Compact flat manifolds and reducibility

Andrzej Derdzinski[0000–0001–9015–0886] and
Paolo Piccione[0000–0002–8929–5938]

Abstract Hiss and Szczepański proved in 1991 that the holonomy group of any compact flat Riemannian manifold, of dimension at least two, acts reducibly on the rational span of the Euclidean lattice associated with the manifold via the first Bieberbach theorem. Geometrically, their result states that such a manifold must admit a nonzero proper parallel distribution with compact leaves. We study algebraic and geometric properties of the sublattice-spanned holonomy-invariant rational vector subspaces that exist due to the above theorem, and of the resulting compact-leaf foliations of compact flat manifolds. The class consisting of the former subspaces, in addition to being closed under spans and intersections, also turns out to admit (usually nonorthogonal) complements. As for the latter foliations, we provide descriptions, first – and foremost – of the intrinsic geometry of their generic leaves in terms of that of the original flat manifold and, secondly – as an essentially obvious afterthought – of the leaf-space orbifold. The general conclusions are then illustrated by examples in the form of generalized Klein bottles.

1 Introduction

As shown by Hiss and Szczepański [10, the corollary in Sect. 1], on any compact flat Riemannian manifold $\mathcal{M}$ with $\dim \mathcal{M} = n \geq 2$ there exists a parallel distribution D of dimension k, where $0 < k < n$, such that the leaves of D are all compact. In the Appendix we reproduce the original algebraic phrasing of their result and mention a stronger version of it, established more recently by Lutowski [12], which

---

Andrzej Derdzinski
Department of Mathematics, The Ohio State University, 231 W. 18th Avenue, Columbus, OH 43210, USA, e-mail: andrzej@math.ohio-state.edu

Paolo Piccione
Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, CEP 05508-900, São Paulo, SP, Brazil, e-mail: piccione@ime.usp.br

implies that – unless $\mathcal{M}$ is a flat torus – there exist at least two distributions D with the above properties, having nonequivalent irreducible holonomy representations.

The present paper deals with geometric consequences of Hiss and Szczepański's theorem. We do not pursue analogous ramifications of Lutowski's generalization.

Our main results are Theorems 1, 2, 3, 4, 5 and Corollary 2.

Theorem 1 describes geometries of the individual leaves $\mathcal{M}'$ of a distribution D on $\mathcal{M}$ having the properties mentioned above, in terms of the short exact sequence $L \to \Pi \to H$ formed by the lattice L, holonomy group H and Bieberbach group Π associated with $\mathcal{M}$, and its analog $L' \to \Pi' \to H'$ for $\mathcal{M}'$. Specifically, according to our Theorem 1(ii), Π' (or, L') may be treated as a subgroup of Π (or, of a certain Euclidean vector space $\mathcal{V}$), and H' as a homomorphic image of a subgroup of H.

In Theorem 2 we establish the existence, on every compact flat Riemannian manifold $\mathcal{M}$ of dimension $n \geq 2$, of *two* proper parallel distributions D and $\hat{D}$ with compact leaves, complementary to each other in the sense that $T\mathcal{M} = D \oplus \hat{D}$.

Corollary 2 states that, in any manifold $\mathcal{M}$ as above, the class of parallel distributions with compact leaves is closed under spans and intersections.

Theorem 3 addresses the particularly simple form of the sequence $L' \to \Pi' \to H'$ described by Theorem 1(ii), arising in the case of leaves $\mathcal{M}'$ which we call *generic*. The union of all generic leaves is an open dense subset of $\mathcal{M}$, they all have the same triple L', Π', H', and are mutually isometric. When all leaves of D happen to be generic, they form a locally trivial bundle with compact flat manifolds serving both as the base and the fibre (the *fibration case*).

Theorems 4 and 5 describe the intersection numbers of the leaves of the two mutually complementary foliations resulting from Theorem 2.

Aside from the holonomy group H' of each individual leaf $\mathcal{M}'$ of D, forming a part of its intrinsic (submanifold) geometry, $\mathcal{M}'$ also gives rise to two "extrinsic" holonomy groups, one arising since $\mathcal{M}'$ is a leaf of the foliation $F_{\mathcal{M}}$ of $\mathcal{M}$ tangent to D, the other coming from the normal connection of $\mathcal{M}'$. Due to the flatness of the normal connection, the two extrinsic holonomy groups coincide, and are trivial for all generic leaves. In Sect. 12 we briefly discuss the leaf space $\mathcal{M}/F_{\mathcal{M}}$, pointing out that (not surprisingly!) $\mathcal{M}/F_{\mathcal{M}}$ is a flat compact orbifold which, in the fibration case mentioned above, constitutes the base manifold of the bundle.

Our results are illustrated by examples (generalized Klein bottles, Sect. 15), where both the fibration and non-fibration cases occur, depending on the choice of D.

In Sect. 16 we discuss certain analogs of the correspondence between Bieberbach groups and compact flat manifolds: one provided by almost-Bieberbach groups and infra-nilmanifolds, the other by the group $\mathrm{Spin}(m,1)$ acting on the orthonormal-frame bundle of the hyperbolic m-space, leading to quotient manifolds that include some compact locally symmetric pseudo-Riemannian Einstein manifolds.

An earlier version [7] of this paper is cited by [2], and therefore still available on the arXiv. The presentation in [7] was – as we eventually realized – rather far from reader-friendly, which prompted us to thoroughly rewrite the whole text.

2 Preliminaries

Manifolds, mappings (except in Lemma 2) and tensor fields, such as bundle and covering projections, submanifold inclusions, and Riemannian metrics, are by definition of class C^∞. Submanifolds need not carry the subset topology, and a manifold may be disconnected (although, being required to satisfy the second countability axiom, it must have at most countably many connected components). Connectedness/compactness of a submanifold always refers to its own topology, and implies the same for its underlying set within the ambient manifold. Thus, a compact submanifold is always endowed with the subset topology.

By a *distribution* on a manifold $\mathcal{N}$ we mean, as usual, a (smooth) vector subbundle D of the tangent bundle $T\mathcal{N}$. An *integral manifold* of D is any submanifold $\mathcal{L}$ of $\mathcal{N}$ with $T_x\mathcal{L} = D_x$ whenever $x \in \mathcal{L}$. The maximal connected integral manifolds of D will also be referred to as the *leaves* of D. If D is integrable, its leaves form the foliation associated with D. We call D *projectable* under a mapping $\psi : \mathcal{N} \to \hat{\mathcal{N}}$ onto a distribution $\hat{D}$ on the target manifold $\hat{\mathcal{N}}$ if $d\psi_x(D_x) = \hat{D}_{\psi(x)}$ for all $x \in \mathcal{N}$.

Remark 1 The following facts are well known: (c) is the compact case of Ehresmann's fibration theorem [8, Corollary 8.5.13]; (b) follows from (c). For (a), see [11, pp. 43–44 and 61–62] – note that finiteness trivially implies proper discontinuity.

a. Free diffeomorphic actions of finite groups on manifolds are properly discontinuous, leading to covering projections onto the resulting quotient manifolds.
b. Any locally-diffeomorphic mapping from a compact manifold into a connected manifold is a (surjective) finite covering projection.
c. More generally, the phrases 'locally-diffeomorphic mapping' and 'finite covering projection' in (b) may be replaced with *submersion* and *fibration*.

Lemma 1 *Given manifolds $\hat{\mathcal{M}}$ and $\mathcal{M}$ with distributions $\hat{D}$ and D, let $\hat{D}$ be projectable onto D under a locally diffeomorphic surjective mapping $\psi : \hat{\mathcal{M}} \to \mathcal{M}$.*

i. *The ψ-image of any leaf of $\hat{D}$ is a connected integral manifold of D.*
ii. *Integrability of $\hat{D}$ implies that of D.*
iii. *For any compact leaf $\mathcal{L}$ of $\hat{D}$, the image $\mathcal{L}' = \psi(\mathcal{L})$ is a compact leaf of D, and the restriction $\psi : \mathcal{L} \to \mathcal{L}'$ constitutes a covering projection.*
iv. *If the leaves of $\hat{D}$ are all compact, so are those of D.*

Proof. Assertion (i) is immediate from the definitions of a leaf and projectability, while (i) yields (ii) as integrability amounts to the existence of an integral manifold through every point. Remark 1(b) and (i) give (iii). Now (iv) follows. □

Lemma 2 *Suppose that F is a mapping from a manifold $\mathcal{W}$ into any set. If for every $x \in \mathcal{W}$ there exists a diffeomorphic identification of a neighborhood $\mathcal{B}_x$ of x in $\mathcal{W}$ with a unit open Euclidean ball centered at 0 under which x corresponds to 0 and F becomes constant on each open straight-line interval of length 1 in the open Euclidean ball having 0 as an endpoint, then F is locally constant on some open dense subset of $\mathcal{W}$.*

Proof. We use induction on $n = \dim \mathcal{W}$. The case $n = 1$ being trivial, let us suppose that the assertion holds in dimension $n-1$ and consider a mapping F from an n-dimensional manifold $\mathcal{W}$, satisfying our hypothesis, along with an embedded open Euclidean ball $\mathcal{B}_x \subseteq \mathcal{M}$ "centered" at a fixed point x, as in the statement of the lemma. The constancy of F along the fibres of the normalization projection $\mu : \mathcal{B}_x \smallsetminus \{x\} \to \mathcal{S}$ onto the unit $(n-1)$-sphere $\mathcal{S}$ gives rise to a mapping G with the domain $\mathcal{S}$ and $F = G \circ \mu$. Let us now fix $s \in \mathcal{S}$, any $y \in \mathcal{B}_x \smallsetminus \{x\}$ with $\mu(y) = s$, and an embedded open Euclidean ball $\mathcal{B}_y$ "centered" at y, such that F is constant on each radial open interval in $\mathcal{B}_y$. The obvious submersion property of μ allows us to pass from $\mathcal{B}_y$ to a smaller concentric ball and then choose a codimension-one open Euclidean ball $\mathcal{B}'_y$ arising as a union of radial intervals within this smaller version of $\mathcal{B}_y$, for which $\mu : \mathcal{B}'_y \to \mathcal{S}$ is an embedding. The assumption of the lemma thus holds when $\mathcal{W}$ and F are replaced by $\mathcal{S}$ and G, leading to the local constancy of G (and F) on a dense open set in $\mathcal{S}$ (and, respectively, in $\mathcal{B}_x \smallsetminus \{x\}$). Since the union of the latter sets over all x is obviously dense in $\mathcal{W}$, our claim follows. ◻

We have the following well-known consequence of the inverse mapping theorem combined with the Gauss lemma for submanifolds.

