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October 1983

AN EASY CONSTRUCTION OF NEW COMPACT RIEMANNIAN MANIFOLDS WITH HARMONIC CURVATURE

(preliminary report)

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§ 1. Introduction.

For any Riemannian manifold (M,g) , the second Bianchi identity implies the relation

$$\delta R = - d r \tag{1}$$

(in local coordinates : $\nabla^s R_{sijk} = \nabla_j r_{ik} - \nabla_k r_{ij}$), R being the curvature tensor and r the Ricci tensor of (M,g) with $r_{ij} = R_{isj}{}^s$. Similarly, we have

$$\delta W = - \frac{n-3}{n-2} d\left(r - \frac{u}{2(n-1)} g\right) , \tag{2}$$

where $n = \dim M \geq 3$, W is the Weyl conformal tensor and $u = g^{ij} r_{ij}$ is the scalar curvature of (M,g) . One says that (M,g) has *harmonic curvature* if $\delta R = 0$, i.e. (cf. (1)), if the Ricci tensor r satisfies the *Codazzi equation* $d r = 0$ ([4], [5]).

The following examples of *compact* Riemannian manifolds with harmonic curvature are known :

(i) Compact manifolds with parallel Ricci tensor, including compact Einstein spaces.

(ii) Compact conformally flat manifolds with constant scalar curvature (cf. (2) and (1)). For existence results, see [7].

(iii) Certain bundles with fibre N over S^1 endowed with suitable *twisted warped product* metrics, where N is an arbitrary compact manifold admitting an Einstein metric with positive scalar curvature (see [5]). If the Einstein metric of N is not of constant curvature (and hence $\dim N \geq 4$), these examples are not of type (i) or (ii).

(iv) Compact manifolds locally isometric to produ

The aim of this note is to describe some new examples of compact Riemannian manifolds with harmonic curvature. More precisely, we prove the following existence result.

THEOREM . *Let* (M,g) *be a compact Riemannian manifold with constant curvature* $K < 0$. *Suppose that* p *is an integer with*

$$p > n - K^{-1}\lambda_1 ,$$

where $n = \dim M > 2$ *and* λ_1 *is the lowest positive eigenvalue of the Laplacian of* (M,g) , *acting on functions.*

Then, for any compact p*-dimensional Einstein manifold* (N,h) *with positive scalar curvature, other than a space of constant curvature, the product manifold* $M \times N$ *admits a metric* G *with harmonic curvature, which is not locally isometric to any of the examples described in* (i), (ii), (iii), (iv) *above.*

Note that the compact manifolds with harmonic curvature obtained in this way have infinite fundamental groups and are of dimensions $n + p \geq 7$.

The proof of the theorem is a trivial consequence of the following special case of a result of H. Yamabe : *On any compact* n*-dimensional Riemannian manifold, the equation*

$$\Delta f + af = cf^{q-1}$$

with real a, q, c *such that* $2 < q < 2n(n-2)^{-1}$, $a > \lambda_1(q-2)^{-1}$ *and* $c > 0$, *is satisfied by some non-constant positive function* f . The required metric G is obtained from a product metric on $M \times N$ by a conformal deformation involving such a function f on (M,g) .

Throughout this note, considering two or more Riemannian metrics, we shall endow the symbols of their geometric quantities with appropriate subscripts or superscripts (like ∇^h , W_h , $\delta_h R_h$ for a metric h). The Laplacian $\Delta = \Delta_g : C^\infty(M) \to C^\infty(M)$ of (M,g) is always defined to have negative principal symbol : $\Delta = -g^{ij}\nabla_i\nabla_j$, and $\lambda_1 = \lambda_1(M,g)$ denotes its lowest positive eigenvalue.

§ 2. Proof of the theorem.

LEMMA 1 .(cf. [3], Chapter XVI). *Suppose that we are given a space* (M,g) *of constant curvature* K , *a* p*-dimensional Einstein manifold* (N,h) *with scalar curvature* $u_h = -p(p-1)K$, *and an arbitrary positive* C^∞ *function* f *on* M . *Then, for any real* q , *the metric*

$$G = f^{q-2}\cdot(g \times h)$$

on $M \times N$ *satisfies the condition* $\delta_G W_G = 0$. *Moreover,*

(i) G *is not conformally flat unless* h *is of constant curvature.*

(ii) *If* M *is compact,* $K < 0$, $q \neq 2$ *and* f *is non-constant, then* $(M \times N, G)$ *admits no parallel symmetric 2-tensor fields other than constant multiples of* G .

PROOF. For conformally related metrics γ , G in dimension m , we have

$$\delta_G W_G = \delta_\gamma W_\gamma - (m-3) W_\gamma(\nabla^\gamma\sigma, \cdot, \cdot, \cdot)$$

whenever $G = e^{2\sigma}\cdot\gamma$. Our product metric $\gamma = g \times h$ on $M \times N$ has $\delta_\gamma W_\gamma = 0$ (cf. (2)) and its Weyl tensor annihilates $\nabla^\gamma(\log f)$, since f is constant along N . Therefore $\delta_G W_G = 0$.

