \mathbb{R} \times \mathcal{E}$ and a lattice in the vector space $\{0\} \times \mathbb{R} \times \mathcal{L}$ while, due to self-adjointness of each $B(t)$,

$$\Omega(u,w) = 0 \quad \text{whenever} \quad u, w \in \mathcal{L}. \tag{7.4}$$

Finally, we denote by Γ the subgroup of G generated by Σ and the element $(1,0,0)$. Then Γ, as a subset of $\mathrm{G} = \mathbb{Z} \times \mathbb{R} \times \mathcal{E}$, equals $\mathbb{Z} \times \mathbb{Z}\theta \times \Lambda$, and

$$\text{each } (k, \ell\theta, u) \in \Gamma = \mathbb{Z} \times \mathbb{Z}\theta \times \Lambda \text{ acts on } \widehat{M} = \mathbb{R}^2 \times V \text{ by} \\ (k,\ell\theta,u) \cdot (t,s,v) = (t+kp,\, s+\ell\theta - \langle \dot{u}(t), 2v+u(t)\rangle,\, v+u(t)). \tag{7.5}$$

In fact, due to (2.4) and T-invariance of Λ, the conjugation by $(1,0,0)$ maps Σ onto itself, and any element of Γ, being a finite product of factors from the set $\Sigma \cup \{(1,0,0),(1,0,0)^{-1}\}$, *equals a power of* $(1,0,0)$ *times an element of* Σ. However, by (2.2), $(k,0,0)\cdot(\ell,0,0) = (k+\ell,0,0)$, and so $(1,0,0)^k = (k,0,0)$ if $k \in \mathbb{Z}$. The last italicized phrase, combined with (2.2) and (2.3), now yields (7.5).

The action (7.5) is free: if $(k,\ell\theta,u)\cdot(t,s,v) = (t,s,v)$, the resulting equalities $kp = \ell\theta - \langle \dot{u}(t), 2v+u(t)\rangle = 0$ and $u(t) = 0$ give $k = 0$, while the first-order linear differential equation $\dot{u} = Bu$ implies that $u = 0$, and so $\ell = 0$ as well.

In view of the regular-dependence theorem for ordinary differential equations,

$$\mathbb{R}\times\mathcal{L}\ni(t,w)\mapsto(t,w(t))\in\mathbb{R}\times V \text{ is a diffeomorphism,} \tag{7.6}$$

since that theorem guarantees smoothness of the inverse of (7.6). We now use Remark 1.1 to conclude that (7.5) is properly discontinuous: if the sequences (t_j,s_j,v_j) and $(k_j,\ell_j\theta,u_j)\cdot(t_j,s_j,v_j)$ both converge, (7.5) gives convergence of k_j and $u_j(t_j)$. The former makes k_j eventually constant, the latter leads, by (7.6), to convergence of u_j, and hence its ultimate constancy (implying via (7.5) the same for ℓ_j), as $u_j\in\Lambda$ and the lattice $\Lambda\subseteq\mathcal{L}$ is discrete, cf. (1.1).

Finally, compactness of the quotient manifold $M=\widehat{M}/\Gamma$ in (7.1) follows since $\widehat{M}$ has a compact subset K intersecting every orbit of Γ. Namely, we may set $K=\{(t,s,v):s\in[0,\theta]\text{ and }(t,v)\in K'\}$, where K' is the image under (7.6) of $[0,p]\times\widehat{K}$, with a compact set $\widehat{K}\subseteq\mathcal{L}$ chosen so as to intersect every orbit of the lattice Λ acting on $\mathcal{L}$ by vector-space translations. Any $(t,s,v)\in\widehat{M}$ can be successively modified by elements of Γ acting on it, so as to eventually end up in K. First, $(k,0,0)\in\Gamma$ with $kp\in[-t,p-t]$, applied to (t,s,v), allows us to assume that $t\in[0,p]$. For the pair (t,w) arising as the preimage under (7.6) of the (t,v) component of this new (t,s,v), and suitably selected $u\in\Lambda$, one has $w+u\in\widehat{K}$, due to our choice of $\widehat{K}$. Now, by (7.5) with $v=w(t)$,

