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Curvature-Homogeneous Indefinite Einstein Metrics in Dimension Four: the Diagonalizable Case

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§0. Introduction

A pseudo-Riemannian manifold (M,g) is called *curvature-homogeneous* if the algebraic type of its metric/curvature pair (g,R) is the same at all points, i.e., if for any $x,y\in M$ some isomorphism $T_xM\to T_yM$ sends $g(x),R(x)$ to $g(y),R(y)$.

Every locally homogeneous pseudo-Riemannian manifold is, obviously, curvature-homogeneous. The converse proposition fails; counterexamples with positive-definite metrics were first found by Takagi [**17**] and, on compact manifolds, by Ferus, Karcher and Münzner [**11**]; see also [**2**]. Analogous examples with indefinite metrics have been known even longer ([**3**] – [**7**], [**13**]).

The present paper provides a classification, up to local isometries, of all those curvature-homogeneous pseudo-Riemannian four-manifolds (M,g) which are Einstein and have, at some (or every) point x, a *complex-diagonalizable* curvature operator $R(x):[T_xM]^{\wedge 2}\to[T_xM]^{\wedge 2}$. (The last condition means that the complex-linear extension of $R(x)$ to the complexification of the bivector space $[T_xM]^{\wedge 2}$ is diagonalizable.) It turns out that all such manifolds are locally homogeneous and, in fact, either locally symmetric, or locally isometric to a Lie group with a left-invariant indefinite metric of a specific type; see Theorems 5.1, 6.1 and 7.1. In those theorems we assume constancy of eigenvalues of the curvature operator, which sounds weaker than curvature-homogeneity, but, in the complex-diagonalizable case, is actually equivalent to it; cf. [**10**], p. 701 and Remark 6.19 on p. 472.

The metric g can have any signature. Using a sign change, we may assume that g is *Riemannian*, *neutral* or *Lorentzian*, that is, has one of the sign patterns

$$++++\,,\qquad --++\,,\qquad -+++\,. \tag{1}$$

Two known families of curvature-homogeneous Einstein four-manifolds, one Lorentzian [**3**] and one neutral ([**10**], p. 705), give rise to infinite-dimensional spaces of local-isometry types. By contrast, for the manifolds classified here, the analogous space is clearly finite-dimensional (see above). Also, our diagonalizability assumption always holds for *Riemannian* manifolds, as the curvature operator is

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self-adjoint, and for Riemannian metrics our theorem becomes the result of [**10**], mentioned below. In the Lorentzian case, the complex-diagonalizability condition means that the curvature is of the Petrov type I at each point, cf. [**10**], p. 659.

Some types of curvature-homogeneous Einstein four-manifolds have already been classified. This includes locally symmetric spaces ([**8**], [**9**]; cf. [**10**], pp. 662–663); Brans's classification [**3**] of Lorentzian Einstein metrics representing the Petrov type III at every point (a condition that implies curvature-homogeneity); as well as the Riemannian case ([**10**], Corollary 7.2, p. 476), in which the metrics in question are all locally symmetric (see also §7).

The text is organized as follows. In sections 2 – 4 we introduce our "model spaces", using a construction basically due to Petrov [**15**]. The classification result is stated in sections 5 – 7 and then proved in sections 8 – 13.

§1. Preliminaries

Our conventions about the curvature tensor $R = R^\nabla$ of any connection ∇ in a real/complex vector bundle $\mathcal{E}$ over a manifold M, its Ricci tensor Ric when $\mathcal{E}$ is the tangent bundle TM, and the scalar curvature s in the case where ∇ is the Levi-Civita connection of a given pseudo-Riemannian metric g on M, are

$$(2)\qquad \begin{aligned} &\text{i)}\quad R(u,v)\psi \,=\, \nabla_v\nabla_u\psi \,-\, \nabla_u\nabla_v\psi \,+\, \nabla_{[u,v]}\psi\,,\\ &\text{ii)}\quad \mathrm{Ric}\,(u,w) \,=\, \mathrm{Trace}\,[v\mapsto R(u,v)w]\,, \qquad \mathrm{s} \,=\, \mathrm{Trace}_g\mathrm{Ric}\,,\end{aligned}$$

for any (local) C^2 sections u,v,w of TM and ψ of $\mathcal{E}$.

A pseudo-Riemannian manifold (M,g) with $\dim M = n$ is said to be an *Einstein manifold* [**1**] if $n \ge 3$ and $\mathrm{Ric} = \mathrm{s}g/n$, while, if $n \ge 4$, formulae $\sigma = \mathrm{Ric} - (2n-2)^{-1}\mathrm{s}g$ and $W = R - (n-2)^{-1}g\wedge\sigma$ define the *Schouten tensor* σ and *Weyl tensor* W of (M,g). Here $\wedge$ is the exterior product of 1-forms valued in 1-forms, obtained using the valuewise multiplication which is also provided by $\wedge$, so that the result is a 2-form valued in 2-forms.

For (M,g) as above, we denote $[TM]^{\wedge 2}$ the vector bundle of bivectors over M, with the fibres $[T_xM]^{\wedge 2}$, $x\in M$. There exists a unique pseudo-Riemannian fibre metric $\langle\,,\rangle$ in $[TM]^{\wedge 2}$ such that $\langle v\wedge u, v'\wedge u'\rangle = g(v,v')\,g(u,u') - g(v,u')\,g(u,v')$ for any $x\in M$ and $v,u,v',u'\in T_xM$. Both R, W are four-times covariant tensor fields on M sharing the (skew)symmetry properties of the curvature tensor, which allows us to treat them as morphisms acting on bivectors and self-adjoint relative to $\langle\,,\rangle$ at each point; in this way, R gives rise to the *curvature operator*

$$(3)\qquad R:[TM]^{\wedge 2}\to[TM]^{\wedge 2}\quad\text{with}\quad \langle R(u\wedge v), w\wedge w'\rangle \,=\, g(R(u,v)w,w')$$

for $x\in M$ and $u,v,w,w'\in T_xM$. When (M,g) is four-dimensional and oriented, another important morphism $[TM]^{\wedge 2}\to[TM]^{\wedge 2}$ is the *Hodge star* $*$, given by $*(e_1\wedge e_2) = \varepsilon_3\varepsilon_4\, e_3\wedge e_4$ for any $x\in M$ and any positive-oriented orthonormal basis $e_1,\dots,e_4$ of T_xM, where $\varepsilon_a = g(e_a,e_a)\in\{1,-1\}$ (no summation). This well-known description of $*$ (cf. [**10**], formula (37.13) on p. 639) is clearly equivalent to its more common definition $\alpha\wedge\beta = \langle *\alpha,\beta\rangle\,\mathrm{vol}$ for any bivectors α,β, where vol is the *volume four-vector*, equal to $e_1\wedge\ldots\wedge e_4$ for any $e_1,\dots,e_4$ as above.

Let (M,g) be an oriented pseudo-Riemannian 4-manifold. Then $[W,*]=0$, that is, the morphisms $W,*:[TM]^{\wedge 2}\to[TM]^{\wedge 2}$ commute (cf. [**16**]), while our formula for $*$ gives $*^2 = \mathrm{Id}$ if g is Riemannian $(++++)$ or neutral $(--++)$, and $*^2 = -\mathrm{Id}$ when g is Lorentzian $(-+++)$. In the Lorentzian case, this turns $[TM]^{\wedge 2}$ into a *complex vector bundle* of fibre dimension 3, in which $*$ is the multiplication by i, and, as $[W,*]=0$, the Weyl tensor W is a *complex-linear bundle morphism* $[TM]^{\wedge 2}\to[TM]^{\wedge 2}$. In the Riemannian and neutral cases, the self-adjoint involution $*:[TM]^{\wedge 2}\to[TM]^{\wedge 2}$ gives rise to the orthogonal decomposition $[TM]^{\wedge 2}=\Lambda^+M\oplus\Lambda^-M$, where $\Lambda^\pm M$, the (± 1)-eigenspace bundles of $*$, are real vector bundles of fibre dimension 3, called the bundles of *self-dual* and *anti-self-dual* bivectors in (M,g). As $[W,*]=0$, both $\Lambda^\pm M$ are W-invariant, which leads to the restrictions $W^\pm:\Lambda^\pm M\to\Lambda^\pm M$ of W, called the *self-dual* and *anti-self-dual* Weyl tensors of (M,g). See [**16**] and [**10**], pp. 637–651.

Remark 1.1. For a pseudo-Riemannian *Einstein* manifold (M,g) of dimension $n\geq 4$, the difference $R-W:[TM]^{\wedge 2}\to[TM]^{\wedge 2}$ of the morphisms R,W clearly equals the constant $\mathrm{s}/[n(n-1)]$ times $\mathrm{Id}=(g\wedge g)/2$. If $n=4$ and M is oriented, relation $[W,*]=0$ thus gives $[R,*]=0$, i.e., in the Lorentzian case the curvature operator $R:[TM]^{\wedge 2}\to[TM]^{\wedge 2}$ is complex-linear, while in the Riemannian and neutral cases both $\Lambda^\pm M$ are R-invariant; we will call the restriction $R^+:\Lambda^+M\to\Lambda^+M$ of R the *self-dual curvature operator* of (M,g).

Remark 1.2. Let x be a point in an oriented pseudo-Riemannian 4-manifold (M,g) having one of the sign patterns (1), and let $u\in T_xM$ be a vector such that $g(u,u)\neq 0$ and the subspace $u\wedge u^\perp$ of $[T_xM]^{\wedge 2}$ formed by all $u\wedge v$ with $v\in u^\perp$ is invariant under the Weyl tensor $W(x):[T_xM]^{\wedge 2}\to[T_xM]^{\wedge 2}$. Then

(a) In the Riemannian and neutral cases, the restriction $u\wedge u^\perp\to\Lambda^+_xM$ of the orthogonal projection $[T_xM]^{\wedge 2}\to\Lambda^+_xM$ is a linear isomorphism under which $W(x):u\wedge u^\perp\to u\wedge u^\perp$ corresponds to $W^+(x):\Lambda^+_xM\to\Lambda^+_xM$.

(b) In the Lorentzian case, the real subspace $u\wedge u^\perp$ spans $[T_xM]^{\wedge 2}$ as a complex vector space, and the Weyl tensor $W(x):[T_xM]^{\wedge 2}\to[T_xM]^{\wedge 2}$ is the unique complex-linear extension of $W(x):u\wedge u^\perp\to u\wedge u^\perp$.

In fact, our formula for $*$ applied to $e_1,\dots,e_4$ with $u=re_1$ for some $r>0$ shows that $\mathcal{H}=u\wedge u^\perp$ and its $*$-image $*\mathcal{H}$ together span $[T_xM]^{\wedge 2}$, and so, for dimensional reasons, $\mathcal{H}\cap *\mathcal{H}=\{0\}$. This gives (b). Now let g be Riemannian or neutral. As $\mathcal{H}\cap *\mathcal{H}=\{0\}$, the space $\mathcal{H}$ contains no nonzero (anti)self-dual bivectors. The projection $[T_xM]^{\wedge 2}\to\Lambda^+_xM$, which has the kernel Λ^-_xM, is therefore injective on $\mathcal{H}$, i.e., constitutes an isomorphism $\mathcal{H}\to\Lambda^+_xM$. Finally, since $\Lambda^\pm M$ are W-invariant, the projection commutes with $W(x)$, and (a) follows.

§2. One particular family of metrics

The construction described here goes back to Petrov; see [**15**], p. 185.