If G is conformally flat, then so is $g \times h$, which easily implies that h has constant curvature. This proves (i).

Assume now that the hypotheses of (ii) are satisfied and E is a symmetric 2-tensor field on $M \times N$ with $\nabla^G E = 0$. In a local product coordinate system $x^1,\dots,x^n;\ x^{n+1},\dots,x^{n+p}$ for $M \times N$, $n = \dim M$, this implies

$$(3)\qquad \nabla^g_k E_{i\alpha} = 2\,\partial_k\sigma\cdot E_{i\alpha} + \partial_i\sigma\cdot E_{k\alpha} - g^{\ell s}\,\partial_\ell\sigma\cdot E_{s\alpha}g_{ik} \quad ,$$

where $\sigma = \frac{1}{2}(q-2)\log f$. Here and in the sequel, the ranges of indices are : $i,j,k,\ell,s = 1,\dots,n$, $\alpha,\beta,\mu,\nu = n+1,\dots,n+p$. For a fixed α , $v = e^{-3\sigma}E_{i\alpha}dx^i$ is a well-defined *global* 1-form on M . By (3), $\nabla^g v = v \wedge d\sigma - v(\nabla^g\sigma)\cdot g$. Hence $\nabla^g v + (\nabla^g v)^* = -\,2v(\nabla^g\sigma)\cdot g$ and so v represents a conformal vector field on (M,g) . Since M is compact and $K < 0$, this implies $v = 0$ and

$$(4)\qquad E_{i\alpha} = 0 \quad \text{for} \quad 1 \le i \le n < \alpha \le n+p\ .$$

Other consequences of $\nabla^G E = 0$ are $\nabla^h_\beta E_{i\alpha} = \partial_i\sigma\cdot E_{\alpha\beta} - g^{jk}\,\partial_j\sigma\cdot E_{ki}h_{\alpha\beta}$, $\nabla^g_k E_{\alpha\beta} = 2\partial_k\sigma\cdot E_{\alpha\beta}$ and $\nabla^h_\mu E_{\alpha\beta} = -\,g^{ij}\partial_i\sigma\cdot E_{j\alpha}h_{\beta\mu} - g^{ij}\partial_i\sigma\cdot E_{j\beta}h_{\alpha\mu}$. By (4), $\nabla^h_\beta E_{i\alpha} = \nabla^h_\mu E_{\alpha\beta} = 0$ and hence these relations give

$$(5)\qquad E_{\alpha\beta} = Ce^{2\sigma}\cdot h_{\alpha\beta} \quad , \quad g^{ij}\,\partial_i\sigma\cdot E_{jk} = Ce^{2\sigma}\cdot\partial_k\sigma \ , \quad 1 \le k \le n < \alpha,\beta \le n+p,$$

with a constant C . Moreover, $\nabla^G E = 0$ gives, by (4), $\nabla^h_\alpha E_{ij} = 0$ and $\nabla^g_k E_{ij} = 2\partial_k\sigma\cdot E_{ij} + \partial_i\sigma\cdot E_{jk} + \partial_j\sigma\cdot E_{ik} - g^{\ell s}\,\partial_\ell\sigma\cdot E_{si}g_{jk} - g^{\ell s}\,\partial_\ell\sigma\cdot E_{sj}g_{ik}$. The symmetric 2-tensor field $P = P_{ij}dx^i dx^j$ with

$$P_{ij} = e^{-2\sigma}E_{ij} - Cg_{ij}$$

is, clearly, well-defined everywhere in M . In view of (5), the preceding formula gives

$$(6)\qquad a)\quad \nabla^g_k P_{ij} = \partial_i\sigma\cdot P_{jk} + \partial_j\sigma\cdot P_{ik}$$

$$\qquad\quad b)\quad g^{jk}\,\partial_j\sigma\cdot P_{ki} = 0\ .$$

Transvecting (6)*a*) with g^{ij} , g^{jk} or with $g^{ik}g^{j\ell}P_{k\ell}$, respectively, we see from (6)*b*) that

$$(7)\qquad dT = dQ = 0\ , \quad g^{jk}\nabla^g_j P_{ki} = T\cdot\partial_i\sigma\ ,$$

where $T = \mathrm{trace}_g P = g^{ij}P_{ij}$ and $Q = g(P,P) = g^{ik}g^{j\ell}P_{ij}P_{k\ell}$. Moreover, (6)*b*) and (7) give

$$(8)\qquad g(P,\nabla^g d\sigma) = -\,T\cdot g(d\sigma,d\sigma)\ .$$

Applying to P the contracted Ricci identity (Weitzenböck formula) and using (6), we obtain

$$T\cdot(\nabla^g d\sigma - d\sigma \otimes d\sigma) + P \circ \nabla^g d\sigma - \nabla^g d\sigma \circ P + (\Delta_g\sigma + g(d\sigma,d\sigma))\cdot P =$$
$$= K\cdot(Tg - nP)\ .$$

Suppose now that P does not vanish identically, i.e., the constant $Q = g(P,P)$ (cf. (7)) is positive. Taking the inner product of the last formula with P we then have, in view of (6)*b*), (7) and (8), the relation $\Delta_g\sigma = c_1 g(d\sigma,d\sigma) + c_2$ for some constants c_1 , c_2 , which obviously contradicts the non-constancy of σ on the compact manifold (M,g) . Therefore $P = 0$ everywhere, i.e., $E_{ij} = Ce^{2\sigma}\cdot g_{ij}$ for $1 \le i,j \le n$. Together with (4) and (5), this gives $E = C\cdot G$, completing the proof.