Lemma 3 *Given a compact submanifold $\mathcal{M}'$ of a Riemannian manifold $\mathcal{M}$, every sufficiently small $\delta \in (0, \infty)$ has the following properties.*

a. *The normal exponential mapping restricted to the radius δ open-disk subbundle $\mathcal{N}_\delta$ of the normal bundle of $\mathcal{M}'$ constitutes a diffeomorphi sm $\mathrm{Exp}^\perp : \mathcal{N}_\delta \to \mathcal{M}'_\delta$ onto the open submanifold $\mathcal{M}'_\delta$ of $\mathcal{M}$ equal to the preimage of $[0, \delta)$ under the function $\mathrm{dist}(\mathcal{M}', \cdot\,)$ of metric distance from $\mathcal{M}'$.*
b. *Each $x \in \mathcal{M}'_\delta$ has a unique point $y \in \mathcal{M}'$ nearest to x, which is simultaneously the unique point y of $\mathcal{M}'$ joined to x by a geodesic in $\mathcal{M}'_\delta$ normal to $\mathcal{M}'$ at y, and the resulting assignment $\mathcal{M}'_\delta \ni x \mapsto y \in \mathcal{M}'$ coincides with the composition of the inverse diffeomorphism of $\mathrm{Exp}^\perp : \mathcal{N}_\delta \to \mathcal{M}'_\delta$ followed by the normal-bundle projection $\mathcal{N}_\delta \to \mathcal{M}'$.*
c. *The $\mathrm{Exp}^\perp$ images of length δ radial line segments emanating from the zero vectors in the fibres of $\mathcal{N}_\delta$ coincide with the length δ minimizing geodesic segments in $\mathcal{M}'_\delta$ emanating from $\mathcal{M}'$ and normal to $\mathcal{M}'$. They are also normal to all the levels of $\mathrm{dist}(\mathcal{M}', \cdot\,)$ in $\mathcal{M}'_\delta$, and realize the minimum distance between any two such levels within $\mathcal{M}'_\delta$.*

Lemma 4 *In a complete metric space, any countable union of closed sets with empty interiors has an empty interior.*

Proof. This is Baire's theorem [9, p. 187] stating, equivalently, that the intersection of countably many dense open subsets is dense. ◻

3 Free Abelian groups

The following well-known facts, cf. [1, p. 2], are gathered here for easy reference.

For a finitely generated Abelian group G, being torsion-free amounts to being free, in the sense of having a $\mathbb{Z}$-basis, by which one means an ordered n-tuple $e_1, \ldots, e_n$ of elements of G such that every $x \in G$ can be uniquely expressed as an integer combination of $e_1, \ldots, e_n$. The integer $n \geq 0$, also denoted by $\dim_{\mathbb{Z}} G$, is an algebraic invariant of G, called its *rank*, or *Betti number*, or $\mathbb{Z}$*-dimension*.

Any finitely generated Abelian group G is isomorphic to the direct sum of its (necessarily finite) torsion subgroup S and the free group G/S. We then set $\dim_{\mathbb{Z}} G = \dim_{\mathbb{Z}}[G/S]$. A subgroup G' (or, a homomorphic image G') of such G, in addition to being again finitely generated and Abelian, also satisfies the inequality $\dim_{\mathbb{Z}} G' \leq \dim_{\mathbb{Z}} G$, strict unless G/G' is finite (or, respectively, the homomorphism in question has a finite kernel).

Lemma 5 *A subgroup G' of a finitely generated free Abelian group G constitutes a direct summand of G if and only if the quotient group G/G' is torsion-free.*

In fact, more generally, given a surjective homomorphism $\chi : P \to P'$ between Abelian groups P, P' and elements $x_j, y_a \in P$ (with j, a ranging over finite sets), such that x_j and $\chi(y_a)$ happen to form $\mathbb{Z}$-bases of $\operatorname{Ker} \chi$ and, respectively, of P', the system consisting of all x_j and y_a is a $\mathbb{Z}$-basis of P (and so P must be free). This is clear as every element of P' (or, of P) then can be uniquely expressed as an integer combination of $\chi(y_a)$ (or, consequently, of x_j and y_a).

Lemma 6 *For each finitely generated subgroup G of the additive group of a finite-dimensional real vector space $\mathcal{V}$, the intersection $G \cap \mathcal{V}'$ with any vector subspace $\mathcal{V}' \subseteq \mathcal{V}$ forms a direct-summand subgroup of G. Furthermore, the class of direct-summand subgroups of G is closed under intersections, finite or not.*

Both claims are obvious from Lemma 5. Next, we have a straightforward exercise:

Lemma 7 *If normal subgroups G', G'' of a group G intersect trivially and every $\gamma' \in G'$ commutes with every $\gamma'' \in G''$, then $G'G'' = \{\gamma'\gamma'' : (\gamma', \gamma'') \in G' \times G''\}$ is a normal subgroup of G, and the assignment $(\gamma', \gamma'') \mapsto \gamma'\gamma''$ defines an isomorphism $G' \times G'' \to G'G''$.*

4 Lattices and vector subspaces

Throughout this section $\mathcal{V}$ denotes a fixed finite-dimensional real vector space, and

$$\text{subspaces } \mathcal{V}', \mathcal{V}'' \subseteq \mathcal{V} \text{ with } \mathcal{V} = \mathcal{V}' \oplus \mathcal{V}'' \text{ are called } \textit{complementary} \tag{1}$$

to each other. As usual, we define a (full) *lattice* in $\mathcal{V}$ to be any subgroup L of the additive group of $\mathcal{V}$ generated by a basis of $\mathcal{V}$ (which must consequently also be a $\mathbb{Z}$-basis of L). The quotient Lie group $\mathcal{V}/L$ then is a torus, and we use the term *subtori* when referring to its compact connected Lie subgroups.

Definition 1 Given a lattice L in $\mathcal{V}$, by an *L-subspace* of $\mathcal{V}$ we will mean any vector subspace $\mathcal{V}'$ of $\mathcal{V}$ spanned by $L \cap \mathcal{V}'$. One may equivalently require $\mathcal{V}'$ to be the span of just a subset of L, rather than specifically of $L \cap \mathcal{V}'$.

Lemma 8 *For a lattice L in $\mathcal{V}$, the parallel distribution on $\mathcal{V}$ tangent to any prescribed vector subspace $\mathcal{V}'$ projects onto a parallel distribution D on the torus group $\mathcal{V}/L$. The leaves of D must be either all compact, or all noncompact, and they are compact if and only if $\mathcal{V}'$ is an L-subspace, in which case the leaf of D through zero is a subtorus of $\mathcal{V}/L$.*

Proof. The projectability assertion is obvious from the general fact, here applied to the projection $\mathcal{V} \to \mathcal{V}/L$, that projectability of distributions under covering projections amounts to their deck-transformation invariance. The first claim about the leaves of D follows as the leaves are one another's translation images. For the second, let $\mathcal{N}$ be the leaf of D through zero. Requiring $\mathcal{V}'$ to be (or, not to be) an L-subspace makes $L \cap \mathcal{V}'$, by Lemma 6, a direct-summand subgroup of L spanning $\mathcal{V}'$ or, respectively, yields the existence of a nonzero linear functional f on $\mathcal{V}'$, the kernel of which contains $L \cap \mathcal{V}'$. In the former case $\mathcal{N}$ is a factor of a product-of-tori decomposition of $\mathcal{V}/L$, in the latter f descends to an unbounded function on $\mathcal{N}$. $\square$

Lemma 9 *Given a lattice L in $\mathcal{V}$, the span and intersection of any family of L-subspaces are L-subspaces. The same is true if one replaces the phrase 'L-subspaces' with 'H-invariant L-subspaces' for any fixed group H of linear automorphisms of $\mathcal{V}$ sending L into itself.*

Proof. The assertion about spans follows from the case of two L-subspaces, obvious in turn due to the second sentence of Definition 1. Next, the intersection of the family of subtori in $\mathcal{V}/L$, arising via Lemma 8 from the given family of L-subspaces, constitutes a compact Lie subgroup of $\mathcal{V}/L$, so that it is the union of finitely many cosets of a subtorus $\mathcal{N}$. Since subtori are totally geodesic relative to the translation-invariant flat affine connection on $\mathcal{V}/L$, while the projection $\mathcal{V} \to \mathcal{V}/L$ is locally diffeomorphic, the tangent space of $\mathcal{N}$ at zero equals the intersection of the tangent spaces of the subtori forming the family, and each tangent space corresponds to an L-subspace from our family. The conclusion is now immediate from Lemma 8. $\square$

Remark 2 For a lattice L in $\mathcal{V}$ generated by a basis $e_1, \ldots, e_n$ of $\mathcal{V}$, the translational action of L on $\mathcal{V}$ has an obvious compact *fundamental domain* (a compact subset of $\mathcal{V}$ intersecting all orbits of L): the parallelepiped formed by all the combinations $t_1 e_1 + \ldots + t_n e_n$ with $t_1, \ldots, t_n$ ranging over $[0, 1]$.