Let $\mathcal{X}$ be a real vector space of any dimension $n\geq 3$ with a codimension-one subspace $V\subset\mathcal{X}$ and an element $u\in\mathcal{X}\smallsetminus V$, and let $\langle\,,\rangle$ be a nondegenerate symmetric bilinear form in V. If a linear operator $F:V\to V$ is *self-adjoint* relative to $\langle\,,\rangle$, that is, $\langle Fv,v'\rangle=\langle v,Fv'\rangle$ for all $v,v'\in V$, then, choosing any $\delta\in\{1,-1\}$, we define a Lie-algebra multiplication $[\,,]$ in $\mathcal{X}$ and a nondegenerate symmetric bilinear form g in $\mathcal{X}$ by

$$(4)\qquad \begin{aligned}&\text{i)}\quad [u,v]=Fv\,,\qquad [v,v']=0\qquad \text{whenever}\quad v,v'\in V,\\ &\text{ii)}\quad g(u,u)=\delta,\quad g(u,v)=0,\quad g(v,v')=\langle v,v'\rangle\quad \text{for all}\quad v,v'\in V.\end{aligned}$$

Let there also be given an n-dimensional real manifold M such that $\mathcal{X}$, rather than being just an abstract Lie algebra, is a *simply transitive Lie algebra of vector* *fields* on M, as defined below in the appendix. An explicit description of such M is given in the last paragraph of this section; another option is to choose M to be the underlying manifold of a Lie group G whose Lie algebra of left-invariant vector fields is isomorphic to $\mathcal{X}$. Formula (4.ii) now defines a pseudo-Riemannian metric g on M such that $g(u,v)$ is constant whenever $u,v\in\mathcal{X}$, or, in Lie-group terms, g is invariant under left translations in G. If ∇, R and Ric denote the Levi-Civita connection, curvature tensor and Ricci tensor of this metric g, and $v,v',w\in V$ are treated, along with u, as vector fields on M, then

$$
(5)\quad
\begin{aligned}
&\text{a)}\ \ \nabla_u u \,=\, \nabla_u v \,=\, 0\,,\quad \nabla_v u \,=\, -\,Fv\,,\quad \nabla_v w \,=\, \delta\langle Fv,w\rangle u\,,\\
&\text{b)}\ \ R(u,v)u \,=\, -\,F^2 v\,,\quad R(v,w)u \,=\, 0\,,\quad R(u,w)v \,=\, \delta\langle F^2 w,v\rangle u\,,\\
&\text{c)}\ \ R(v,v')w = \delta\langle Fv',w\rangle Fv - \delta\langle Fv,w\rangle Fv',\qquad \text{d)}\ \ R(u\wedge v) = -\delta u\wedge F^2 v,
\end{aligned}
$$

with $R(u\wedge v)$ as in (3). Namely, by (4) with $\langle Fv,v'\rangle=\langle v,Fv'\rangle$, the connection ∇ in TM *defined* by (5.a) is torsionfree and $\nabla g=0$, so that ∇ must be the Levi-Civita connection of g. Next, (5.b, c) follow from (2.i), (5.a) and (4.i), while (5.d) is clear since, by (3) and (5.b, c), $R(u\wedge v)+u\wedge F^2 v$ is orthogonal to $u\wedge w$ and $v'\wedge w$ for $v',w\in V$ (cf. §1), and hence to all bivectors at every point. Also,

$$
(6)\quad \mathrm{Ric}\,(u,u) = -\,\mathrm{Trace}\,F^2,\quad \mathrm{Ric}\,(u,v) \,=\, 0,\quad \mathrm{Ric}\,(v,w) = -\delta\langle Fv,w\rangle\,\mathrm{Trace}\,F,
$$

for $v,w\in V$. In fact, by (2.ii) and (4.ii), $\mathrm{Ric}\,(u,u)$, $\mathrm{Ric}\,(v,w)\,-\,\delta g(R(u,v)u,w)$ and $\mathrm{Ric}\,(u,v)$ are the traces of the operators $V\to V$ given by $v\mapsto R(u,v)u$, $v'\mapsto R(v,v')w$, and $w\mapsto \mathrm{pr}[R(u,w)v]$, where $\mathrm{pr}:\mathcal{X}\to V$ is the orthogonal projection, so that (6) is immediate from (5.b, c).

An n-dimensional manifold M, admitting a simply transitive Lie algebra $\mathcal{X}$ of vector fields (cf. the appendix) with a vector subspace $V\subset\mathcal{X}$ and a linear operator $F:V\to V$ such that $\dim V=n-1$ and the Lie bracket in $\mathcal{X}$ satisfies (4.i) for some $u\in\mathcal{X}\smallsetminus V$, can be constructed as follows. We fix V and $F:V\to V$, then set $M=V\times(0,\infty)$ and let $\mathcal{X}=V+\mathbf{R}u$ be the space of vector fields on M spanned by u and V, where u is the linear vector field with $u(x,t)=(-Fx,t)$ for $(x,t)\in M$, and each $v\in V$ is identified with the constant vector field $(v,0)$. Now (4.i) follows since $[v,w]=d_v w-d_w v$ for vector fields v,w on any open subset U of a finite-dimensional vector space $\mathcal{X}$, treated as functions $U\to\mathcal{X}$.

§3. The Einstein case

Let $\mathcal{X},n,V,u,F,\langle\,,\rangle,\delta,M$ have the properties listed in §2, and let g be the pseudo-Riemannian metric with (4.ii) on the n-dimensional manifold M. Then g is Einstein if and only if one of the following conditions holds:

(i) F equals some real scalar λ times the identity.
(ii) $F\neq 0$, while $\mathrm{Trace}\,F=0$ and $F^2=0$.
(iii) $F^2\neq 0$ and $\mathrm{Trace}\,F=\,\mathrm{Trace}\,F^2=0$.

To see this, note that each of (i) – (iii) implies, by (6), that g is Einstein. Conversely, let g be Einstein; then either $\mathrm{Trace}\,F\neq 0$ (and hence (6) for v,w yields (i)), or $\mathrm{Trace}\,F=0$ and so, by (6), $\mathrm{Trace}\,F^2=0$, which in turn gives (i) (when $F=0$), or (ii) (when $F\neq 0$ and $F^2=0$), or, finally, (iii) (when $F^2\neq 0$).

Lemma 3.1. *For* $\mathcal{X},n,V,u,F,\langle\,,\rangle,\delta,M$ *and* g *as above, let* g *be an Einstein metric, so that we have* (i), (ii) *or* (iii). *Then*

In case (i), g *has the constant sectional curvature* $-\delta\lambda^2$.

In case (ii), *under the additional assumption that* $n = 4$, *the metric* g *is flat.* *In case* (iii), g *is Ricci-flat but* not *locally symmetric.*

In fact, the assertion about (i) follows from (5.b, c). Next, if $n = 4$, (ii) gives $F(V) \subset \operatorname{Ker} F \neq V$ and $\dim[F(V)] + \dim[\operatorname{Ker} F] = \dim V = 3$, i.e., $\dim[F(V)] = 1$ and $\dim[\operatorname{Ker} F] = 2$. Thus, the right-hand sides in (5.b, c) both vanish: the former since $F^2 = 0$, the latter in view of skew-symmetry in $Fv, Fv' \in F(V)$ with $\dim[F(V)] = 1$. This proves our claim about (ii).

Finally, in case (iii), $\mathrm{Ric} = 0$ by (6), while $(\nabla_w R)(u,v)v' = \nabla_w[R(u,v)v'] - R(\nabla_w u, v)v' - R(u, \nabla_w v)v' - R(u,v)\nabla_w v'$ for $v, v', w \in V$, and so $\delta(\nabla_w R)(u,v)v' = -\langle F^2 v, v'\rangle Fw + \langle Fv, v'\rangle F^2 w - \langle F^2 w, v'\rangle Fv + \langle Fw, v'\rangle F^2 v$ by (5). If R were parallel, setting $w = v$ and applying $g(\,\cdot\,, v'')$ with any $v'' \in V$ (see §1) we would get $Fv \wedge F^2 v = 0$ for all $v \in V$. Every $Fv \in F(V) \smallsetminus \{0\}$ thus would be an eigenvector of F, making F^2 a multiple of F, contrary to (iii) (cf. Remark 3.2). □

Remark 3.2. If $\operatorname{Trace} F = 0$ for an operator $F : V \to V$ in a finite-dimensional vector space V and F^2 equals a nonzero scalar times F, then $F = 0$. In fact, let $rF^2 = 2F$ and $r \neq 0$. Then $A^2 = \mathrm{Id}$ for the operator $A = \mathrm{Id} - rF$ in V, and so $A = \pm\,\mathrm{Id}$ on some subspaces $V_\pm$ with $V = V_+ \oplus V_-$. Hence $\operatorname{Trace} A = n_+ - n_-$, where $n_\pm = \dim V_\pm$, while $\operatorname{Trace} A = \dim V = n_+ + n_-$ as $A = \mathrm{Id} - rF$ and $\operatorname{Trace} F = 0$. Thus, $n_- = 0$, i.e., $V = V_+$, so that $A = \mathrm{Id}$ and $F = 0$.

§4. The curvature operator

Given a fixed sign $\pm$, formulae

(a) $V = \mathbf{C} \times \mathbf{R}$ and $\langle (z,t), (z',t')\rangle = \operatorname{Im} zz' \pm tt'$ for $(z,t), (z',t') \in V$,

(b) $F(z,t) = (pqz, pt)$, with $q = e^{2\pi i/3}$ and any fixed $p \in \mathbf{R} \smallsetminus \{0\}$,

define a real vector space V with $\dim V = 3$, a nondegenerate symmetric bilinear form $\langle\,,\rangle$ in V with the sign pattern $-\pm+$, and a self-adjoint operator $F : V \to V$ satisfying (iii) in §3. (See also the last paragraph of this section.)

In fact, $\langle F(z,t), (z',t')\rangle = \langle (pqz, pt), (z',t')\rangle = p(\operatorname{Im} qzz' \pm tt')$ is symmetric in $(z,t), (z',t')$, while (iii) holds for F since $F^2(z,t) = (p^2q^2 z, p^2 t)$, $q = (\sqrt{3}\,i - 1)/2$ and $q^2 = q^{-1} = \overline{q}$.

Remark 4.1. Let $B : V \to V$ be a linear operator in an n-dimensional real vector space V. As in §0, we call B *complex-diagonalizable* if its complex-linear extension $B : V^{\mathbf{C}} \to V^{\mathbf{C}}$ to the complexification of V is diagonalizable. Clearly, (a) if B is diagonalizable, it is complex-diagonalizable; (b) B is complex-diagonalizable whenever its characteristic polynomial has n distinct complex roots; (c) if V is the underlying real space of a complex vector space in which B acts complex-linearly, then complex-diagonalizability of B is equivalent to diagonalizability of B as a complex-linear operator.

Example 4.2. Let a four-manifold M and an indefinite metric g on M be chosen as in §2 using $n = 4$, some $\delta \in \{1, -1\}$, and $V, \langle\,,\rangle, F$ defined in (a), (b) above for any fixed sign $\pm$ and $p \in \mathbf{R} \smallsetminus \{0\}$. According to Lemma 3.1, g is Ricci-flat but not locally symmetric. By (4.ii), the sign pattern of g is $-\pm++$ (when $\delta = 1$) or $--\pm+$ (when $\delta = -1$). We consider two cases:

(i) $\delta = 1$ and the sign $\pm$ is $+$. Thus, g is a Ricci-flat Lorentzian metric.

(ii) $\delta = \mp 1$, so that g is a neutral $(--++)$ Ricci-flat metric.

In both cases, (M, g) is locally homogeneous and, by (4.ii), locally isometric to a Lie group with a left-invariant metric. (See Corollary A.3 in the appendix.)

Also, the curvature operator R of (M,g) is complex-diagonalizable at every point. Namely, by (5.d), R leaves invariant the subbundle $\mathcal{H}$ of $[TM]^{\wedge 2}$ spanned by all $u\wedge v$ with $v\in V$. Also, again by (5.d), $R:\mathcal{H}\to\mathcal{H}$ is, at every point, algebraically equivalent to $-\delta F^2 : V\to V$. On the other hand, F^2 is complex-diagonalizable by Remark 4.1(b), since F^2/p^2 has the characteristic roots $1, q, \overline{q}$, and our assertion follows, in case (i), from Remark 1.2(b) for any fixed orientation of M, combined with Remark 4.1(c), and, in case (ii), from Remark 1.2(a) applied to both orientations of M.