REMARK. The metric G required in our theorem can now be constructed as follows. If (M,g) is compact and has constant curvature K and (N,h) is Einstein, with scalar curvature $u_h = -p(p-1)K$, where $\dim M = n > 2$, $\dim N = p$ and $n + p \geq 4$, then $(M \times N, G)$ has $\delta_G W_G = 0$, $G = G_f$ being defined as in Lemma 1 with any positive function f on M and with

$$(9) \qquad q = \frac{2(n+p)}{n+p-2} .$$

The scalar curvature u_G of $G = G_f$ is constant along N and one easily verifies (cf. [1], p. 126) that, on M,

$$(10) \qquad \Delta_g f + \tfrac{1}{4}(n-p)(n+p-2)Kf = \tfrac{1}{4} u_G (n+p-2)(n+p-1)^{-1} f^{q-1}$$

with q given by (9). If f is such that u_G is constant, then, by (2) and (1), $\delta_G R_G = 0$. Therefore, all we need in order to construct, in this way, a (new) metric G with $\delta_G R_G = 0$, is to find a *non-constant* positive solution f of (10) with (9) on (M,g), for some *real number* u_G and an integer $p > 0$. When $K \geq 0$, such an f does not exist, which one can easily prove using integration by parts. On the other hand, if $K < 0$ and $p > n - K^{-1}\lambda_1$, $\lambda_1 = \lambda_1(M,g)$, the existence of such an f is immediate from the following result.

LEMMA 2 (H. Yamabe [8], cf. [1], p. 115-119). *Let* (M,g) *be a compact Riemannian manifold,* $\dim M = n > 2$. *Then, for any real* a, q, c *with* $2 < q < 2n(n-2)^{-1}$, $a(q-2) > \lambda_1 = \lambda_1(M,g)$ *and* $c > 0$, *there exists a non-constant positive* C^∞ *function* f *on* M *such that*

$$(11) \qquad \Delta f + af = cf^{q-1} .$$

PROOF. The required function f can be found using Yamabe's method (see [1], p. 115-119), which consists in minimizing the functional

$$I_q(f) = \left(\int_M |\nabla f|^2 + a\int_M f^2\right)\cdot\left(\int_M f^q\right)^{-2/q}$$

in the class of all non-negative functions f in the first Sobolev space $L^2_1(M)$ which do not vanish identically. A minimum f of I_q exists if $2 < q < 2n(n-2)^{-1}$ and it is a positive C^∞ function on M, satisfying (11) with some $c > 0$; by rescaling f, any $c > 0$ can be attained. Suppose now that this minimum f is a positive constant. Calculating the second variation of I_q at f, we then obtain

$$\int_M |\nabla\phi|^2 \geq a(q-2)\int_M \phi^2$$

for each C^∞ function ϕ with $\int_M \phi = 0$. Hence $a(q-2) \leq \lambda_1$ (see [2], p. 186), which contradicts our hypothesis. Therefore f is non-constant, which completes the proof.

Proof of the theorem. Let (M,g), p and (N,h) satisfy our hypotheses. By rescaling h, we may assume it has scalar curvature $u_h = -p(p-1)K$. Fix a non-constant positive function f satisfying (10) with (9) on (M,g), for a constant $u_G > 0$, which exists by Lemma 2. According to the remark preceding Lemma 2, the metric $G = f^{q-2}\cdot(g \times h)$ on $M \times N$ then has harmonic curvature, q being given by (9). Moreover, G is neither conformally flat ((i) of Lemma 1), nor locally reducible (by a recent result of D. DeTurck and H. Goldschmidt [6], all metrics with harmonic curvature are analytic in suitable coordinates, so that local reducibility of $(M \times N, G)$ would contradict (ii) of Lemma 1). If G had parallel Ricci tensor, it would be Einstein in view of local irreducibility. Therefore, $\pi_1(M \times N)$ would be finite (since G has scalar curvature $u_G > 0$), which would contradict the negativity of the curvature of (M,g). Thus, $(M \times N, G)$ is not, even locally, of the type represented by examples (i), (ii) or (iv) of §1. If it were locally of type (iii), then, in view of analyticity, $M \times N$ and a compact bundle over S^1, whose fibre has finite fundamental group, would have homeomorphic universal covering spaces, which is obviously impossible as $\dim M \geq 2$. This completes the proof.

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