The next lemma is immediate from the first part of Lemma 6 and the well-known fact [3, Chap. VII, Théorème 2] that lattices in $\mathcal{V}$ are precisely the same as discrete subgroups of $\mathcal{V}$, spanning $\mathcal{V}$.

Lemma 10 *For a lattice L in $\mathcal{V}$, a vector subspace $\mathcal{V}' \subseteq \mathcal{V}$, and $L' = L \cap \mathcal{V}'$,*

a. *L' is a lattice in the vector subspace spanned by it, and*

b. *L' constitutes a direct-summand subgroup of L.*

Lemma 11 *Let $\mathcal{W}$ be the rational vector subspace of a finite-dimensional real vector space $\mathcal{V}$, spanned by a fixed lattice L in $\mathcal{V}$. Then the four sets consisting, respectively, of all*

i. *L-subspaces $\mathcal{V}'$ of $\mathcal{V}$,*
ii. *direct-summand subgroups L' of L,*
iii. *rational vector subspaces $\mathcal{W}'$ of $\mathcal{W}$,*
iv. *subtori $\mathcal{N}'$ of the torus group $\mathcal{V}/L$, that is, its compact connected Lie subgroups,*

stand in mutually consistent, natural bijective correspondences with one another, obtained by declaring $\mathcal{V}'$ to be the real span of both L' and $\mathcal{W}'$ as well as the identity component of the preimage of $\mathcal{N}'$ under the projection homomorphism $\mathcal{V} \to \mathcal{V}/L$. Furthermore, $\mathcal{W}'$ equals $\mathcal{W} \cap \mathcal{V}'$ and, simultaneously, is the rational span of L', while $\mathcal{N}' = \mathcal{V}'/L'$ and $L' = L \cap \mathcal{V}' = L \cap \mathcal{W}'$. Finally, $\dim_{\mathbb{R}} \mathcal{V}' = \dim_{\mathbb{Z}} L' = \dim_{\mathbb{Q}} \mathcal{W}' = \dim \mathcal{N}'$.

'Mutual consistency' means here that the above finite set of bijections is closed under the operations of composition and inverse.

Proof. The mappings (ii) $\to$ (i) and (iii) $\to$ (i), as well as (iv) $\to$ (i), defined in the three lines following (iv), are all bijections, with the inverses given by $(L', \mathcal{W}', \mathcal{N}') = (L \cap \mathcal{V}', \mathcal{W} \cap \mathcal{V}', \mathcal{V}'/L')$. Namely, each of the three mappings and their purported inverses takes values in the correct set, and each of the six mapping-inverse compositions is the respective identity. To be specific, the claim about the values follows from Lemma 8 for (iv) $\to$ (i) and (i) $\to$ (iv), from Definition 1 and Lemma 6 for (ii) $\to$ (i) and (i) $\to$ (ii), while it is obvious for (i) $\to$ (iii) and, for (iii) $\to$ (i), immediate from Definition 1, since we are free to assume that

$$(L, \mathcal{W}, \mathcal{V}) = (\mathbb{Z}^n, \mathbb{Q}^n, \mathbb{R}^n), \quad \text{where } n = \dim \mathcal{V}, \tag{2}$$

and every rational vector subspace of $\mathbb{Q}^n$ has a basis contained in $\mathbb{Z}^n$. Next, the compositions (ii) $\to$ (i) $\to$ (ii) and (i) $\to$ (ii) $\to$ (i) are the identity mappings – the former due to the fact that $L \cap \operatorname{span}_{\mathbb{R}} L' \subseteq L'$ (which one sees extending a $\mathbb{Z}$-basis of L' to a $\mathbb{Z}$-basis of L) – the opposite inclusion being obvious; the latter, as Definition 1 gives $\mathcal{V}' = \operatorname{span}_{\mathbb{R}}(L \cap \mathcal{V}')$. Similarly for (iii) $\to$ (i) $\to$ (iii) and (i) $\to$ (iii) $\to$ (i), as long as one replaces the letters L and $\mathbb{Z}$ with $\mathcal{W}$ and $\mathbb{Q}$, using (2) and the line following it. Finally, (iv) $\to$ (i) $\to$ (iv) and (i) $\to$ (iv) $\to$ (i) are the identity mappings as a consequence of Lemma 8, and the dimension equalities become obvious if one, again, chooses a $\mathbb{Z}$-basis of L containing a $\mathbb{Z}$-basis of L'. □

In the next theorem, as H is finite, all $A \in H$ must have $\det A = \pm 1$, and so the L-preserving property of H means that $AL = L$ (rather than just $AL \subseteq L$).

Proposition 1 *For a lattice L in a finite-dimensional real vector space $\mathcal{V}$, a finite group H of L-preserving linear automorphisms of $\mathcal{V}$, and an H-invariant L-subspace $\mathcal{V}'$ of $\mathcal{V}$, there exists an H-invariant L-subspace $\mathcal{V}''$ of $\mathcal{V}$, complementary to $\mathcal{V}'$ in the sense of* (1).

Proof. Let $\mathcal{W}' = \mathcal{W} \cap \mathcal{V}'$, where $\mathcal{W}$ is the rational span of L (see Lemma 11). Restricted to $\mathcal{W}$, the elements of H act by conjugation on the rational affine space $\mathcal{P}$ of all $\mathbb{Q}$-linear projections $\mathcal{W} \to \mathcal{W}'$ (by which we mean linear operators $\mathcal{W} \to \mathcal{W}'$ equal to the identity on $\mathcal{W}'$). The average of any orbit of the action of H on $\mathcal{P}$ is an H-invariant projection $\mathcal{W} \to \mathcal{W}'$ with a kernel $\mathcal{W}''$ corresponding via Lemma 11 to our required $\mathcal{V}''$. □

Corollary 1 *If* $L, \mathcal{V}, H$ *satisfy the hypotheses of Proposition* 1*, then every nonzero* H*-invariant* L*-subspace* $\mathcal{V}_0'$ *of* $\mathcal{V}$ *can be decomposed into a direct sum of one or more nonzero* H*-invariant* L*-subspaces, each of which is minimal in the sense of not containing any further nonzero proper* H*-invariant* L*-subspace.*

Proof. Induction on the possible values of $\dim \mathcal{V}_0'$. The case $\dim \mathcal{V}_0' = 1$ is trivial. Assuming the claim true for subspaces of dimensions less than $\dim \mathcal{V}_0'$, along with non-minimality of $\mathcal{V}_0'$, we fix a nonzero proper H-invariant L-subspace $\mathcal{V}'$ of $\mathcal{V}$, contained in $\mathcal{V}_0'$, and choose an H-invariant complement $\mathcal{V}''$ of $\mathcal{V}'$, guaranteed to exist by Proposition 1. Since $\mathcal{V}''$ intersects every coset of $\mathcal{V}'$ in $\mathcal{V}$, including cosets within $\mathcal{V}_0'$, the subspace $\mathcal{V}_0' \cap \mathcal{V}''$ is an H-invariant complement of $\mathcal{V}'$ in $\mathcal{V}_0'$, as well as an L-subspace (due to Lemma 9). We may now apply the inductive assumption to both $\mathcal{V}'$ and $\mathcal{V}_0' \cap \mathcal{V}''$. □

For geometric consequences of Corollary 1, see the end of Sect. 9, where we also point out that a decomposition into minimal summands is in general nonunique.

Given a lattice L in a finite-dimensional real vector space $\mathcal{V}$ and an L-subspace $\mathcal{V}'$ of $\mathcal{V}$, the restriction to L of the quotient-space projection $\mathcal{V} \to \mathcal{V}/\mathcal{V}'$ has the kernel $L' = L \cap \mathcal{V}'$, and so it descends to an injective group homomorphism $L/L' \to \mathcal{V}/\mathcal{V}'$, the image of which is a (full) lattice in an $\mathcal{V}/\mathcal{V}'$ (which follows if one uses a $\mathbb{Z}$-basis of L containing a $\mathbb{Z}$-basis of L'). From now on we will treat L/L' as a subset of $\mathcal{V}/\mathcal{V}'$. The discreteness of the lattice $L/L' \subseteq \mathcal{V}/\mathcal{V}'$ clearly implies the existence of an open subset $\mathcal{U}'$ of $\mathcal{V}$, containing $\mathcal{V}'$ and forming a union of cosets of $\mathcal{V}'$, such that $L \cap \mathcal{U}' = L'$.

5 Affine spaces

In this section all the affine and vector spaces are real and finite-dimensional, we denote by $\operatorname{End} \mathcal{V}$ the space of linear endomorphisms of a given real vector space $\mathcal{V}$, and scalars stand for the corresponding multiples of identity, so that the identity itself becomes $1 \in \operatorname{End} \mathcal{V}$.