Remark 4.3. As we just saw, for (M,g) obtained in Example 4.2, the curvature operator R (case (i)), or its self-dual restriction R^+, for either orientation (case (ii)), has the complex eigenvalues $\lambda, \lambda e^{2\pi i/3}, \lambda e^{4\pi i/3}$ with $\lambda\in\mathbf{R}\smallsetminus\{0\}$. Also, for every locally symmetric pseudo-Riemannian Einstein 4-manifold with a complex-diagonalizable curvature operator, R (or, R^+) has a multiple eigenvalue ([**10**], pp. 662–663). Finally, according to sections 5 – 7 below, Example 4.2 describes, locally, all possible 4-dimensional curvature-homogeneous pseudo-Riemannian Einstein manifolds with the sign patterns (1), which are not locally symmetric. Thus, the algebraic types of curvature operators realized by curvature-homogeneous pseudo-Riemannian Einstein 4-manifolds are quite special, in analogy with the result of [**14**] for curvature-homogeneous *Riemannian* manifolds of dimension 4.

The claim made in the three lines following (a), (b) above remains valid if one replaces (b) with $F(z,t) = (\pm it, \operatorname{Re} z)$. This leads, as in Example 4.2, to another locally homogeneous Ricci-flat pseudo-Riemannian 4-manifold (M,g), except that, for analogous reasons, its curvature operator is *not* complex-diagonalizable.

§5. A classification theorem for the Lorentzian case

In the following theorem, proved in §13, the diagonalizability assumption about the curvature operator amounts to its complex-diagonalizability; see Remark 4.1(c).

Theorem 5.1. *Let (M,g) be an oriented four-dimensional Lorentzian Einstein manifold whose curvature operator, treated as a complex-linear vector bundle morphism $R:[TM]^{\wedge 2}\to[TM]^{\wedge 2}$, is diagonalizable at every point and has complex eigenvalues that form constant functions $M\to\mathbf{C}$. Then (M,g) is locally homogeneous, and one of the following three cases occurs:*

(a) *(M,g) is a space of constant curvature.*

(b) *(M,g) is locally isometric to the Riemannian product of two pseudo-Riemannian surfaces having the same constant Gaussian curvature.*

(c) *(M,g) is locally isometric to Petrov's Ricci-flat manifold of* Example 4.2(i). *Furthermore, (M,g) is locally symmetric in cases* (a) – (b), *but not in* (c), *and in case* (c) *it is locally isometric to a Lie group with a left-invariant metric.*

§6. A classification theorem in the neutral case

The next theorem will be proved at the end of §13. For the definitions of R^+ and complex-diagonalizability, see Remarks 1.1 and 4.1.

Theorem 6.1. *Let the self-dual curvature operator $R^+:\Lambda^+M\to\Lambda^+M$ of an oriented four-dimensional Einstein manifold (M,g) of the metric signature $--++$ be complex-diagonalizable at every point, with complex eigenvalues forming* *constant functions* $M \to \mathbf{C}$. *If* $\nabla R^+ \neq 0$ *somewhere in* M, *then* (M, g) *is locally homogeneous, namely, locally isometric to a Lie group with a left-invariant metric.*

More precisely, (M, g) *then is locally isometric to one of Petrov's Ricci-flat manifolds, described in* Example 4.2(ii).

Theorem 6.1 sounds much stronger than its Riemannian analogue, i.e., Theorem 7.2 in §7: an assumption about R^+ yields a complete local description of the metric in the former result, but only an assertion about R^+ in the latter. However, if the clause "$\nabla R^+ \neq 0$ somewhere" were to be included among the hypotheses of Theorem 7.2, as it is in Theorem 6.1, the conclusion of Theorem 7.2 would become an equally strong nonexistence statement.

§7. The Riemannian case

For Riemannian metrics, our assertion amounts to the following theorem, in which the assumption of complex-diagonalizability is redundant, as the curvature operator is self-adjoint at every point; cf. Remark 4.1(a). See also [**12**].

Theorem 7.1 ([**10**], Corollary 7.2 on p. 476). *If the curvature operator of a four-dimensional Riemannian Einstein manifold* (M, g), *acting on bivectors, has the same eigenvalues at every point* $x \in M$, *then* (M, g) *is locally symmetric.*

This is immediate from the next result, proved in §13 (and, originally, in [**10**]):

Theorem 7.2 ([**10**], p. 476, Theorem 7.1). *If* (M, g) *is an oriented Riemannian Einstein four-manifold and its self-dual curvature operator* $R^+ : \Lambda^+ M \to \Lambda^+ M$ *has the same eigenvalues at every point, then* R^+ *is parallel.*

§8. Further basics

Unless stated otherwise, all tensor fields are of class C^∞. For 1-forms ξ, η, vector fields u, v, w and a pseudo-Riemannian metric g on any manifold, $\xi \wedge \eta$, $d\xi$ and the Lie derivative $\mathcal{L}_w g$ are given by $(\xi \wedge \eta)(u, v) = \xi(u)\eta(v) - \xi(v)\eta(u)$, $(d\xi)(u, v) = d_u[\xi(v)] - d_v[\xi(u)] - \xi([u, v])$ and, with $[\ ,\ ]$ denoting the Lie bracket,

$$(\mathcal{L}_w g)(u, v) = d_w[g(u, v)] - g([w, u], v) - g(u, [w, v]). \tag{7}$$

On a pseudo-Riemannian manifold (M, g) we use the same symbol, such as u, for a vector field and the corresponding 1-form $g(u, \cdot)$. Similarly, a vector-bundle morphism $\alpha : TM \to TM$ is treated as a twice-contravariant tensor field, and as a twice-covariant one with $\alpha(u, v) = g(\alpha u, v)$ for vector fields u, v. In particular, a bivector field α (such as $v \wedge u$) is also regarded as a differential 2-form, or a morphism $\alpha : TM \to TM$ with $\alpha^* = -\alpha$ (i.e., skew-adjoint at each point). Specifically, for bivector fields α, α' and C^1 vector fields u, v, w,

$$\begin{aligned}
&\text{a) } v \wedge u = v \otimes u - u \otimes v, \qquad dw = P - P^*, \quad \text{where } P = \nabla w, \\
&\text{b) } (v \otimes u)w = \langle v, w\rangle u, \qquad (v \wedge u)w = \langle v, w\rangle u - \langle u, w\rangle v, \\
&\text{c) } \langle \alpha, v \wedge u\rangle = g(\alpha v, u), \qquad \langle \alpha, \alpha'\rangle = -\operatorname{Trace}(\alpha \circ \alpha')/2, \\
&\text{d) } \alpha \circ (v \wedge u) = v \otimes (\alpha u) - u \otimes (\alpha v), \quad \text{e) } \operatorname{Trace}[\alpha \circ (v \wedge u)] = -2g(\alpha v, u), \\
&\text{f) } 2P^* w = d\langle w, w\rangle, \qquad \text{where } P = \nabla w \text{ and } \langle w, w\rangle = g(w, w),
\end{aligned} \tag{8}$$

with $\langle v, w\rangle = g(v, w)$, $\langle u, w\rangle = g(u, w)$ in b). (Cf. §1.) Here d) follows from b), and implies e), as $\operatorname{Trace}(v \otimes u) = g(v, u)$, while $P = \nabla w : TM \to TM$ in a), f) acts by $Pv = \nabla_v w$, and so $2\langle v, P^* w\rangle = 2\langle Pv, w\rangle = d_v\langle w, w\rangle$, which gives f).

Remark 8.1. Let (M, g) be a pseudo-Riemannian Einstein manifold. Then $\operatorname{div} W = 0$. If, in addition, M is oriented, $\dim M = 4$, and the sign pattern of g is $++++$ or $--++$, then also $\operatorname{div} W^+ = \operatorname{div} W^- = 0$.

In fact, these are well-known consequences of the second Bianchi identity (cf. [**10**], pp. 460, 468). Here $\operatorname{div}\alpha$, for any covariant tensor field α, is the g-contraction of $\nabla\alpha$ involving the first argument of α and the differentiation argument.

By a *complex vector field* on a real manifold M we mean a section w of its complexified tangent bundle. Sections of the ordinary ("real") tangent bundle of M may be referred to as *real vector fields* on M. Thus, $w = u + iv$ with real vector fields $u = \operatorname{Re} w$, $v = \operatorname{Im} w$. Complex *bivector* fields are defined similarly.

All real-multilinear operations involving real vector/bivector fields will, without further comment, be extended to complex vector (or, bivector) fields v, w (or, α, α'), so as to become complex-linear in each argument. This includes the Lie bracket $[v, w]$, the covariant derivative $\nabla_v w$ relative to any connection in the tangent bundle, the inner product $g(v, w)$, the Lie derivative $\mathcal{L}_w g$ for any given pseudoRiemannian metric g, the composite $\alpha \circ \alpha'$, as well as $\langle \alpha, \alpha' \rangle$, $*\alpha$ and αv (cf. (8.c)). Note that $g(v, w)$ and $\langle \alpha, \alpha' \rangle$ are *complex-bilinear* (not sesquilinear!) in v, w or α, α', while a C^∞ complex vector field w is a Killing field, i.e., $\mathcal{L}_w g = 0$, if and only if its real and imaginary parts both are real Killing fields.

Although the bivector bundle $[TM]^{\wedge 2}$ of an oriented Lorentzian four-manifold (M, g) is a complex vector bundle with the multiplication by i provided by $*$ (see §1), it is also convenient to use the complexification $([TM]^{\wedge 2})^{\mathbf{C}}$ of its underlying real vector bundle. Then $([TM]^{\wedge 2})^{\mathbf{C}} = \Lambda^+ M \oplus \Lambda^- M$, where $\Lambda^\pm M$ are, this time, the complex vector bundles of fibre dimension 3, obtained as the $(\pm i)$-eigenspace bundles of $*$. This is clear since $*^2 = -\operatorname{Id}$, cf. §1, and the complex-conjugation antiautomorphism $([TM]^{\wedge 2})^{\mathbf{C}} \to ([TM]^{\wedge 2})^{\mathbf{C}}$ obviously sends $\Lambda^+ M$ onto $\Lambda^- M$.

§9. A unified treatment of all three cases

Throughout this section (M, g) stands for a fixed oriented pseudo-Riemannian four-manifold with a metric g of one of the sign patterns (1), while $\mathcal{E}$ is a complex vector bundle of of fibre dimension 3 over M, and $W^{(+)}$ is a complex-linear bundle morphism $\mathcal{E} \to \mathcal{E}$. Our choices of $\mathcal{E}$ and $W^{(+)}$ are quite specific. Namely, when g is Riemannian or neutral, $\mathcal{E} = [\Lambda^+ M]^{\mathbf{C}}$ is the complexification of the subbundle $\Lambda^+ M$ of $[TM]^{\wedge 2}$ (§1) and $W^{(+)}$ is the unique $\mathbf{C}$-linear extension of $W^+ : \Lambda^+ M \to \Lambda^+ M$ to $[\Lambda^+ M]^{\mathbf{C}}$, while, if g is Lorentzian, $\mathcal{E} = \Lambda^+ M \subset ([TM]^{\wedge 2})^{\mathbf{C}}$ (see end of §8) and $W^{(+)}$ is the restriction to $\mathcal{E}$ of the $\mathbf{C}$-linear extension of $W : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$ to $([TM]^{\wedge 2})^{\mathbf{C}}$. (The latter extension leaves $\mathcal{E}$ invariant, since $[W, *] = 0$, cf. §1.)