$$(0,0,u)\cdot(t,s,v)=(t,s-\langle\dot{u}(t),2v+u(t)\rangle,w(t)+u(t)),$$

that is, $(0,0,u)\cdot(t,s,v)=(t',s',v')$ for some $(t',v')\in K'$ and $s'\in\mathbb{R}$. Choosing $\ell\in\mathbb{Z}$ such that $s'+\ell\theta\in[0,\theta]$, we obtain $(0,\ell\theta,0)\cdot(t',s',v')\in K$.

8 Proof of Theorem A: further conclusions

For $\widehat{M},p$ and $\Gamma=\mathbb{Z}\times\mathbb{Z}\theta\times\Lambda$ as in the last section, the surjective submersion $\widehat{M}=\mathbb{R}^2\times V\to\mathbb{R}$ sending (t,s,v) to t/p is, by (7.5), equivariant relative to the homomorphism $\Gamma=\mathbb{Z}\times\mathbb{Z}\theta\times\Lambda\ni(k,\ell\theta,u)\to k\in\mathbb{Z}$ along with the actions of Γ on $\widehat{M}$ and $\mathbb{Z}$ on $\mathbb{R}$, so that it descends to a surjective submersion $M=\widehat{M}/\Gamma\to\mathbb{R}/\mathbb{Z}=S^1$ which, according to Remark 1.2, is a bundle projection. Since the homomorphism $\Gamma\to\mathbb{Z}$ has the kernel $\Sigma=\{0\}\times\mathbb{Z}\theta\times\Lambda$, the fibre of this projection $M\to S^1$ over the $\mathbb{Z}$-coset of t/p may be identified with the quotient $\widehat{M}_t/\Sigma$, where $\widehat{M}_t=\{t\}\times\mathbb{R}\times V$. Restricted to Σ and $\widehat{M}_t$, (7.5) is given by

$$(0,\ell\theta,u)\cdot(t,s,v)=(t,s+\ell\theta-\langle\dot{u}(t),2v+u(t)\rangle,v+u(t)), \tag{8.1}$$

with fixed $t\in\mathbb{R}$. By (7.6), the restriction of (2.6) to $\{t\}\times\mathbb{R}\times\mathcal{L}$ is a diffeomorphism onto $\widehat{M}_t$, and so its Σ-equivariance, immediate from G-equivariance, means that, when we use it to identify $\widehat{M}_t$ with $\{t\}\times\mathbb{R}\times\mathcal{L}$, and hence also with $\mathrm{H}'=\{0\}\times\mathbb{R}\times\mathcal{L}$ (a subgroup of G containing Σ), the restriction of (2.3) to $\mathrm{H}'\times\widehat{M}_t$ becomes the action of H' on itself via left translations. By (2.2) with $k=\ell=0$ and (7.4), H' is an Abelian subgroup of G, and the resulting group operation in H' coincides with addition in the vector space $\{0\} \times \mathbb{R} \times \mathcal{L}$. Since $\Sigma = \{0\} \times \mathbb{Z}\theta \times \Lambda$ is a lattice in $\{0\} \times \mathbb{R} \times \mathcal{L}$, cf. the lines preceding (7.4), this shows that the fibre $\widehat{M}_t/\Sigma$ is a torus, which makes M, with the projection $M \to S^1$ described above, *a torus bundle over the circle.*

The torus bundle $M \to S^1$ is nontrivial: (7.3) combined with [1, Theorem 5.1(f)], implies that Γ has no Abelian subgroup of finite index, so that M cannot be diffeomorphic to a torus, or even covered by a torus.

Geodesic completeness of M, and its lack of local homogeneity, are immediate from (3.1), while the claim about an infinite-dimensional moduli space of the local-isometry types is an obvious consequence of Remark 3.1 and (7.2).

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