For an affine space $\mathcal{E}$ with the translation vector space $\mathcal{V}$, let $\operatorname{Aut} \mathcal{E}$ be the group of all affine transformations (automorphisms) of $\mathcal{E}$, and $\operatorname{Aff} \mathcal{E}$ the set of all (possibly nonbijective) affine mappings $\mathcal{E} \to \mathcal{E}$. We have the inclusions $\operatorname{Aut} \mathcal{E} \subseteq \operatorname{Aff} \mathcal{E}$ and $\mathcal{V} \subseteq \operatorname{Aut} \mathcal{E}$, the latter expressing the fact that $\operatorname{Aut} \mathcal{E}$ contains the normal subgroup consisting of all translations. Any vector subspace $\mathcal{V}'$ of $\mathcal{V}$ gives rise to a foliation of $\mathcal{E}$, with the leaves formed by affine subspaces $\mathcal{E}'$ parallel to $\mathcal{V}'$, meaning

$$\text{the orbits of the translational action of } \mathcal{V}' \text{ on } \mathcal{E}, \tag{3}$$

which will also be referred to as the *cosets* of $\mathcal{V}'$ in $\mathcal{E}$. The resulting leaf (quotient) space $\mathcal{E}/\mathcal{V}'$ constitutes an affine space having the translation vector space $\mathcal{V}/\mathcal{V}'$. Clearly, for cosets $\mathcal{E}'$ and $\mathcal{E}''$ of vector subspaces $\mathcal{V}', \mathcal{V}''$ in a vector space $\mathcal{V}$,

$$\text{the complementarity condition (1) implies that } \mathcal{E}' \cap \mathcal{E}'' \text{ is a one-point set.} \tag{4}$$

A fixed inner product in $\mathcal{V}$ turns $\mathcal{E}$ into a *Euclidean affine space*, with the isometry group $\mathrm{Iso}\,\mathcal{E} \subseteq \mathrm{Aut}\,\mathcal{E}$. If $\delta \in (0,\infty)$, we define the *δ-neighborhood* of an affine subspace $\mathcal{E}'$ of $\mathcal{E}$ to be the set of points in $\mathcal{E}$ lying at distances less that δ from $\mathcal{E}'$. Clearly, the δ-neighborhood of $\mathcal{E}'$ is a union of cosets of a vector subspace $\mathcal{V}'$ of $\mathcal{V}$ (one of them being $\mathcal{E}'$ itself), as well as the preimage, under the projection $\mathcal{E} \to \mathcal{E}/\mathcal{V}'$, of the radius δ open ball centered at the point $\mathcal{E}'$ in the quotient Euclidean affine space $\mathcal{E}/\mathcal{V}'$ (for the obvious inner product on $\mathcal{V}/\mathcal{V}'$).

Given a Euclidean affine space $\mathcal{E}$ with the translation vector space $\mathcal{V}$ and an affine subspace $\mathcal{E}' \subseteq \mathcal{E}$ parallel, as in (3), to a vector subspace $\mathcal{V}' \subseteq \mathcal{V}$, (affine) self-isometries ζ of $\mathcal{E}$ such that $\zeta(x) = x$ for all $x \in \mathcal{E}'$ are in an obvious one-to-one correspondence with linear self-isometries A of the orthogonal complement of $\mathcal{V}'$. In this case, for easy later reference (in the proof of Lemma 14),

$$\text{we will call } \zeta \text{ the affine extension of } A \text{ centered on } \mathcal{E}'. \tag{5}$$

Definition 2 In an affine space $\mathcal{E}$ having the translation vector space $\mathcal{V}$, given an affine mapping $\gamma \in \mathrm{Aff}\,\mathcal{E}$ with the linear part $A \in \mathrm{End}\,\mathcal{V}$, we define the *translational-part coset* of γ to be the subset $b + \hat{\mathcal{V}}$ of $\mathcal{V}$, where $\hat{\mathcal{V}}$ denotes the image of $A-1$, and $b \in \mathcal{V}$ is the translational part of γ relative to a fixed origin $o \in \mathcal{E}$, in the sense that $\gamma(o+v) = o+Av+b$ for all $v \in \mathcal{V}$. The coset $b + \hat{\mathcal{V}}$ is clearly independent of the choice of an origin o, as a new origin $o + w$ results in the replacement of b with $b + (A-1)w$.

For an affine transformation $\gamma \in \mathrm{Aut}\,\mathcal{E}$ of an affine space $\mathcal{E}$ with the translation vector space $\mathcal{V}$, and a vector subspace $\mathcal{V}'$ of $\mathcal{V}$, consider this condition:

$$\begin{array}{l}\text{the linear part } A \text{ of } \gamma \text{ leaves } \mathcal{V}' \text{ invariant and descends to}\\ \text{the identity transformation of } \mathcal{V}/\mathcal{V}', \text{ that is, } (A-1)(\mathcal{V}) \subseteq \mathcal{V}'.\end{array} \tag{6}$$

Lemma 12 *If, in Lemma 3, $\mathcal{M}'$ is a compact leaf of a parallel distribution D on a complete flat Riemannian manifold $\mathcal{M}$, we get the following additional conclusions.*

a. *Every level of* $\mathrm{dist}(\mathcal{M}', \cdot)$ *in $\mathcal{M}'_\delta$, and $\mathcal{M}'_\delta$ itself, is a union of leaves of D.*
b. *Restrictions of $\mathcal{M}'_\delta \ni x \mapsto y \in \mathcal{M}'$ to leaves of D in $\mathcal{M}'_\delta$ are locally isometric.*
c. *The local inverses of all the above locally-isometric restrictions correspond via the diffeomorphism* $\mathrm{Exp}^\perp$ *to all local sections of the normal bundle of $\mathcal{M}'$ obtained by restricting to $\mathcal{M}'$ local parallel vector fields of lengths $r \in [0,\delta)$ that are tangent to $\mathcal{M}$ and normal to $\mathcal{M}'$, with r equal to the value of* $\mathrm{dist}(\mathcal{M}', \cdot)$ *on the leaf in question.*

This trivially follows from the fact the pullback of D to the Euclidean affine space $\mathcal{E}$ constituting the Riemannian universal covering space of $\mathcal{M}$ is a distribution whose integral manifolds are the affine subspaces parallel to $\mathcal{V}'$, in the sense of (3), for some vector subspace $\mathcal{V}'$ of the translation vector space $\mathcal{V}$ of $\mathcal{E}$.

6 Bieberbach groups and flat manifolds

Let $\mathcal{E}$ be a Euclidean affine n-space with the translation vector space $\mathcal{V}$. By a *Bieberbach group* [4, p. 4, Definition 1.7] in $\mathcal{E}$ one means any torsion-free discrete subgroup Π of $\mathrm{Iso}\,\mathcal{E}$ for which there exists a compact fundamental domain (Remark 2). Using the linear-part homomorphism $\alpha : \mathrm{Aut}\,\mathcal{E} \to \mathrm{Aut}\,\mathcal{V} \cong \mathrm{GL}(n, \mathbb{R})$, one defines the *lattice subgroup* L of Π and its *holonomy group* $H \subseteq \mathrm{Iso}\,\mathcal{V} \cong \mathrm{O}(n)$ by

$$L = \Pi \cap \mathcal{V}, \qquad H = \alpha(\Pi). \tag{7}$$

Thus, L is the set of all translations lying in Π (which also makes it the kernel of the restriction $\alpha : \Pi \to H$), and H consists of the linear parts of elements of Π. Note that $L \subseteq \mathcal{V}$ is a (full) lattice in the usual sense [4, p. 17, Theorem 3.1(ii)], as defined in Sect. 4. The relations involving Π, L and H are conveniently summarized by the short exact sequence

$$L \to \Pi \to H, \quad \text{where the arrows are the inclusion homomorphism and } \alpha. \tag{8}$$

As the normal subgroup L of Π is Abelian, the action of Π on L by conjugation descends to an action on L of the quotient group Π/L, identified via (8) with H.

> This last action clearly coincides with the ordinary linear action of H on $\mathcal{V}$, restricted to the lattice $L \subseteq \mathcal{V}$, and so L is H-invariant. (9)

Remark 3 The action of a Bieberbach group Π on the Euclidean affine space $\mathcal{E}$ being always free and properly discontinuous [4, p. 3, Proposition 1.1], the quotient $\mathcal{M} = \mathcal{E}/\Pi$, with the projected metric, forms a compact flat Riemannian manifold, while H must be finite [4, p. 17, Theorem 3.1(i)].

Remark 4 The assignment of $\mathcal{M} = \mathcal{E}/\Pi$ to Π establishes a well-known bijective correspondence [4, p. 65, Theorem 5.4] between equivalence classes of Bieberbach groups and isometry types of compact flat Riemannian manifolds. Bieberbach groups Π and $\hat{\Pi}$ in Euclidean affine spaces $\mathcal{E}$ and $\hat{\mathcal{E}}$ are called *equivalent* here if some affine isometry $\mathcal{E} \to \hat{\mathcal{E}}$ conjugates Π onto $\hat{\Pi}$. Furthermore, Π and H in (8) serve as the fundamental and holonomy groups of $\mathcal{M}$, while Π also acts via deck tranformations on the Riemannian universal covering space of $\mathcal{M}$, isometrically identified with $\mathcal{E}$.

Since the lattice subgroup L of the fundamental group Π of $\mathcal{M}$ gives rise to a covering projection $\mathcal{E}/L \to \mathcal{M}$, Lemmas 8 and 9 combined with Remark 4 have the following obvious consequence.