We will use the symbol ∇ for the connection in $\mathcal{E}$ induced by the Levi-Civita connection of g, and let h stand for the complex-bilinear fibre metric in $\mathcal{E}$ which, in the Riemannian/neutral (or, Lorentzian) case is the unique complex-bilinear extension of $\langle\, ,\rangle$ (see §1) from $\Lambda^+ M$ to $[\Lambda^+ M]^{\mathbf{C}}$ (or, respectively, the restriction to $\mathcal{E} = \Lambda^+ M$ of the complex-bilinear extension of $\langle\, ,\rangle$ from $[TM]^{\wedge 2}$ to $([TM]^{\wedge 2})^{\mathbf{C}}$). Note that, in all cases, $\nabla h = 0$ and $\Lambda^+ M$ is a ∇-parallel subbundle of $[TM]^{\wedge 2}$ or $([TM]^{\wedge 2})^{\mathbf{C}}$, since the Levi-Civita connection of g makes both g and $*$ parallel.

In this and the next two sections, the indices j, k, l always vary in the range $\{1, 2, 3\}$ and repeated indices are summed over, unless explicitly stated otherwise.

Given $M, g, \mathcal{E}, W^{(+)}, \nabla, h$ as above, let us now fix any C^∞ local sections α_j of $\mathcal{E}$ which trivialize $\mathcal{E}$ on an open set $U \subset M$. This gives rise to complex-valued functions h_{jk} and 1-forms ξ_j^k with

$$(9)\qquad \begin{aligned} &\text{a) } \nabla\alpha_j = \xi_j^l \otimes \alpha_l\,, \ \text{ i.e., } \ \nabla_v\alpha_j = \xi_j^l(v)\alpha_l \ \text{ for every tangent vector field } v,\\ &\text{b) } dh_{jk} = \xi_{jk} + \xi_{kj}\,, \quad \text{where } \ \xi_{jk} = \xi_j^l h_{lk} \ \text{ and } \ h_{jk} = h(\alpha_j, \alpha_k). \end{aligned}$$

Thus, h_{jk} are the component functions of the fibre metric h and ξ_j^k are the connection forms of ∇, relative to the α_j, while (9.b) states that $\nabla h = 0$. For div as in Remark 8.1 and all tangent vectors v we have, with summation over k,

$$(10)\qquad \begin{aligned} &\text{i) } [\nabla_v W]\alpha_j = \theta_j^k(v)\alpha_k\,, && \text{ii) } [\operatorname{div} W]\alpha_j = \alpha_k\,\theta_j^k\,, \quad \text{where}\\ &\text{iii) } W\alpha_j = W_j^k\alpha_k \quad \text{and} && \text{iv) } \theta_j^l = dW_j^l + W_j^k\xi_k^l - W_k^l\xi_j^k\,. \end{aligned}$$

In fact, ∇_v applied to iii) gives i) (by (9.a)), and contracting i) we get ii) . (Here W might be replaced by $W^{(+)}$, as $\nabla W^{\pm}$ are the $\Lambda^{\pm}M$ components of ∇W.)

Remark 9.1. If g is Riemannian or neutral, $W^{(+)}$ is the **C**-linear extension of W^+ to $[\Lambda^+M]^{\mathbf{C}}$. Thus, in the Riemannian case, the eigenvalues of $W^{(+)}$ at every point are all real, as $W : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$ is self-adjoint.

If g is Lorentzian, $W^{(+)}$ is, at each point, algebraically equivalent to W acting in $[TM]^{\wedge 2}$, since $\alpha \mapsto \alpha - i[*\alpha]$ is an isomorphism $[TM]^{\wedge 2} \to \Lambda^+M$ of complex vector bundles, sending W onto $W^{(+)}$ (as $[W, *] = 0$, cf. §1).

§10. Calculations in a local orthonormal frame

As in §9, the indices j, k, l vary in the set $\{1, 2, 3\}$. The *Ricci symbol* ε_{jkl} will always stand for the signum of the permutation (j, k, l) of $(1, 2, 3)$, if $j \neq k \neq l \neq j$, while $\varepsilon_{jkl} = 0$ if $j = k$ or $k = l$ or $l = j$. From now on we assume (cf. Remark 10.3 below) that, for our α_j and some $\varepsilon_j \in \mathbf{R}$,

$$(11)\qquad \begin{aligned} &\text{i) } \varepsilon_j\alpha_j \circ \alpha_j = -\mathrm{Id} \ \text{ and } \ \alpha_j \circ \alpha_k = \varepsilon_l\alpha_l = -\alpha_k \circ \alpha_j \ \text{ if } \varepsilon_{jkl} = 1,\\ &\text{ii) } \varepsilon_1\varepsilon_2\varepsilon_3 = 1 \quad \text{and} \quad \varepsilon_j \in \{1, -1\}\,, \ j = 1, 2, 3. \end{aligned}$$

(No summing over j, l.) For a complex vector field w, the complex vector fields

$$(12)\qquad v_j = \alpha_j w\,, \qquad j = 1, 2, 3\,,$$

satisfy, in view of (11) and skew-adjointness of the α_j, the relations

$$(13)\qquad \begin{aligned} &\text{a) } \langle v_j, v_k\rangle = \varepsilon_j\langle w, w\rangle\,\delta_{jk} \ \text{ (no summation)}, \ \langle w, v_j\rangle = 0, \ \ j, k = 1, 2, 3,\\ &\text{b) } \alpha_j v_k = -\alpha_k v_j = \varepsilon_l v_l, \ \ \alpha_j v_j = -\varepsilon_j w \ \text{ (no summing) if } \varepsilon_{jkl} = 1, \end{aligned}$$

with $\langle\,,\rangle$ standing for $g(\ ,\ )$. From (11), (9.b) and (8.c), $h_{jk} = 2\varepsilon_j\delta_{jk}$ (no summing), and so, again by (9.b), the ξ_{jk} are skew-symmetric in j, k. Therefore, as $\varepsilon_k\varepsilon_l = \varepsilon_j$ when $\{j, k, l\} = \{1, 2, 3\}$ (by (11.ii)), we have, from (9.a), (11.ii),

$$(14)\qquad \begin{aligned} &\text{i) } \xi_j^j = 0 \text{ (no summing) and } \xi_j^k = \varepsilon_j\xi_l\,, \ \xi_j^l = -\varepsilon_j\xi_k \ \text{ if } \varepsilon_{jkl} = 1\,,\\ &\text{ii) } \varepsilon_j\nabla\alpha_j = \xi_l \otimes \alpha_k - \xi_k \otimes \alpha_l \ \text{ (no summing) whenever } \varepsilon_{jkl} = 1\,, \end{aligned}$$

with the 1-forms ξ_j defined by $\xi_j = \varepsilon_j\xi_{kl}$ if $\varepsilon_{jkl} = 1$. Next, we define complex-valued functions λ_j, μ_j, $j = 1, 2, 3$, by

$$(15)\qquad \lambda_j = W_j^j\,, \quad \text{and} \quad \mu_j = \varepsilon_l W_k^l \ \text{ if } \ \{j, k, l\} = \{1, 2, 3\} \ \text{ (no summing).}$$

We always have $\operatorname{Trace} W^{(+)} = 0$ ([**10**], p. 650); thus, for any function s on U,

$$\text{(16)}\qquad \begin{aligned} &\text{a)}\ \lambda_1 + \lambda_2 + \lambda_3 = 0\,, \\ &\text{b)}\ L_1 + L_2 + L_3 = 0 \ \text{ if } \ L_j = (\lambda_k - \lambda_l)(\lambda_j + \mathrm{s}/12) \ \text{ whenever } \ \varepsilon_{jkl} = 1\,, \end{aligned}$$

Remark 10.1. By (10.iv), (14.i) and (15), $\theta_j^j = d\lambda_j + 2\mu_k\xi_k - 2\mu_l\xi_l$, $\theta_j^k = \varepsilon_k d\mu_l + \varepsilon_j(\lambda_j - \lambda_k)\xi_l + \varepsilon_j\varepsilon_k\mu_j\xi_k - \mu_k\xi_j$, $\theta_j^l = \varepsilon_l d\mu_k + \varepsilon_j(\lambda_l - \lambda_j)\xi_k - \varepsilon_j\varepsilon_l\mu_j\xi_l - \mu_l\xi_j$ (no summing), if $\varepsilon_{jkl} = 1$. Hence, from (10.ii) and (11.i), $[\operatorname{div} W]\,\alpha_j = \alpha_j\,[d\lambda_j + w_k - w_l]$ whenever $\varepsilon_{jkl} = 1$ (no summing), for the complex vector fields w_j given by $w_j = 2\mu_j\xi_j + \alpha_j\,[d\mu_j + \varepsilon_k\varepsilon_l(\lambda_k - \lambda_l)\xi_j + \varepsilon_k\mu_k\xi_l - \varepsilon_l\mu_l\xi_k]$ if $\varepsilon_{jkl} = 1$ (no summing).

Consequently, if $\operatorname{div} W^{(+)} = 0$ and the λ_j are all constant, then there exists a complex vector field w with $w_1 = w_2 = w_3 = w$. This is clear if one applies α_j to the above formula for $[\operatorname{div} W]\,\alpha_j$ and uses (11.i).

Remark 10.2. For $M, g, \mathcal{E}, W^{(+)}, U, \alpha_j, \varepsilon_j, \xi_j$ as above, with (11),

(i) $dv_j = \varepsilon_j\,\xi_l \wedge v_k - \varepsilon_j\,\xi_k \wedge v_l + P_j$ if $\varepsilon_{jkl} = 1$, for v_j, P_j given by (12) and

$$\text{(17)}\qquad P_j = \alpha_j \circ P + P^* \circ \alpha_j \qquad \text{with} \quad P = \nabla w\,,$$

where w is any given complex C^∞ vector field defined on U.

(ii) If W_j^k in (10.iii) are constant, the following two conditions are equivalent:

a) $\nabla W^{(+)} = 0$ everywhere in U.

b) $\mu_1\xi_1 = \mu_2\xi_2 = \mu_3\xi_3$ and $(\lambda_k - \lambda_l)\xi_j + \varepsilon_l\mu_k\xi_l - \varepsilon_k\mu_l\xi_k = 0$ for λ_j, μ_j given by (15) and any j, k, l with $\{j, k, l\} = \{1, 2, 3\}$.

(iii) If g is Einstein and $\varepsilon_{jkl} = 1$, then $d\xi_j + \varepsilon_j\,\xi_k \wedge \xi_l = -(W + \mathrm{s}/12)\,\alpha_j$.

In fact, $\nabla v_j = \varepsilon_j\,\xi_l \otimes v_k - \varepsilon_j\,\xi_k \otimes v_l + \alpha_j \circ P$ if $\varepsilon_{jkl} = 1$, by (14.ii), and so (8.a) yields (i). Next, (ii) is clear from (10.i), (15) and the formulae for $\theta_j^j, \theta_j^k, \theta_j^l$ in Remark 10.1. Finally, let g be Einstein. For fixed j, k, l with $\varepsilon_{jkl} = 1$, (2.i), (14.ii) and the formulae for $\xi \wedge \eta, d\xi$ in §8 give $-\varepsilon_k\varepsilon_l\,\omega/2 = d\xi_j + \varepsilon_j\,\xi_k \wedge \xi_l$, where ω is the complex-valued 2-form with $\omega(u, v) = h(R^\nabla(u, v)\alpha_k, \alpha_l)$ for any vector fields u, v, with R^∇ denoting the curvature of our connection ∇ in $\mathcal{E}$ (§9). However, $R^\nabla(u, v)\alpha_k = [R(u, v), \alpha_k]$, where $[\,,\,]$ also stands for the commutator of bundle morphisms $[TM]^{\mathbf{C}} \to [TM]^{\mathbf{C}}$, and $R(u, v) : [TM]^{\mathbf{C}} \to [TM]^{\mathbf{C}}$ is defined as in (2.i); this is easily seen using (2.i) and the Leibniz-rule equality $\nabla_u\alpha = [\nabla_u, \alpha]$ for such morphisms α (with the commutator applied, this time, to operators acting on vector *fields*). Hence, by Lemma 5.3 on p. 460 of [**10**], $\omega = R[\alpha_k, \alpha_l]$ with $[\alpha_k, \alpha_l] = \alpha_k \circ \alpha_l - \alpha_l \circ \alpha_k$, i.e., as $[\alpha_k, \alpha_l] = 2\varepsilon_j\alpha_j$ (by (11.i)) and $R = W + \mathrm{s}/12$ (Remark 1.1), we have $\omega = 2\varepsilon_j(W + \mathrm{s}/12)\alpha_j$, and (iii) follows.