Corollary 2 *In any compact flat Riemannian manifold, the class of parallel distributions with compact leaves is closed under spans and intersections.*

7 Lattice-reducibility

A Bieberbach group Π in a Euclidean affine space $\mathcal{E}$ (or, the compact flat Riemannian manifold $\mathcal{M} = \mathcal{E}/\Pi$ corresponding to Π, in the sense of Remark 4, will be called *lattice-reducible* if, for $\mathcal{V}, H$ and L associated with $\mathcal{E}$ and Π as in Sect. 6, there exists $\mathcal{V}'$ such that

$$\mathcal{V}' \text{ is a nonzero proper } H\text{-invariant } L\text{-subspace of } \mathcal{V}. \tag{10}$$

(See Definition 1.) To emphasize the role of $\mathcal{V}'$ in (10), we also say that

$$\text{the lattice-reducibility condition (10) holds for } (\mathcal{V}, H, L, \mathcal{V}'). \tag{11}$$

As shown by Hiss and Szczepański [10], *every compact flat Riemannian manifold of dimension greater than one is lattice-reducible*. For details, see the Appendix.

Given a Bieberbach group Π in a Euclidean affine space $\mathcal{E}$ and an affine subspace $\mathcal{E}'$ of $\mathcal{E}$ parallel, as in (3), to a vector subspace $\mathcal{V}'$ of its translation space $\mathcal{V}$, satisfying (10) – (11), we denote by Σ' the *stabilizer group of* $\mathcal{E}'$ *in* Π, so that

$$\Sigma' \text{ consists of all the elements of } \Pi \text{ mapping } \mathcal{E}' \text{ onto itself.} \tag{12}$$

Let $\gamma \in \Pi$. As the foliation of $\mathcal{E}$ formed by the cosets of $\mathcal{V}'$ is Π-invariant, cf. (10),

$$\gamma \in \Sigma' \text{ if and only if } \gamma(\mathcal{E}') \text{ intersects } \mathcal{E}'. \tag{13}$$

Theorem 1 *For a lattice-reducible Bieberbach group* Π *in a Euclidean affine space* $\mathcal{E}$ *and a vector subspace* $\mathcal{V}'$ *of* $\mathcal{V}$ *with* (10)*, the following conclusions hold.*

i. *The affine subspaces of dimension* $\dim \mathcal{V}'$ *in* $\mathcal{E}$*, parallel to* $\mathcal{V}'$ *in the sense of* (3)*, are the leaves of a foliation* $F_{\mathcal{E}}$ *on* $\mathcal{E}$*, projectable under the covering projections* $\mathrm{pr} : \mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$ *and* $\mathcal{E} \to \mathcal{T} = \mathcal{E}/L$ *onto foliations* $F_{\mathcal{M}}$ *of* $\mathcal{M}$ *and* $F_{\mathcal{T}}$ *of the torus* $\mathcal{T} = \mathcal{E}/L$*, both of which have compact totally geodesic leaves, tangent to a parallel distribution.*
ii. *The leaves* $\mathcal{M}'$ *of* $F_{\mathcal{M}}$ *coincide with the* pr*-images of the leaves* $\mathcal{E}'$ *of* $F_{\mathcal{E}}$*, and the restrictions* $\mathrm{pr} : \mathcal{E}' \to \mathcal{M}'$ *are covering projections. The same remains true if one replaces* $\mathcal{M}$ *and* pr *with* $\mathcal{T}$ *and the projection* $\mathcal{E} \to \mathcal{T}$*. Any such* $\mathcal{M}'$*, being a compact flat Riemannian manifold, corresponds via Remark* 4 *to a Bieberbach group* Π' *in the Euclidean affine space* $\mathcal{E}'$*. For* L', H' *appearing in the* $\mathcal{M}'$*-analog* $L' \to \Pi' \to H'$ *of* (8)*, with* Σ' *defined by* (12)*,*

a. Π' *consists of the restrictions to* $\mathcal{E}'$ *of all the elements of* Σ',
b. H' *is formed by the restrictions to* $\mathcal{V}'$ *of the linear parts of elements of* Σ',
c. $L' = \Pi' \cap \mathcal{V}'$, *as in* (7), *and* $L \cap \mathcal{V}' \subseteq L'$.

iii. *The restriction homomorphism* $\Sigma' \to \Pi'$ *of* (ii-a) *is an isomorphism.*

The last inclusion of (ii-c) may be proper; see the end of Sect. 15.

We chose to format the proof Theorem 1 as the whole next section, since some parts of it are of independent interest, and can in this way be more comfortably cited later in the text (Sections 10 – 11, 13 – 14, 16).

8 Proof of Theorem 1

The projectability of the foliation $F_{\mathcal{E}}$ under both covering projections $\mathrm{pr} : \mathcal{E} \to \mathcal{M}$ and $\mathcal{E} \to \mathcal{T}$ follows as a trivial consequence from the fact that, due to the H-invariance of $\mathcal{V}'$,

$$F_{\mathcal{E}} \text{ is } \Pi\text{-invariant and, obviously, } L\text{-invariant,} \tag{14}$$

while Lemma 1(ii) implies the integrability of the image distribution. Next,

$$\begin{array}{l}\mathrm{pr} \text{ is the composition } \mathcal{E} \to \mathcal{T} \to \mathcal{M} \text{ of two mappings: the} \\ \text{universal-covering projection of the flat torus } \mathcal{T} = \mathcal{E}/L, \\ \text{and the quotient projection for the action of } \Pi \text{ on } \mathcal{T},\end{array} \tag{15}$$

the latter action clearly becoming free if one replaces Π with $\Pi/L \cong H$. Both factor mappings, $\mathcal{E} \to \mathcal{T}$ and $\mathcal{T} \to \mathcal{M}$, are covering projections – the first since L is a lattice in $\mathcal{V}$, the second due to Remark 1(a). Parts (iii)–(iv) of Lemma 1, along with Lemma 8, may now be applied to the foliations $F_{\mathcal{T}}$ and $F_{\mathcal{M}}$ of the torus $\mathcal{T}$ and of $\mathcal{M}$ obtained as projections of $F_{\mathcal{E}}$, proving the last (compact-leaves) claim of (i), as well as the first two sentences of (ii).

We now fix a leaf $\mathcal{E}'$ of $F_{\mathcal{E}}$, and choose a leaf $\mathcal{M}'$ of $F_{\mathcal{M}}$ containing $\mathrm{pr}(\mathcal{E}')$, cf. Lemma 1(i). It follows that

$$\mathrm{pr} : \mathcal{E}' \to \mathcal{M}' \text{ is a (surjective) covering projection,} \tag{16}$$

since (15) decomposes $\mathrm{pr} : \mathcal{E}' \to \mathcal{M}'$ into the composition $\mathcal{E}' \to \mathcal{T}' \to \mathcal{M}'$, in which the first mapping is the universal-covering projection of the torus $\mathcal{T}' = \mathcal{E}'/L'$, and the second one must be a covering due to Remark 1(b).

Two points of $\mathcal{E}'$ have the same pr-image if and only if one is transformed into the other by an element of the group Π' described in assertion (ii-a); namely, the 'only if' part follows since, given $x, y \in \mathcal{E}'$ with $\mathrm{pr}(x) = \mathrm{pr}(y)$ in $\mathcal{M} = \mathcal{E}/\Pi$, the element of Π sending x to y must lie in Σ' by (13). Furthermore, Π' acts on $\mathcal{E}'$ freely since Π does so on $\mathcal{E}$ (Remark 3). Thus, Π' coincides with the deck transformation group for the universal covering projection (16), and satisfies (ii-a). Next, (ii-b) and (ii-c) are consequences of the definitions of H' and L'.

Finally, (iii) follows since nontrivial elements of Σ', being fixed-point free (Remark 3), have nontrivial restrictions to $\mathcal{E}'$.

9 Geometric consequences of Lemma 8 and Proposition 1

Hiss and Szczepański's result mentioned in the Introduction, combined with Remark 4, Proposition 1 and Theorem 1, has the following immediate consequence.

Theorem 2 *Every compact flat Riemannian manifold $\mathcal{M}$ of dimension $n \geq 2$ admits two proper parallel distributions D and $\hat{D}$ with compact leaves, which are complementary to each other in the sense that $T\mathcal{M} = D \oplus \hat{D}$.*

Theorem 1(i), Lemma 8 and Corollary 1 also easily imply that the tangent bundle $T\mathcal{M}$ of any compact flat Riemannian manifold $\mathcal{M}$ admits a maximal direct-sum decomposition into parallel subbundles (distributions) with compact leaves, maximality meaning that none of the summand subbundles can be further decomposed in the same manner. Hiss and Szczepański's result [10] guarantees that, unless $\dim \mathcal{M} < 2$, at least two such summand distributions are present.

Decompositions just mentioned may be quite far from unique: when $\mathcal{M}$ is an n-torus, they stand in a bijective correspondence with decompositions of a n-dimensional rational vector space into a direct sum of lines. This is why one probably should not expect them to have interesting general properties.

10 Geometries of individual leaves

Throughout this section we adopt the assumptions and notation of Theorem 1. The Π-invariance of the foliation $F_{\mathcal{E}}$, cf. (14), trivially gives rise to the obvious

$$\text{isometric action of } \Pi \text{ on the quotient Euclidean affine space } \mathcal{E}/\mathcal{V}' \tag{17}$$

(that is, on the leaf space of $F_{\mathcal{E}}$, whose points coincide with the affine subspaces $\mathcal{E}'$ of $\mathcal{E}$ parallel to $\mathcal{V}'$). Whenever $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ is fixed, its stabilizer group Σ' in (12) obviously coincides with the isotropy group of $\mathcal{E}'$ for (17). The action (17) is not effective, as the kernel of the corresponding homomorphism $\Pi \to \mathrm{Iso}\,[\mathcal{E}/\mathcal{V}']$ clearly contains the group $L' = L \cap \mathcal{V}'$ forming a lattice in $\mathcal{V}'$, cf. Definition 1 and Lemma 10(a). Now the H-invariance of L – see (9) – combined with the H-invariance of $\mathcal{V}'$ shows that $L' = L \cap \mathcal{V}'$ is a normal subgroup of Π, which leads to a further homomorphism $\Pi/L' \to \mathrm{Iso}\,[\mathcal{E}/\mathcal{V}']$ (still in general noninjective, cf. (24) below). Let pr again stand for the covering projection $\mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$.

Given $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ and a vector $v \in \mathcal{V}$ orthogonal to $\mathcal{V}'$, we set $\mathcal{M}'_v = \mathrm{pr}(\mathcal{E}'+v)$, so that, according to (16), $\mathcal{M}'_0 = \mathcal{M}'$. Again by (16),

$$\mathrm{pr} : \mathcal{E}'+v \to \mathcal{M}'_v \text{ is a locally-isometric universal-covering projection.} \tag{18}$$

and $\mathcal{M}'_v$ must be a (compact) leaf of $F_{\mathcal{M}}$. We also

$$\begin{array}{l}\text{choose } \delta \text{ as in Lemma 3 and Lemma 12 for the submanifold}\\ \mathcal{M}' = \mathrm{pr}(\mathcal{E}') \text{ with (16) of the compact flat manifold } \mathcal{M} = \mathcal{E}/\Pi,\\ \text{and denote by } \Sigma'_v \subseteq \Pi \text{ the stabilizer group of } \mathcal{E}' + v, \text{ cf. (12).}\end{array} \tag{19}$$

Lemma 13 *Under the above hypotheses, for any $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ there exists $\delta \in (0, \infty)$ such that, whenever $u \in \mathcal{V}$ is a unit vector orthogonal to $\mathcal{V}'$ and $r, s \in (0, \delta)$, the isometries $\mathcal{E}'+ru \to \mathcal{E}'+su$ and $\mathcal{E}'+ru \to \mathcal{E}'$ acting via translations by the vectors $(s-r)u$ and, respectively, $-ru$, descend under the universal-covering projections* (18)*, with v equal to ru, su or* 0*, to an isometry $\mathcal{M}'_{ru} \to \mathcal{M}'_{su}$ or, respectively, a k-fold covering projection $\mathcal{M}'_{ru} \to \mathcal{M}'$, where the integer $k = k(u) \ge 1$ may depend on u – see the end of Sect.* 15 *– but not on r.*

Proof. For δ selected in (19) and any $c \in [0, 1]$, let $\psi_c : \mathcal{M}_\delta \to \mathcal{M}_\delta$ correspond, via the $\mathrm{Exp}^\perp$-diffeomorphic identification of Lemma 3(a), to the mapping $\mathcal{N}_\delta \to \mathcal{N}_\delta$ which multiplies vectors normal to $\mathcal{M}'$ by the scalar c. With ϕ denoting our isometry $\mathcal{E}'+ru \to \mathcal{E}'+su$ (or, $\mathcal{E}'+ru \to \mathcal{E}'$) we now have $\mathrm{pr} \circ \phi = \psi_c \circ \mathrm{pr}$ on $\mathcal{E}'+ru$, where $c = s/r$ (or, respectively, $c = 0$) since, given $x \in \mathcal{E}'$, the pr-image of the line segment $\{x + tu : t \in [0, \delta)\}$ in $\mathcal{E}$ is the length δ minimizing geodesic segment in $\mathcal{M}'_\delta$ emanating from the point $y = \mathrm{pr}(x) \in \mathcal{M}'$ in a direction normal to $\mathcal{M}'$, and $\mathrm{pr} \circ \phi$ sends $x + tu$, in both cases, to $\mathrm{pr}(x + ctu) = \psi_c(\mathrm{pr}(x + tu))$. The pr-image of $\phi(z)$, for any $z \in \mathcal{E}' + ru$, thus depends only on $\mathrm{pr}(z)$ (by being its ψ_c-image), and so both original isometries ϕ descend to (necessarily locally-isometric) mappings $\mathcal{M}'_{ru} \to \mathcal{M}'_{su}$ and $\mathcal{M}'_{ru} \to \mathcal{M}'$, which constitute finite coverings (Remark 1(b)). The former is also bijective, its inverse arising when one switches r and s. As the composition $\mathcal{M}'_{su} \to \mathcal{M}'_{ru} \to \mathcal{M}'$ clearly equals the analogous covering projection $\mathcal{M}'_{su} \to \mathcal{M}'$ (with s rather than r), the coverings $\mathcal{M}'_{ru} \to \mathcal{M}'$ and $\mathcal{M}'_{su} \to \mathcal{M}'$ have the same multiplicity, which completes the proof. ◻

Remark 5 Replacing δ of (19) with $1/4$ times its original value, we can also require it to have the following property: *if $\gamma \in \Pi$ and $x \in \mathcal{E}$ are such that both x and $\gamma(x)$ lie in the δ-neighborhood of $\mathcal{E}'$, defined as in Sect.* 5*, then $\gamma \in \Sigma'$ for the stabilizer group Σ' of $\mathcal{E}'$ given by* (12). In fact, letting $\mathcal{E}''$ be the leaf of $F_{\mathcal{E}}$ through x, we see from (14) that its γ-image $\gamma(\mathcal{E}'')$ is also a leaf of $F_{\mathcal{E}}$, while both leaves are within the distance δ from $\mathcal{E}'$, which yields $\mathrm{dist}(\mathcal{E}'', \gamma(\mathcal{E}'')) < 2\delta$ and so, due to the triangle inequality, $\mathrm{dist}(\mathcal{E}', \gamma(\mathcal{E}')) \le \mathrm{dist}(\mathcal{E}', \mathcal{E}'') + \mathrm{dist}(\mathcal{E}'', \gamma(\mathcal{E}'')) + \mathrm{dist}(\gamma(\mathcal{E}''), \gamma(\mathcal{E}')) < \delta + 2\delta + \delta = 4\delta$. Thus, $x + ru \in \gamma(\mathcal{E}')$ for some $x \in \mathcal{E}'$, some unit vector $u \in \mathcal{V}$ orthogonal to $\mathcal{V}'$, and $r = \mathrm{dist}(\mathcal{E}', \gamma(\mathcal{E}')) \in [0, 4\delta)$. Assuming now (19) with δ replaced by 4δ, one gets $r = 0$, that is, $\gamma(\mathcal{E}') = \mathcal{E}'$ and $\gamma \in \Sigma'$. Namely, the pr-image of the curve $[0, 4\delta) \ni t \mapsto x + tu$ is a geodesic in the image of the diffeomorphism $\mathrm{Exp}^\perp$ of Lemma 3(a), which intersects $\mathcal{M}'$ only at $t = 0$, while $\mathcal{M}' = \mathrm{pr}(\mathcal{E}') = \mathrm{pr}(\gamma(\mathcal{E}'))$, since $\mathcal{M} = \mathcal{E}/\Pi$.

Lemma 14 *Let there be given $\mathcal{V}', \mathcal{E}'$ as in Lemma* 13*, δ having the additional property of Remark* 5*, any $r \in (0, \delta)$, and any unit vector $u \in \mathcal{V}$ orthogonal to $\mathcal{V}'$.*

a. *The stabilizer group* Σ'_{ru} *in* (19) *does not depend on* $r \in (0,\delta)$.
b. *The linear part of each element of* Σ'_{ru} *keeps* u *fixed.*
c. Σ'_{ru} *is a subgroup of* Σ'_0 *with the finite index* $k = k(u) \geq 1$ *of Lemma* 13*,*
d. $\mathrm{pr} : \mathcal{E} \to \mathcal{M}$ *maps the* δ*-neighborhood* $\mathcal{E}_\delta$ *of* $\mathcal{E}'$ *in* $\mathcal{E}$ *onto* $\mathcal{M}'_\delta$ *of Lemma* 3(a).
e. $\mathcal{E}_\delta$ *and* $\mathcal{M}'_\delta$ *are unions of leaves of, respectively,* $F_{\mathcal{E}}$ *and* $F_{\mathcal{M}}$.
f. *The preimage under* $\mathrm{pr} : \mathcal{E}_\delta \to \mathcal{M}_\delta$ *of the leaf* $\mathcal{M}'_{ru} = \mathrm{pr}(\mathcal{E}' + ru)$ *of* $F_{\mathcal{M}}$ *equals the union of the images* $\gamma(\mathcal{E}' + ru)$ *over all* $\gamma \in \Sigma'_0$.