Remark 10.3. Let $M, g, \mathcal{E}, W^{(+)}$ be as in §9. If $W^{(+)}(x) : \mathcal{E}_x \to \mathcal{E}_x$ is diagonalizable for every $x \in M$ and the set of its eigenvalues does not depend on x, then a suitable connected neighborhood U of any given point of M admits C^∞ local trivializing sections α_j of $\mathcal{E}$, $j = 1, 2, 3$, satisfying conditions (11) along with (14.ii) for suitable ε_j, ξ_j, and such that the corresponding complex-valued functions λ_j, μ_j in (15) are all constant, with $\mu_j = 0$. Thus, $W\alpha_j = \lambda_j\alpha_j$ (no summing), for $j = 1, 2, 3$, i.e., the λ_j then are the (constant) eigenvalues of $W^{(+)}$.

Namely, by Lemma 6.15(ii),(iii) of [**10**], p. 468, $W\alpha_j = \lambda_j\alpha_j$ for some C^∞ sections α_j trivializing $\mathcal{E}$ on such a set U, and constants λ_j. As $W^{(+)}$ is self-adjoint relative to h, cf. §1, while h is nondegenerate, the α_j may be chosen so that $h_{jk} = 2\varepsilon_j\delta_{jk}$ (no summing) with $\varepsilon_j \in \{1, -1\}$. Next, for sections α, β

of $\mathcal{E}$, the anticommutator $\{\alpha,\beta\} = \alpha\circ\beta+\beta\circ\alpha$ equals $-h(\alpha,\beta)$ times Id, and the commutator $[\alpha,\beta] = \alpha\circ\beta-\beta\circ\alpha$ is a section of $\mathcal{E}$. (For $\{\alpha,\beta\}$ one can verify this, in the Lorentzian case, using a basis of $\Lambda^+_x M$, $x\in M$, obtained by replacing each α by $\alpha - i[*\alpha]$ in a basis of $[TM]^{\wedge 2}$ of the form (37.28) in [**10**], p. 642; about the Riemannian and neutral cases, and for $[\alpha,\beta]$, see [**10**], formulae (37.31), (37.29) on pp. 642, 643.) Thus, by (8.c), $\varepsilon_j\alpha_j\circ\alpha_j = -\mathrm{Id}$, $j=1,2,3$, and $\alpha_j\circ\alpha_k = \delta_l\varepsilon_l\alpha_l = -\alpha_k\circ\alpha_j$ whenever $\varepsilon_{jkl}=1$, with some $\delta_j\in\{1,-1\}$, $j=1,2,3$. (In view of (8.c), $\alpha_j\circ\alpha_k = [\alpha_j,\alpha_k]/2$ is h-orthogonal to α_j,α_k.) Now, as $(\alpha_j\circ\alpha_k)\circ\alpha_l = \alpha_j\circ(\alpha_k\circ\alpha_l)$, we get $\delta_l=\delta_j$. Consequently, $\delta_1=\delta_2=\delta_3$. Applying an odd permutation to the α_j and/or replacing them by $-\alpha_j$, if necessary, we now obtain (11).

§11. The main structure theorem

The following result is a crucial step in our classification argument. We establish it using a refined version of the proof of Theorem 7.1 in [**10**] (pp. 477–479).

THEOREM 11.1. *Suppose that (M,g) is an oriented pseudo-Riemannian Einstein four-manifold with one of the sign patterns* (1), *such that $W^{(+)}:\mathcal{E}\to\mathcal{E}$, defined as in* §9, *is diagonalizable at every point and has constant eigenvalues.*

(i) *If g is positive definite, the self-dual Weyl tensor W^+ is parallel.*

(ii) *If g is Lorentzian $(-+++)$ or neutral $(--++)$ and $W^{(+)}$ is not parallel, then any given point of M has a neighborhood U with C^∞ complex vector fields w,v_1,v_2,v_3 which are linearly independent at every point of U, commute with every real Killing field defined on any open subset of U, and satisfy the inner-product and Lie-bracket relations*

$$\begin{aligned}&g(w,w)=g(v_j,v_j)=\gamma\ \text{(no summing)},\ \ g(w,v_j)=g(v_j,v_k)=0\ \text{ if } j\neq k,\\ &[w,v_j]=\rho_j v_j\quad\text{(no summing)},\qquad [v_j,v_k]=0\quad\text{for all } a,b\in\{1,2,3\},\end{aligned}\tag{18}$$

for some $\gamma\in\mathbf{C}\smallsetminus\{0\}$, where $\rho_j\in\mathbf{C}$ are the three cubic roots of γ^2, and both $g,[\,,\,]$ act complex-bilinearly on complex vector fields.

PROOF. Given $x\in M$, let us choose $\mathcal{E}, W^{(+)}, U, \alpha_j, \varepsilon_j, \xi_j, \lambda_j$ as in Remark 10.3, with $x\in U$. Since (M,g) is Einstein, $\operatorname{div} W^{(+)}=0$ (Remark 8.1) and so, by Remark 10.1, $w_1=w_2=w_3=w$ for some complex vector field w, where the w_j are as in Remark 10.1 with $\mu_1=\mu_2=\mu_3=0$. Now

(a) $v_j=(\lambda_l-\lambda_k)\xi_j$ if $\varepsilon_{jkl}=1$, for the complex vector fields $v_j=\alpha_j w$,

(b) $d\xi_j + \varepsilon_j\xi_k\wedge\xi_l = -(\lambda_j+\mathrm{s}/12)\alpha_j$ whenever $\varepsilon_{jkl}=1$,

s being the scalar curvature; in fact, (a) follows if one applies α_j to the formula for w_j in Remark 10.1 (with $\mu_j=0$ and $w_j=w$), using (11.i, iii), while (b) is obvious from Remark 10.2(iii) with $W\alpha_j=\lambda_j\alpha_j$. We now define a constant ϕ by

$$\phi = (\lambda_j-\lambda_k)(\lambda_k-\lambda_l)(\lambda_l-\lambda_j)\qquad\text{whenever}\quad \varepsilon_{jkl}=1\,.\tag{19}$$

Throughout this proof we will write $\langle\,,\rangle$ instead of $g(\ ,\ )$. For P_j given by (17),

$$\begin{aligned}&\text{i)}\ \ (\lambda_j-\lambda_k)(\lambda_j-\lambda_l)P_j + 2\varepsilon_j(\lambda_k-\lambda_l)v_k\wedge v_l = -(\lambda_j+\mathrm{s}/12)\,\phi\,\alpha_j\,,\\ &\text{ii)}\ \ (\lambda_j-\lambda_k)(\lambda_j-\lambda_l)\operatorname{div} w + 2(\lambda_k-\lambda_l)\langle w,w\rangle = -(2\lambda_j+\mathrm{s}/6)\,\phi\,,\end{aligned}\tag{20}$$

if $\varepsilon_{jkl}=1$. In fact, since the λ_j are constant, multiplying (b) above by ϕ and using (a) we obtain $(\lambda_j-\lambda_k)(\lambda_j-\lambda_l)\,dv_j+\varepsilon_j(\lambda_k-\lambda_l)v_k\wedge v_l = -(\lambda_j+\mathrm{s}/12)\phi\alpha_j$, if $\varepsilon_{jkl}=1$. In view of Remark 10.2(i), (a) above, and (19), this is nothing else than

(20.i). Also, as $\operatorname{div} w = \operatorname{Trace}_{\mathbf{C}} \nabla w$, taking the complex trace of the composites of both sides of (20.i) with α_j, we obtain (20.ii) from (8.e), (13) and (11.i), since, by (17), $2\operatorname{Trace} P = -\varepsilon_j \operatorname{Trace}(\alpha_j \circ P_j)$ (no summation). Next, for ϕ as in (19),

$$\text{(21)} \qquad \phi = 0 \qquad \text{if and only if} \qquad \nabla W^{(+)} = 0 \quad \text{identically.}$$

Namely, if $\phi = 0$, by (19), (a) above, (12) and (11.i), $w = 0$ and $(\lambda_l - \lambda_k)\xi_j = 0$ whenever $\varepsilon_{jkl} = 1$, and so (as $\mu_j = 0$, $j = 1, 2, 3$), Remark 10.2(ii) yields $\nabla W^{(+)} = 0$. Conversely, let $\nabla W^{(+)} = 0$. Remark 10.2(ii) with $\mu_j = 0$ now gives $(\lambda_l - \lambda_k)\xi_j = 0$ whenever $\varepsilon_{jkl} = 1$. Hence $\phi = 0$, for if we had $\phi \neq 0$, the last relation and (19) would imply $\xi_j = 0$, $j = 1, 2, 3$, i.e., from (b) above, $\lambda_1 = \lambda_2 = \lambda_3 = -\mathrm{s}/12$, and, by (19), ϕ would be zero anyway.

Since our assertion is immediate when $\nabla W^{(+)} = 0$, we now assume that

$$\text{(22)} \qquad \phi \neq 0\,, \qquad \text{i.e.,} \quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1\,.$$

(Cf. (19), (21).) We may treat (20.ii) as a system of three linear equations with two unknowns: $\operatorname{div} w$ and $\langle w, w\rangle$. This system's matrix has the 2×2 subdeterminants equal, by (16.a), to $\pm 6\lambda_j(\lambda_k - \lambda_l)^2$, $\varepsilon_{jkl} = 1$. They cannot be all zero, or else (16.a) would give $0 = \lambda_j(\lambda_k - \lambda_l) = -(\lambda_k + \lambda_l)(\lambda_k - \lambda_l) = \lambda_l^2 - \lambda_k^2$, if $\varepsilon_{jkl} = 1$, i.e., any two of the λ_j would coincide up to a sign, proving that, with the λ_j suitably rearranged, $\lambda_2 = \lambda_3 = \pm\lambda_1$, contrary to (22). Hence the system (20.ii) has rank two, and can be solved for $\operatorname{div} w$ and $\langle w, w\rangle$ using determinants. Thus, $\langle w, w\rangle$ is constant since so are the coefficients of (20.ii) (cf. (19)); in addition, $\langle w, w\rangle \neq 0$. Namely, if $\langle w, w\rangle$ were zero, we would have $\operatorname{div} w = (\lambda_k - \lambda_l)(2\lambda_j + \mathrm{s}/6)$, $\varepsilon_{jkl} = 1$, by (20.ii), (19) and (22); summed over $j = 1, 2, 3$, this would yield $\operatorname{div} w = 0$ (cf. (16.b)); hence $(\lambda_k - \lambda_l)(2\lambda_j + \mathrm{s}/6) = 0$, $\varepsilon_{jkl} = 1$, which, in view of (22), would imply that $2\lambda_j = -\mathrm{s}/6$ for $j = 1, 2, 3$, contrary to (22). Next,

$$\text{(23)} \qquad \begin{aligned} &\text{i)}\ \ \nabla_{v_j} w = \lambda_j(\lambda_k - \lambda_l)\, v_j \qquad \text{whenever} \quad \varepsilon_{jkl} = 1\,,\\ &\text{ii)}\ \ \operatorname{div} w = 0\,, \qquad \text{iii)}\ \ \mathrm{s} = 0\,,\ \text{ i.e., } (M, g) \text{ is Ricci-flat.}\end{aligned}$$