Proof. By (16) and (19), $\mathcal{M}'_v = (\mathcal{E}' + v)/\Pi'_v$, if one lets Π'_v denote the image of Σ'_v under the injective homomorphism of restriction to $\mathcal{E}'$, cf. Theorem 1(iii). Fixing $s \in [0,\delta)$ and $r \in (0,\delta)$ we therefore conclude from Lemma 13 and (13) that, whenever $x \in \mathcal{E}' + ru$ and $\gamma \in \Sigma'_{ru}$, there exists $\hat\gamma \in \Sigma'_{su}$ satisfying the condition

$$\gamma(x) + v = \hat\gamma(x+v), \quad \text{where } v = (s-r)u, \quad \text{and } \hat\gamma = \gamma \text{ when } s = r, \tag{20}$$

the last clause being obvious since $\gamma, \hat\gamma \in \Pi$ and the action of Π is free. With u and γ fixed as well, for each given $\hat\gamma \in \Sigma'_{su}$ the set of all $x \in \mathcal{E}' + ru$ having Property (20) is closed in $\mathcal{E}' + ru$ while, as we just saw, the union of these sets over all $\hat\gamma \in \Sigma'_{su}$ equals $\mathcal{E}' + ru$. Thus, by Baire's theorem (Lemma 4), some $\hat\gamma \in \Sigma'_{su}$ satisfies (20) with all x from some nonempty open subset of $\mathcal{E}' + ru$, and hence – by real-analyticity – for all $x \in \mathcal{E}' + ru$. In terms of the translation τ_v by the vector v, we consequently have $\hat\gamma = \tau_v \circ \gamma \circ \tau_v^{-1}$ on $\mathcal{E}' + su$, so that, due to Theorem 1(iii), γ uniquely determines $\hat\gamma$, while the assignment $\gamma \mapsto \hat\gamma$ is a homomorphism $\Sigma'_{ru} \to \Sigma'_{su} \subseteq \Pi$, and $\zeta = \hat\gamma \circ \tau_v \circ \gamma^{-1} \circ \tau_v^{-1}$ equals the identity on $\mathcal{E}' + su$. If we now allow s to vary from r to 0, the resulting curve $s \mapsto \zeta$ consists of affine extensions, defined as in (5), of linear self-isometries of the orthogonal complement of $\mathcal{V}'$, and $\hat\gamma = \zeta \circ \tau_v \circ \gamma \circ \tau_v^{-1}$ on $\mathcal{E}$. As Π is discrete, the curve $s \mapsto \hat\gamma \in \Pi$, with $v = (s-r)u$, must be constant, and can be evaluated by setting $s = r$ (or, $v = 0$). Thus, $\hat\gamma = \gamma$ on $\mathcal{E}$ from the last clause of (20), and so $\Sigma'_{ru} \subseteq \Sigma'_{su}$. For $s > 0$, switching r with s we get the opposite inclusion, and (a) follows. Also, taking the linear part of the resulting relation $\gamma = \zeta \circ \tau_v \circ \gamma \circ \tau_v^{-1}$, we see that ζ equals the identity, for all s. Hence $\gamma = \tau_v \circ \gamma \circ \tau_v^{-1}$, that is, γ commutes with τ_v which, by (9), amounts to (b). Setting $s = 0$, we obtain the first part of (c): $\Sigma'_{ru} \subseteq \Sigma'_0$. Assertion (d) is clear as pr, being locally isometric, maps line segments onto geodesic segments. Lemma 12(a) for $D = F_{\mathcal{M}}$ gives (e). With $\mathrm{pr} : \mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$ in Theorem 1(i), the additional property of δ (Remark 5) yields (f). Finally, for $k = k(u)$, the geodesic $[0,r] \ni t \mapsto \mathrm{pr}(x + tu)$, normal to $\mathcal{M}'$ at $y = \mathrm{pr}(x)$, is one of k such geodesics $[0,r] \ni t \mapsto \mathrm{pr}(x + tw)$, joining y to points of its preimage under the projection $\mathcal{M}'_{ru} \to \mathcal{M}'$ of Lemma 13, where w ranges over a k-element set $\mathcal{R}$ of unit vectors in $\mathcal{V}$, orthogonal to $\mathcal{V}'$. The union of the corresponding subset $C = \{\mathcal{E}' + rw : w \in \mathcal{R}\}$ of the leaf space of $F_{\mathcal{E}}$ equals the preimage in (f) – and hence an orbit for the action of Σ'_0 – as every leaf in the preimage contains a point nearest x. Due to the already-established inclusion $\Sigma'_{ru} \subseteq \Sigma'_0$ and (19), Σ'_{ru} is the isotropy group of $\mathcal{E}' + ru$ relative to the transitive action of Σ'_0 on C, and so k, the cardinality of C, equals the index of Σ'_{ru} in Σ'_0, which proves the second part of (c). ◻

11 The generic stabilizer group

Given a Bieberbach group Π in a Euclidean affine space $\mathcal{E}$ with the translation vector space $\mathcal{V}$, let us fix a vector subspace $\mathcal{V}'$ of $\mathcal{V}$ satisfying (10) for L, H introduced in (8). As long as $\dim \mathcal{E} \geq 2$, such $\mathcal{V}'$ always exists (Sect. 7). An element $\mathcal{E}'$ of $\mathcal{E}/\mathcal{V}'$, that is, a coset of $\mathcal{V}'$ in $\mathcal{E}$, will be called *generic* if its stabilizer group $\Sigma' \subseteq \Pi$, defined by (12), equals

$$\text{the kernel of the homomorphism } \Pi \to \operatorname{Iso}[\mathcal{E}/\mathcal{V}'] \text{ corresponding to (17).} \tag{21}$$

The pr-images of generic cosets of $\mathcal{V}'$ will be called generic leaves of $F_{\mathcal{M}}$.

Still using the symbols L, H and pr for the groups appearing in (7) – (8) and the universal-covering projection $\mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$, let us also

denote by $K' \subseteq H$ the normal subgroup consisting of all elements of H that act on the orthogonal complement of $\mathcal{V}'$ as the identity, and by $\mathcal{U}'$ the subset of $\mathcal{E}/\mathcal{V}'$ formed by all generic cosets of $\mathcal{V}'$ in $\mathcal{E}$. (22)

Theorem 3 *For* Σ' *equal to* (21)*, under the assumptions preceding* (21)*, with the notation of* (22)*, one has the following conclusions.*

i. $\mathcal{U}'$ *in* (22) *constitutes an open dense subset of* $\mathcal{E}/\mathcal{V}'$.
ii. *The normal subgroup* Σ' *of* Π *is contained as a finite-index subgroup in the stabilizer group of every* $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ *for the action* (17)*, and equal to this stabilizer group if* $\mathcal{E}' \in \mathcal{U}'$.
iii. *The* pr-*images* $\mathcal{M}', \mathcal{M}''$ *of any* $\mathcal{E}', \mathcal{E}'' \in \mathcal{U}'$ *are isometric to each other.*
iv. *If one identifies* $\mathcal{E}$ *with its translation vector space* $\mathcal{V}$ *via a choice of an origin,* Σ' *becomes the set of all the elements of* Π *having, for* K' *given by* (22)*, the form*

$$\mathcal{V} \ni x \mapsto Ax + b \in \mathcal{V}, \text{ in which } b \in \mathcal{V}' \text{ and the linear part } A \text{ lies in } K'. \tag{23}$$

v. *Whenever* $\mathcal{E}' \in \mathcal{U}'$*, the homomorphism which restricts elements of the generic stabilizer group* Σ' *to* $\mathcal{E}'$ *is injective, and the resulting isomorphic image* Π' *of* Σ' *constitutes a Bieberbach group in the Euclidean affine space* $\mathcal{E}'$*. The lattice subgroup of* Π' *and its holonomy group* H' *are the intersection* $L' = L \cap \mathcal{V}'$ *and the image* H' *of the group* K' *defined in* (22) *under the injective homomorphism of restriction to* $\mathcal{V}'$.

Proof. Lemma 14(a) states that the assumptions of Lemma 2 are satisfied by the Euclidean affine space $\mathcal{W} = \mathcal{E}/\mathcal{V}'$ and the mapping F that sends $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ to its stabilizer group Σ' with (12). The assignment $\mathcal{E}' \mapsto \Sigma'$ is thus *locally constant* on some open dense set $\mathcal{U}' \subseteq \mathcal{E}/\mathcal{V}'$. Letting Σ' be the constant value of this assignment assumed on a nonempty connected open subset $\mathcal{W}'$ of $\mathcal{U}'$, and fixing $\gamma \in \Sigma'$, we obtain $\gamma(\mathcal{E}') = \mathcal{E}'$ for all $\mathcal{E}' \in \mathcal{W}'$, and hence, from real-analyticity, for all $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$. Thus, our Σ' is contained in the stabilizer group of every $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$. Since the same applies also to another constant value Σ'' assumed on a nonempty connected open set, $\Sigma'' = \Sigma'$ and the phrase ‘locally constant’ may be replaced with *constant*. By Lemma 14(c), any such Σ' must be a finite-index subgroup of each stabilizer group. As Σ' consists of the elements of Π preserving every $\mathcal{E}' \in \mathcal{U}'$, real-analyticity implies that they preserve all $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$, and so Σ' coincides with (21), which also shows that Σ' is a normal subgroup of Π, and (i) – (ii) follow.

Next, (iv) holds since the set of all $\gamma \in \mathrm{Aut}\,\mathcal{E}$ satisfying (6) clearly constitutes a subgroup of $\mathrm{Aut}\,\mathcal{E}$ containing, as a normal subgroup, the set of all γ with (6) such that translational-part coset of γ (see Definition 2) is contained in $\mathcal{V}'$. To verify this claim, note that the latter set is a normal subgroup, being the kernel of the obvious homomorphism from the subgroup of all γ having the property (6) into the translation subgroup $\mathcal{V}/\mathcal{V}'$ of $\mathrm{Aut}\,[\mathcal{E}/\mathcal{V}']$. More precisely, γ represented by the pair (A, b) (as in Definition 2) preserves each element of $\mathcal{E}/\mathcal{V}'$ if and only if $Av+b$ differs from v, for every $v \in \mathcal{V}$, by an element of $\mathcal{V}'$ or, equivalently (as one sees setting $v = 0$), $\mathcal{V}'$ contains both b and $(A-1)(\mathcal{V})$.