In fact, both sides of (20.i) may be treated as bundle morphisms $[TM]^{\mathbf{C}} \to [TM]^{\mathbf{C}}$, and hence applied to the complex vector field $v_j = \alpha_j w$, giving, by (17), (13) and (8.b), $(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)\,\alpha_j P v_j = \varepsilon_j(\lambda_j + \mathrm{s}/12)\,\phi w$, whenever $\varepsilon_{jkl} = 1$. (Note that, from (8.b) and (13.a), $(v_k \wedge v_l)v_j = 0$, while, as $\langle w, w\rangle$ is constant, (13.b) and (8.f) yield $P^*\alpha_j v_j = 0$.) Now, applying α_j to both sides of the last equality, we obtain $(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)\, P v_j = -(\lambda_j + \mathrm{s}/12)\phi v_j$ from (11.i) and (12). Thus, since $Pv = \nabla_v w$ for all vectors v, (19) and (22) imply that $\nabla_{v_j} w = (\lambda_k - \lambda_l)(\lambda_j + \mathrm{s}/12)\, v_j$ whenever $\varepsilon_{jkl} = 1$. Also, w, v_1, v_2, v_3 form an orthogonal trivialization of $[TU]^{\mathbf{C}}$ (by (13.a) with $\langle w, w\rangle \neq 0$). Evaluating $\operatorname{div} w$ in that trivialization, we get, from (13.a), $\langle w, w\rangle \operatorname{div} w = \langle w, w\rangle \operatorname{Trace}_{\mathbf{C}} \nabla w = \sum_{j=1}^{3} \varepsilon_j \langle v_j\,, \nabla_{v_j} w\rangle$, since $\langle w, \nabla_w w\rangle = 0$ as $\langle w, w\rangle$ is constant. Therefore, (23.ii) is immediate from the above formula for $\nabla_{v_j} w$ and (13.a), (16.b). Finally, we have $(\lambda_k - \lambda_l)\langle w, w\rangle = -(\lambda_j + \mathrm{s}/12)\,\phi$, $\varepsilon_{jkl} = 1$, from (20.ii) and (23.ii). Summed over j this gives $\mathrm{s}\phi = 0$ (by (16.a)), and so (22) yields (23.iii), while (23.iii) and our formula for $\nabla_{v_j} w$ imply (23.i).

Next, (20.ii) and (23.ii, iii) give $\phi\lambda_j = (\lambda_l - \lambda_k)\langle w, w\rangle$, $\varepsilon_{jkl} = 1$, i.e., by (19) and (22), $\langle w, w\rangle = \lambda_j(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)$ if $\varepsilon_{jkl} = 1$. However, this means that $\langle w, w\rangle = 2\lambda_j^3 + \lambda_1\lambda_2\lambda_3$, $j = 1, 2, 3$. (Note that, whenever $\varepsilon_{jkl} = 1$, (16.a) yields $(\lambda_j - \lambda_k)(\lambda_j - \lambda_l) = \lambda_j^2 - \lambda_j(\lambda_k + \lambda_l) + \lambda_k\lambda_l = 2\lambda_j^2 + \lambda_k\lambda_l$.) Thus, $\lambda_j^3 = \mu$ for some complex number μ, not depending on $j \in \{1, 2, 3\}$ and, by (22), the λ_j are the three cubic roots of μ, so that $\lambda_1\lambda_2\lambda_3 = \mu$. As $\langle w, w\rangle = 2\lambda_j^3 + \lambda_1\lambda_2\lambda_3$, we have $\lambda_j^3 = -\gamma$, $j = 1, 2, 3$, for the constant $\gamma = -\langle w, w\rangle/3 \in \mathbf{C} \smallsetminus \{0\}$.

Hence, by (22), the λ_j cannot be all real. Therefore, according to Remark 9.1, (22) implies that g is not Riemannian, i.e., $\phi = 0$ in the Riemannian case, which, in view of (21), proves assertion (i).

We may now assume that g is Lorentzian or neutral and $\nabla W^{(+)} \neq 0$. Thus,

$$\text{(24)}\quad \begin{array}{l} \text{i) } \lambda_k = z\lambda_j\,,\ \lambda_l = \overline{z}\lambda_j \text{ if } \varepsilon_{jkl} = 1, \quad \text{ii) } \lambda_k - \lambda_l = \pm i\sqrt{3}\,\lambda_j\,,\ \ \varepsilon_{jkl} = 1,\\ \text{iii) } [w, v_j] = \lambda_j(\lambda_l - \lambda_k)\, v_j \ \text{ if } \varepsilon_{jkl} = 1, \quad \text{iv) } [v_j, v_k] = 0 \text{ for all } j,\, k, \end{array}$$

where $z = e^{\pm 2\pi i/3}$ for a suitable sign $\pm$ and $[\,,\,]$ is the Lie bracket. In fact, i) follows since $\lambda_j^3 = -\gamma \neq 0$, while ii) is obvious from i) as $z - \overline{z} = \pm i\sqrt{3}$. Next, applying (20.i) to w and using (17), (12), (8.b), (13.a), (23.ii), (19) and (22), we obtain the formula $\alpha_j\nabla_w w = -(\nabla w)^* v_j + \lambda_j(\lambda_k - \lambda_l)\, v_j$, if $\varepsilon_{jkl} = 1$. By (23.i), $\langle(\nabla w)^* v_j, v_k\rangle = \langle v_j, (\nabla w)v_k\rangle = 0$ whenever $k \neq j$ (cf. (13.a)). Since w, v_1, v_2, v_3 form a complex orthogonal basis at every point, our formula for $\alpha_j\nabla_w w$ thus shows that, at each point, $\alpha_j\nabla_w w$ is a combination of w and v_j, i.e., by (12), (13.b) and (11.i), $\nabla_w w = \psi_j w + \chi_j v_j$ for some functions ψ_j, χ_j. As this is true for all $j \in \{1, 2, 3\}$ and w does not depend on j, we have $\chi_j = 0$, while $\psi_j = 0$ since $\langle w, w\rangle$ is constant. Consequently, $\nabla_w w = 0$. Furthermore, $\nabla_w \alpha_j = 0$ for all j in view of (14.ii) and the relation $\langle \xi_j, w\rangle = 0$ (immediate from (a) above, (13.a) and (22)), and so (12) with $\nabla_w w = 0$ gives $\nabla_w v_j = 0$, $j = 1, 2, 3$. This, combined with (23.i) and the fact that ∇ is torsionfree, proves (24.iii). Next, since the v_j are mutually orthogonal by (13.a), and every ξ_j is a multiple of v_j in view of (a) with (22), we have, by (14.ii), $\nabla_{v_j}\alpha_k = \varepsilon_k\langle \xi_j, v_j\rangle\alpha_l$ (no summation) whenever $\varepsilon_{jkl} = 1$. Hence, by (a) and (13.a), $\nabla_{v_j}\alpha_k = \varepsilon_j\varepsilon_k(\lambda_l - \lambda_k)^{-1}\langle w, w\rangle\alpha_l$, i.e., from (11.ii) and (12), $[\nabla_{v_j}\alpha_k]w = \varepsilon_l(\lambda_l - \lambda_k)^{-1}\langle w, w\rangle v_l$, $\varepsilon_{jkl} = 1$. On the other hand, by (23.i) and (13.b), $\alpha_k(\nabla_{v_j} w) = -\varepsilon_l\lambda_j(\lambda_k - \lambda_l)v_l$. From (12) and our expressions for $[\nabla_{v_j}\alpha_k]w$ and $\alpha_k(\nabla_{v_j} w) = 0$ we now obtain $\nabla_{v_j} v_k = \nabla_{v_j}(\alpha_k w) = [\nabla_{v_j}\alpha_k]w + \alpha_k(\nabla_{v_j} w) = 0$ if $\varepsilon_{jkl} = 1$, as (24.ii) with $\lambda_j^3 = -\gamma = \langle w, w\rangle/3$ gives $(\lambda_l - \lambda_k)^{-1}\langle w, w\rangle = \pm i\sqrt{3}\,\lambda_j^2$. Similarly, $\nabla_{v_j} v_l = 0$ if $\varepsilon_{jkl} = 1$. Thus, $\nabla_{v_j} v_k = 0$ when $j \neq k$, proving (24.iv).

Since, by (13.a), $\langle w, w\rangle = \langle v_j, v_j\rangle = -3\gamma$ (no summing) for $j = 1, 2, 3$, the new complex vector fields $\tilde{w} = \pm iw/\sqrt{3}$ and $\tilde{v}_j = iv_j/\sqrt{3}$, with the same sign $\pm$ as in (24.ii), have $\langle \tilde{w}, \tilde{w}\rangle = \langle \tilde{v}_j, \tilde{v}_j\rangle = \gamma$ and are pairwise orthogonal by (13.a), while, from (24.ii) – (24.iv), $[\tilde{v}_j, \tilde{v}_k] = 0$, and $[\tilde{w}, \tilde{v}_j] = \lambda_j^2\tilde{v}_j$ (no summation). As $\lambda_j^3 = -\gamma$, replacing w, v_j with $\tilde{w}$, $\tilde{v}_j$ and setting $\rho_j = \lambda_j^2$, we now obtain (ii).

Finally, w, v_j and $\tilde{w}, \tilde{v}_j$ commute with all Killing fields since, up to permutations and sign changes, they are invariant under all isometries between connected open subsets of M. Namely, by (22), relations $W\alpha_j = \lambda_j\alpha_j$, (11.i), (14.ii) and (a) above determine the α_j, ξ_j, v_j and w uniquely up to permutations and sign changes. This completes the proof. □

§12. Complex Lie algebras and real manifolds

Given a real/complex vector space $\mathcal{Z}$ of sections of a real/complex vector bundle $\mathcal{E}$ over a manifold M, we will say that $\mathcal{Z}$ *trivializes* $\mathcal{E}$ if it consists of C^∞ sections of $\mathcal{E}$ and, for every $x \in M$, the evaluation operator $\psi \mapsto \psi(x)$ is an isomorphism $\mathcal{Z} \to \mathcal{E}_x$. This amounts to requiring that $\dim \mathcal{Z}$ coincide with the fibre dimension of $\mathcal{E}$ and each $v \in \mathcal{Z}$ be either identically zero, or nonzero at every point of M. Equivalently, a basis of $\mathcal{Z}$ then is a C^∞ trivialization of $\mathcal{E}$.

For instance, a simply transitive Lie algebra of vector fields on a manifold M (see the appendix) is nothing else than a real vector space of vector fields on M, trivializing its (real) tangent bundle, and closed under the Lie bracket.

Let the real/complexified tangent bundle of a manifold M be trivialized by a real/complex vector space $\mathcal{Z}$ of real/complex vector fields on M (cf. end of §8). We will say that a real/complex vector field w defined of any open subset U of M *commutes with* $\mathcal{Z}$, and write $[w, \mathcal{Z}] = \{0\}$, if w is of class C^∞ and $[w, v] = 0$ for every $v \in \mathcal{Z}$. In view of the Jacobi identity, real/complex vector fields $\mathcal{Z}$ defined on a given open set U and commuting with $\mathcal{Z}$ form a Lie algebra.

LEMMA 12.1. *Let $\mathcal{Z}$ be a real/complex Lie algebra of real/complex vector fields on a real manifold M, trivializing its real/complexified tangent bundle. Then, the real/complexified tangent bundle of any sufficiently small connected neighborhood U of any given point x of M is trivialized by the Lie algebra $\mathcal{Y}$ of all real/complex vector fields defined on U and commuting with $\mathcal{Z}$.*

In fact, let D be the unique connection in the real/complexified tangent bundle $\mathcal{T}$ with $\mathrm{D}_v w = [v, w]$ for all $v \in \mathcal{Z}$ and all C^1 sections w of $\mathcal{T}$. Thus, D is flat: by (2.i), $R^{\mathrm{D}}(v, w)u = [w, [v, u]] - [v, [w, u]] + [[v, w], u]$ whenever $v, w, u \in \mathcal{Z}$, which is zero by the Jacobi identity. (As $\mathcal{Z}$ is a Lie algebra, $[v, w] \in \mathcal{Z}$, and so $\mathrm{D}_{[v,w]} u = [[v, w], u]$.) Now $\mathcal{Y}$ consists of all D-parallel sections of $\mathcal{T}$ on U. $\square$

For instance, the real Lie algebra $\mathcal{X}$ of left-invariant vector fields on a Lie group G trivializes its real tangent bundle. A real vector field on an open connected subset U of G commutes with $\mathcal{X}$ if and only if it is the restriction to U of a right-invariant vector field on G. In fact, right-invariant fields w all commute with $\mathcal{X}$, since the flow of w (or, of any $v \in \mathcal{X}$) consist of left (or, right) translations, while left and right translations commute due to associativity. The converse follows since both Lie algebras are of dimension $\dim G$ (Lemma 12.1).

REMARK 12.2. If a real/complex vector space $\mathcal{Z}$ of real/complex vector fields on a manifold M trivializes its real/complex tangent bundle, then any real/complex vector field w on M with $[w, \mathcal{Z}] = \{0\}$ is a real/complex Killing field on (M, g) for any pseudo-Riemannian metric g on M such that $g(u, v)$ is constant whenever $u, v \in \mathcal{Z}$. In fact, (7) then gives $(\mathcal{L}_w g)(u, v) = 0$ for all $u, v \in \mathcal{Z}$.

Let a complex Lie algebra $\mathcal{Z}$ of complex vector fields on a manifold M trivialize its complexified tangent bundle $[TM]^{\mathbf{C}}$. We say that $\mathcal{Z}$ *admits a real form* if $\operatorname{Re} w \in \mathcal{Z}$ for every $w \in \mathcal{Z}$. This is obviously equivalent to the existence of a real Lie algebra $\mathcal{X}$ of real vector fields on M, trivializing its ordinary tangent bundle TM, and such that $\mathcal{Z} = \mathcal{X} + i\mathcal{X}$, i.e., $\mathcal{Z}$ is the complexification of $\mathcal{X}$ (or, $\mathcal{X}$ is a *real form* of $\mathcal{Z}$). Clearly, $\mathcal{X}$ then is uniquely determined by $\mathcal{Z}$, as $\mathcal{X} = \{\operatorname{Re} w : w \in \mathcal{Z}\} = \{w \in \mathcal{Z} : \operatorname{Im} w = 0\}$. Thus, $\mathcal{Z}$ admits a real form if and only if the real vector fields which are elements of $\mathcal{Z}$ form a real Lie algebra trivializing TM.

REMARK 12.3. If a complex Lie algebra $\mathcal{Z}$ of complex vector fields on a manifold M trivializes its complexified tangent bundle and $\dim_{\mathbf{C}} \mathcal{Y} = \dim M$ for the Lie algebra $\mathcal{Y}$ of all C^∞ complex vector fields w on M with $[w, \mathcal{Z}] = \{0\}$, then

(i) $\mathcal{Y}$ trivializes the complexified tangent bundle of M.

(ii) $\mathcal{Z}$ admits a real form whenever $\mathcal{Y}$ does.

To see this, first note that Lemma 12.1 yields (i). Next, let $\mathcal{V}$ be a real form of $\mathcal{Y}$, and let a complex vector field w commute with $\mathcal{Y}$, so that $[w,\mathcal{Y}]=\{0\}$. Since $\mathcal{V}\subset\mathcal{Y}$, we have $[w,\mathcal{V}]=\{0\}$. Therefore $[\mathrm{Re}\,w,\mathcal{V}]=\{0\}$, as $\mathcal{V}$ consists of real vector fields and $[\,,\,]$ is complex-bilinear; this and relation $\mathcal{Y}=\mathcal{V}+i\mathcal{V}$ now give $[\mathrm{Re}\,w,\mathcal{Y}]=\{0\}$. The Lie algebra $\mathcal{Z}'$ of all complex vector fields commuting with $\mathcal{Y}$ thus is closed under the real-part operator Re. However, $\mathcal{Z}\subset\mathcal{Z}'$ and, by Lemma 12.1, $\dim_{\mathbf{C}}\mathcal{Z}'\le\dim M=\dim_{\mathbf{C}}\mathcal{Z}$, so that $\mathcal{Z}'=\mathcal{Z}$, which proves (ii).

LEMMA 12.4. *Let $\mathcal{Z}$ be a complex Lie algebra of complex vector fields on a pseudo-Riemannian manifold (M,g), trivializing the complexified tangent bundle of M and such that $g(u,v)$ is constant for any $u,v\in\mathcal{Z}$. Then*

(a) *(M,g) is locally homogeneous.*
(b) *Under the additional assumption that $[u,v]=0$ for every $u\in\mathcal{Z}$ and every real Killing field v defined on any open subset of M, we have $\mathrm{Re}\,w\in\mathcal{Z}$ whenever $w\in\mathcal{Z}$, i.e., $\mathcal{Z}$ admits a real form.*

In fact, by Lemma 12.1 and Remark 12.2, every vector in T_xM, $x\in M$, is the value at x of some real Killing field on a neighborhood of x, which proves (a) (cf. [**10**], p 546). Now let us fix $x\in M$ and choose $U,\mathcal{Y}$ for $x,\mathcal{Z}$ as in Lemma 12.1. Remark 12.2 and our hypothesis show that $\mathcal{Y}$ then is *precisely* the Lie algebra of all complex Killing fields on U. Thus, $\mathcal{Y}$ is closed under the real-part operator Re, i.e., admits a real form, and Remark 12.3(ii) yields (b). □

§13. Real forms of some specific complex Lie algebras

We use the standard notation Ad for the adjoint representation of any given Lie algebra $\mathcal{X}$, so that $\mathrm{Ad}\,v:\mathcal{X}\to\mathcal{X}$ is, for any $v\in\mathcal{X}$, given by $(\mathrm{Ad}\,v)w=[v,w]$.

LEMMA 13.1. *Let a basis w,v_1,v_2,v_3 of a four-dimensional complex Lie algebra $\mathcal{Z}$ satisfy conditions* (18) *for some complex-bilinear symmetric form g on $\mathcal{Z}$ and a complex number $\gamma\ne0$, where $[\,,\,]$ is the Lie-algebra multiplication of $\mathcal{Z}$ and ρ_1,ρ_2,ρ_3 are the three cubic roots of γ^2. Also, let $\mathcal{X}\subset\mathcal{Z}$ be a four-dimensional real Lie subalgebra with $\mathcal{Z}=\mathcal{X}+i\mathcal{X}$ and $g(\mathcal{X},\mathcal{X})\subset\mathbf{R}$. In other words, $\mathcal{X}$ spans $\mathcal{Z}$ as a complex space and the form g restricted to $\mathcal{X}$ is real-valued.*

Then $w\in\mathcal{X}$ and there exist a three-dimensional real vector subspace V of $\mathcal{X}$, a linear operator $F:V\to V$, and a real-valued bilinear form $\langle\,,\rangle$ on V, satisfying conditions (4) *with $u=|\gamma|^{-1/2}w$ and $\delta=\mathrm{sgn}\,\gamma$, and such that $\langle\,,\rangle,F$ are, for a suitable isomorphic identification $V=\mathbf{C}\times\mathbf{R}$, given by* (a),(b) *in* §4 *with some sign $\pm$ and some $p\in\mathbf{R}\smallsetminus\{0\}$.*

PROOF. We set $V=\mathcal{X}\cap\mathrm{Ker}\,\Psi$, where $\Psi:\mathcal{Z}\to\mathbf{C}$ is the $\mathbf{C}$-linear functional with $\Psi(w)=1$ and $\Psi(v_j)=0$, $j=1,2,3$. For any $u\in\mathcal{Z}\smallsetminus\mathrm{Ker}\,\Psi$,

(a) $\mathrm{Ad}\,u$ has the characteristic roots 0 and $\Psi(w)\rho_j$, $j=1,2,3$.
(b) $\dim_{\mathbf{R}}V=3$ and $\mathrm{Span}_{\mathbf{C}}V=\mathrm{Ker}\,\Psi$.

In fact, by (18), $\mathrm{Ad}\,u:\mathcal{Z}\to\mathcal{Z}$ is diagonalizable with the eigenvalues as in (a) for the eigenvectors u and v_j, which proves (a). Also, as $\dim_{\mathbf{C}}[\mathrm{Ker}\,\Psi]=3$, our $\mathcal{X}$ cannot be contained in $\mathrm{Ker}\,\Psi$, and so the image $\Psi(\mathcal{X})$ is a nontrivial real vector subspace of $\mathbf{C}$. For any fixed $u\in\mathcal{X}\smallsetminus\mathrm{Ker}\,\Psi$, (a) gives $\Psi(w)\rho_j\in\mathbf{R}$ for some $j\in\{1,2,3\}$. (In fact, as $\mathcal{X}$ contains a basis of $\mathcal{Z}$, the characteristic roots of $\mathrm{Ad}\,w:\mathcal{Z}\to\mathcal{Z}$ coincide with those of $\mathrm{Ad}\,w:\mathcal{X}\to\mathcal{X}$, so that the number of nonreal ones among them is 0 or 2.) Thus, $\Psi(\mathcal{X})$ is contained in the union of the real lines $\mathbf{R}\overline{\rho}_j\subset\mathbf{C}$, $j=1,2,3$, i.e., must coincide with one of them, and we may fix $j \in \{1,2,3\}$ with $\Psi(\mathcal{X}) = \mathbf{R}\overline{\rho_j}$. Now $\dim_{\mathbf{R}} V = 3$, since $V = \mathcal{X} \cap \operatorname{Ker}\Psi$ is the kernel of $\Psi : \mathcal{X} \to \mathbf{C}$. Also, $\mathcal{X}$ spans $\mathcal{Z}$, so that vectors in $\mathcal{X}$, linearly independent over $\mathbf{R}$, are also linearly independent over $\mathbf{C}$ in $\mathcal{Z}$. This implies (b): $\operatorname{Span}_{\mathbf{C}} V = \operatorname{Ker}\Psi$ as $\operatorname{Span}_{\mathbf{C}} V \subset \operatorname{Ker}\Psi$ and $\dim_{\mathbf{C}}[\operatorname{Span}_{\mathbf{C}} V] = \dim_{\mathbf{C}}[\operatorname{Ker}\Psi] = 3$.

Since $\dim_{\mathbf{R}} V = 3$, we may choose $u \in \mathcal{X} \smallsetminus \{0\}$ which is g-orthogonal to V. By (b), u then is also g-orthogonal to $\operatorname{Ker}\Psi$. Thus, in view of (18), $u \in \mathbf{C}w$, i.e., $u = \Psi(u)w$ with $\Psi(u) \neq 0$. Also, $\Psi(u)\rho_j$ is real, for j chosen above (as $\Psi(u) \in \mathbf{R}\overline{\rho_j}$), and hence so is its cube $[\Psi(u)]^3\gamma^2$. On the other hand, (18) gives $[\Psi(u)]^2\gamma = g(u,u) \in \mathbf{R}$. Consequently, the numbers $\Psi(u)\gamma$, $\Psi(u)$, ρ_j and γ are all real, while $w \in \mathcal{X}$, as $\mathcal{X}$ contains $u = \Psi(u)w$ and $\Psi(u) \in \mathbf{R} \smallsetminus \{0\}$.

As $\gamma \in \mathbf{R} \smallsetminus \{0\}$, replacing such u by $|\gamma|^{-1/2}w$ and letting $\langle\,,\rangle$ stand for the restriction of g to V, we now obtain $\langle u,u\rangle = \delta$ with $\delta = \operatorname{sgn}\gamma \in \{1,-1\}$.

The real 3-space $V = \mathcal{X} \cap \operatorname{Ker}\Psi$ is $(\operatorname{Ad} u)$-invariant, since so are $\mathcal{X}$ (as $u \in \mathcal{X}$) and $\operatorname{Ker}\Psi$ (by (18) with $u = |\gamma|^{-1/2}w$). The restriction $F : V \to V$ of $\operatorname{Ad} u$ is self-adjoint, since that is the case for $F, V, \langle\,,\rangle$ replaced by $\operatorname{Ad} u, \mathcal{Z}, g$ (as $\operatorname{Ad} u : \mathcal{Z} \to \mathcal{Z}$ is diagonalized by the g-orthogonal basis w, v_1, v_2, v_3, cf. (18)). Combining (a) with our assumptions about the cubes ρ_j^3 and the fact that $\dim_{\mathbf{R}} V = 3$ is odd, we see that F has the characteristic roots $p, pq, p\overline{q}$, where $q = e^{2\pi i/3}$ and $p \in \mathbf{R} \smallsetminus \{0\}$, and we may choose $\xi, \eta, \zeta \in V$ such that ζ and $\xi + i\eta$ are eigenvectors of $\operatorname{Ad} u : \mathcal{Z} \to \mathcal{Z}$ for the eigenvalues p and pq. (Since $pq \notin \mathbf{R}$, this implies that ξ, η are linearly independent over $\mathbf{R}$.) By (18), ζ and $\xi + i\eta$ are complex multiples of v_j, v_k for some j,k. Thus, $g(\zeta,\zeta) \neq 0$, i.e., ζ may be normalized so that $\langle\zeta,\zeta\rangle = \pm 1$ for some sign $\pm$, while $g(\zeta, \xi + i\eta) = 0$, and so $\langle\zeta,\xi\rangle = \langle\zeta,\eta\rangle = 0$, as g is real-valued on $V \subset \mathcal{X}$. Next, $\langle F\xi,\eta\rangle = \langle\xi,F\eta\rangle$ since F is self-adjoint, so that $\langle\xi,\xi\rangle + \langle\eta,\eta\rangle = 0$ in view of the eigenvector relation $F\xi + iF\eta = pq(\xi + i\eta)$ with $q = (\sqrt{3}\,i - 1)/2$. Finally, let c be a complex number with $2\overline{c}^2 = -g(\xi + i\eta, \xi + i\eta)$. Thus, $c \neq 0$, since $g(v_k, v_k) \neq 0$, and it is easy to verify that the isomorphism $V \to \mathbf{C} \times \mathbf{R}$ sending the basis ξ, η, ζ onto $(c,0), (-ic,0), (0,1)$ has the required properties. This completes the proof. □

Proofs of Theorems 5.1, 6.1 and 7.2. In all three cases, $R - W$ is a constant multiple of the identity (Remark 1.1), and so the hypotheses of Theorem 11.1 are satisfied (cf. Remark 9.1). If g is Riemannian, Theorem 11.1(i) yields Theorem 7.2. If g is Lorentzian and $\nabla W = 0$, i.e., $\nabla R = 0$, Theorem 41.5 of [**10**] (pp. 662–663) implies (a) or (b) in Theorem 5.1, as the diagonalizability condition excludes option (c) in [**10**] on p. 663. The only remaining cases now are those named in (ii) of Theorem 11.1, the conclusion of which shows that Lemma 13.1 can be applied to the Lie algebra $\mathcal{Z} = \operatorname{Span}_{\mathbf{C}}\{w, v_1, v_2, v_3\}$ and its real form $\mathcal{X}$ which exists in view of Lemma 12.4(b). As a result, (M,g) is obtained as in Example 4.2(i) or (ii); the situation where $\delta = -1$ and $\pm$ is $-$ cannot occur, as it would lead to the sign pattern $---+$, which is not one of (1). □

Appendix. Simply transitive Lie algebras of vector fields

In this section we prove Corollary A.3, a well-known result included here to provide a convenient, self-contained reference for a conclusion in Example 4.2.

A *simply transitive Lie algebra of vector fields* on a manifold M is any vector space $\mathcal{X}$ of C^∞ (real) vector fields on M, closed under the Lie bracket and such that the evaluation operator $\mathcal{X} \ni w \mapsto w(x) \in T_xM$ is bijective for every $x \in M$. An example is the Lie algebra of left-invariant vector fields on a Lie group.

Given a simply transitive Lie algebra $\mathcal{X}$ of vector fields on a manifold M and a fixed point $y \in M$, the *exponential mapping* $E : U_y \to M$ *for* $\mathcal{X}$, *centered at* y, is given by $E(v) = x(1)$, where U_y is the set of all $v \in \mathcal{X}$ for which an integral curve $t \mapsto x(t)$ of v with $x(0) = y$ can be defined on the whole interval $[0, 1]$. It is clear that U_y is a neighborhood of 0 in $\mathcal{X}$ and, for every $v \in U_y$ and $t \in [0, 1]$, we have $tv \in U_y$ and $x(t) = E(tv)$, with $x(t)$ as above.

Let $Q : \mathbf{C} \to \mathbf{C}$ be the entire function with $Q(z) = (1 - e^{-z})/z$ if $z \neq 0$ and $Q(0) = 1$. Its Maclaurin series defines $Q(A)$ for any linear operator $A : V \to V$ in a vector space V with $\dim V < \infty$. Thus, with Ad as in §13, $Q(\mathrm{Ad}\, v) = \sum_{k=0}^{\infty} (-\mathrm{Ad}\, v)^k/[(k+1)!]$ for a Lie algebra $\mathcal{X}$ with $\dim \mathcal{X} < \infty$ and $v \in \mathcal{X}$.

Proposition A.1. *Let* $\mathcal{X}$ *be a simply transitive Lie algebra of vector fields on a manifold* M*, and let* $dE_v : \mathcal{X} \to T_{E(v)}M$ *be the differential at* $v \in U_y$ *of the exponential mapping of* $\mathcal{X}$ *centered at a point* $y \in M$*, with the usual identification* $T_v\mathcal{X} = \mathcal{X}$*. Then* dE_v *equals the composite mapping in which* $Q(\mathrm{Ad}\, v) : \mathcal{X} \to \mathcal{X}$*, defined above, is followed by the evaluation isomorphism* $\mathcal{X} \to T_{E(v)}M$*.*

Proof. For any C^∞ mapping $(s, t) \mapsto x(s, t) \in M$ of a rectangle $K \subset \mathbf{R}^2$, let $u_s, u_t : K \to \mathcal{X}$ assign to (s, t) the unique elements of $\mathcal{X}$ which coincide, at $x(s, t)$, with $\partial x/\partial s$ and, respectively, $\partial x/\partial t$ (that is, with the velocity at s, or t, of the curve $s \mapsto x(s, t)$ or $t \mapsto x(s, t)$). Using subscripts for partial derivatives of u_s, u_t we thus have $u_{st}, u_{ts}, u_{stt} : K \to \mathcal{X}$ with $u_{st} = \partial u_s/\partial t$, etc.; we also let $[u_s, u_t] : K \to \mathcal{X}$ stand for the valuewise bracket of the Lie-algebra valued functions u_s, u_t. In local coordinates x^j at any given $x_0 = x(s_0, t_0)$, the vector fields $u_s(s, t), u_t(s, t)$ have some component functions $u_s^j(s, t, x), u_t^j(s, t, x)$, also depending on a point x near x_0. Thus, $u_s^j(s, t, x(s, t)) = \partial[x^j(s, t)]/\partial s$ and $u_t^j(s, t, x(s, t)) = \partial[x^j(s, t)]/\partial t$. Applying $\partial/\partial t$ to the first relation, $\partial/\partial s$ to the second, and using equality of mixed partial derivatives for the $x^j(s, t)$, we get $\partial u_s^j/\partial t - \partial u_t^j/\partial s = u_s^k \partial_k u_t^j - u_t^k \partial_k u_s^j$, with $\partial_k = \partial/\partial x^k$, which is the coordinate form of the identity $u_{st} - u_{ts} = [u_s, u_t]$. If $u_{tt} = 0$ for all $(s, t) \in K$, taking $\partial/\partial t$ of that identity, we obtain the *Jacobi equation* $u_{stt} = [u_{st}, u_t]$ (as $u_{tst} = u_{tts} = 0$).

It is clear that $u_{tt} = 0$ identically if and only if $t \mapsto x(s, t)$ is, for each fixed s, an integral curve of some vector field $v(s) \in \mathcal{X}$. Then, obviously, $u_t(s, t) = v(s)$.

Now let $u_{tt} = 0$ for all (s, t), and let K intersect the s-axis $\mathbf{R} \times \{0\}$. The Jacobi equation (see above) reads $\partial u_{st}/\partial t = -[\mathrm{Ad}\, v(s)]\, u_{st}$, with $v(s) = u_t(s, t)$, and so $u_{st}(s, t) = e^{-t\, \mathrm{Ad}\, v(s)}\, w(s)$, where $w(s) = u_{st}(s, 0)$. Since $d[tQ(t\, \mathrm{Ad}\, v)]/dt = e^{-t\, \mathrm{Ad}\, v}$ (cf. our formula for $Q(\mathrm{Ad}\, v)$), we get $u_s(s, t) = u_s(s, 0) + tQ(t\, \mathrm{Ad}\, v(s))\, w(s)$, as both sides satisfy the same initial value problem in the variable t.

Finally, let $K = I \times [0, 1]$ and $x(s, t) = E(tv(s))$ for some interval I and some C^∞ curve $I \ni s \mapsto v(s) \in U_y$. Thus, $u_{tt} = 0$ identically and $u_t(s, t) = v(s)$, so that $u_{ts}(s, t) = \dot v(s)$, with $\dot v = dv/ds$. Also, $x(s, 0) = y$, and hence $u_s(s, 0) = 0$. Evaluating at $(s, 0)$ the identity $u_{st} - u_{ts} = [u_s, u_t]$, established above, and setting $w(s) = u_{st}(s, 0)$ as in the preceding paragraph, we thus get $w(s) = u_{ts}(s, 0) = \dot v(s)$. Writing $v, \dot v$ instead od $v(s), dv/ds$ we now see that $u_s(s, 1)$ equals the preimage of $dE_v \dot v$ under the evaluation isomorphism $\mathcal{X} \to T_{E(v)}M$ (cf. the definition of u_s) while $u_s(s, 1) = Q(\mathrm{Ad}\, v)\dot v$, as one sees setting $t = 1$ in $u_s(s, t) = u_s(s, 0) + tQ(t\, \mathrm{Ad}\, v(s))\, w(s)$. This completes the proof. ☐

Corollary A.2. *Given a simply transitive Lie algebra* $\mathcal{X}$ *of vector fields on a manifold* M *and a point* $y \in M$, *there exists a neighborhood* U *of* 0 *in* $\mathcal{X}$ *such that* $U \subset U_y$ *and the exponential mapping* $E : U_y \to M$ *sends* U *diffeomorphically onto an open subset of* M. *For any* U *with this property,* $Q(\mathrm{Ad}\, v) : \mathcal{X} \to \mathcal{X}$ *is an isomorphism for every* $v \in U$, *and the pullback under* E *of any vector field* $w \in \mathcal{X}$ *is the vector field on* U *given by* $U \ni v \mapsto [Q(\mathrm{Ad}\, v)]^{-1} w$.

In fact, $Q(\mathrm{Ad}\, v)$ is an isomorphism by Proposition A.1, since dE_v is. □

By Corollary A.2, the local diffeomorphism type of a simply transitive Lie algebra of vector fields is determined by its Lie-algebra isomorphism type. Since every finite-dimensional Lie algebra is the Lie algebra of some Lie group, this yields

Corollary A.3. *Given a simply transitive Lie algebra* $\mathcal{X}$ *of vector fields on a manifold* M, *there exists a Lie group* G *with the following property: Every point of* M *has a neighborhood* U *which may be diffeomorphically identified with an open set* $U' \subset G$ *so as to make* $\mathcal{X}$ *restricted to* U *appear as the Lie algebra of the restrictions to* U' *of all left-invariant vector fields on* G. □

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