Finally, Theorem 1(ii)-(iii) yields (v), while (v) implies (iii) via Remark 4. □

The example provided by a compact flat manifold $\mathcal{M} = \mathcal{E}/\Pi$ which is a Riemannian product $\mathcal{M} = \mathcal{M}'\times\mathcal{M}''$ with $\mathcal{E} = \mathcal{E}'\times\mathcal{E}''$ and $\Pi = \Pi'\times\Pi''$ for two Bieberbach groups Π', Π'' in Euclidean affine spaces $\mathcal{E}', \mathcal{E}''$ having the translation vector spaces $\mathcal{V}', \mathcal{V}''$, while $\mathcal{M}'$ is not a torus, shows that, in general,

$$\text{an element of } \Pi \text{ acting trivially on } \mathcal{E}/\mathcal{V}' \text{ need not lie in } L'. \tag{24}$$

Namely, the H-invariant subspace $\mathcal{V}'\times\{0\}$ then gives rise to the $\mathcal{M}'$ factor foliation $F_{\mathcal{E}}$ of the product manifold $\mathcal{M}$, and the action of the group $\Pi'\times\{1\}$ on its leaf space is obviously trivial, even though not all elements of $\Pi'\times\{1\}$ are translations.

Lemma 15 *Using any given $\mathcal{E}' \in \mathcal{U}'$ in Theorem* 3*, where $\mathcal{V}'$ satisfying* (10) *is fixed, let us identify Σ' with Π' and H' with K' via the isomorphisms $\Sigma' \to \Pi'$ and $K' \to H'$ resulting from Theorem* 3(v)*, which turns Π' and H' into subgroups of Π and H. These subgroups $\Pi' \subseteq \Pi$ and $H' \subseteq H$ do not depend on the choice of $\mathcal{E}' \in \mathcal{U}'$, and neither does the mapping degree $d = |H'|$ of the d-fold covering projection $\mathcal{T}' \to \mathcal{M}' = \mathcal{T}'/H'$ analogous to those mentioned in* (15)*.*

Proof. Our claims about Π' and H' are immediate from (21) and (22). □

Any lattice L in the translation vector space $\mathcal{V}$ of a Euclidean affine space $\mathcal{E}$ is, obviously, a Bieberbach group in $\mathcal{E}$. In the case of a fixed vector subspace $\mathcal{V}'$ of $\mathcal{V}$ with (10), all the general facts established about any given Bieberbach group Π in $\mathcal{E}$, the compact flat manifold $\mathcal{M} = \mathcal{E}/\Pi$, and the leaves $\mathcal{M}'$ of $F_{\mathcal{M}}$ (see Theorem 1) thus remain valid for the torus $\mathcal{T} = \mathcal{E}/L$ and the leaves $\mathcal{T}'$ of $F_{\mathcal{T}}$. Every coset of $\mathcal{V}'$ is generic if we declare the lattice L of Π to be the new Bieberbach group.

12 The leaf space

We again adopt the assumptions and notation of Theorem 1. Not surprisingly, the leaf space $\mathcal{M}/F_{\mathcal{M}}$ has the following property (discussed below): $\mathcal{M}/F_{\mathcal{M}}$ forms

$$\text{a flat compact orbifold, canonically identified with the quotient of the torus } [\mathcal{E}/\mathcal{V}']/[L\cap\mathcal{V}'] \text{ under the isometric action of the finite group } H. \tag{25}$$

By a *crystallographic group* [13] in a Euclidean affine space one means a discrete group of isometries having a compact fundamental domain, cf. Remark 2.

Proposition 2 *Under the assumptions listed in the first line of Sect.* 11*, with* Σ' *denoting the normal subgroup* (21) *of* Π*, the quotient group* Π/Σ' *acts effectively by isometries on the quotient Euclidean affine space* $\mathcal{E}/\mathcal{V}'$ *and, when identified with a subgroup of* $\mathrm{Iso}\,[\mathcal{E}/\mathcal{V}']$*, it constitutes a crystallographic group.*

Proof. A compact fundamental domain exists since Π/Σ' contains the lattice subgroup L/L' of $\mathcal{V}/\mathcal{V}'$ (see the end of Sect. 4). To verify the discreteness of Π/Σ', suppose that, on the contrary, some sequence $\gamma_k \in \Pi$, $k = 1, 2 \ldots$, has terms representing mutually distinct elements of Π/Σ' which converge in $\mathrm{Iso}\,[\mathcal{E}/\mathcal{V}']$. As L' is a lattice in $\mathcal{V}'$, fixing $x \in \mathcal{E}$ and suitably choosing $v_k \in L'$ we achieve boundedness of the sequence $\hat{\gamma}_k(x) = \gamma_k(x) + v_k$, while $\hat{\gamma}_k$ represent the same (distinct) elements of Π/Σ' as γ_k. The ensuing convergence of a subsequence of $\hat{\gamma}_k$ contradicts the discreteness of Π. □

The resulting quotient of $\mathcal{E}/\mathcal{V}'$ under the action of Π/Σ' is thus a flat compact orbifold [5], which may clearly be identified both with the leaf space $\mathcal{M}/F_{\mathcal{M}}$, and with the quotient of the torus $[\mathcal{E}/\mathcal{V}']/[L\cap\mathcal{V}']$ mentioned in (25). The latter identification clearly implies the Hausdorff property of the leaf space $\mathcal{M}/F_{\mathcal{M}}$.

On the other hand, for an H-invariant subspace $\mathcal{V}''$ of $\mathcal{V}$ *not* assumed to be an L-subspace, there exists an L-*closure* of $\mathcal{V}''$, meaning the smallest L-subspace $\mathcal{V}'$ of $\mathcal{V}$ containing $\mathcal{V}''$, which is obviously obtained by intersecting all such L-subspaces (Lemma 9). The leaf space $\mathcal{M}/F_{\mathcal{M}}$ corresponding to $\mathcal{V}'$ then forms a natural "Hausdorffization" of the leaf space of $\mathcal{V}''$, and may also be described in terms of Gromov-Hausdorff limits. See the recent paper [2].

13 Intersections of generic complementary leaves

Throughout this section Π is a given Bieberbach group in a Euclidean affine space $\mathcal{E}$ of dimension $n \geq 2$, while $\mathcal{V}', \mathcal{V}''$ are two mutually complementary H-invariant L-subspaces of the translation vector space $\mathcal{V}$ of $\mathcal{E}$, in the sense of (1) and Definition 1, for L and H associated with Π via (7). We also fix generic cosets $\mathcal{E}'$ of $\mathcal{V}'$ and $\mathcal{E}''$ of $\mathcal{V}''$ (see the beginning of Sect. 11), which leads to the analogs $L' \to \Pi' \to H'$ and $L'' \to \Pi'' \to H''$ of (8), described by Theorem 1(ii) and, $\mathcal{E}', \mathcal{E}''$

being generic, Theorem 3(v) yields $L' = L \cap \mathcal{V}'$ and $L'' = L \cap \mathcal{V}''$. Furthermore, for these $\Pi, \Pi', \Pi'', L, L', L'', H, H', H''$,

$$\text{the conclusions of Lemma 7 hold if we replace } G \text{ with } \Pi, L \text{ or } H, \tag{26}$$

since so do the assumptions of Lemma 7, provided that one uses Lemma 15 to treat Π' and Π'' (or, H' and H'') as subgroups of Π (or, respectively, H). In fact, (23) and the description of K' in (22) show that all $A \in K'$ (and, among them, the linear parts of all elements of $\Sigma' = \Pi'$) leave invariant both $\mathcal{V}'$ and $\mathcal{V}''$, and act via the identity on the latter. (We have the obvious isomorphic identifications of $\mathcal{V}/\mathcal{V}'$ with $\mathcal{V}''$ on the one hand, and with the orthogonal complement of $\mathcal{V}'$ in $\mathcal{V}$ on the other, while such A descend to the identity automorphism of $\mathcal{V}/\mathcal{V}'$.) The same is clearly the case if one switches the primed symbols with the double-primed ones, while elements of Σ' now commute with those of Σ'' in view of (23). This yields (26) and, consequently, allows us to form

$$\text{the quotient groups } \hat{\Pi} = \Pi/(\Pi'\Pi''), \quad \hat{L} = L/(L'L''), \quad \hat{H} = H/(H'H''). \tag{27}$$

Finally, let $\mathrm{pr} : \mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$ and $\mathcal{M}', \mathcal{M}'', \mathcal{T}', \mathcal{T}''$ denote, respectively, the covering projection of Theorem 1(i), the pr-image $\mathcal{M}'$ of $\mathcal{E}'$ (or, $\mathcal{M}''$ of $\mathcal{E}''$), and the tori $\mathcal{E}'/L'$ and $\mathcal{E}''/L''$, contained in the torus $\mathcal{T} = \mathcal{E}/L$ of (15). Note that $\mathcal{M}'$ and $\mathcal{M}''$ are (compact) leaves of the parallel distributions arising, due to Theorem 1(i), on $\mathcal{M} = \mathcal{E}/\Pi$, which itself is a compact flat Riemannian manifold (Remark 3).

For a homology interpretation of parts (a) and (c) below, see Theorem 5.

Theorem 4 *Under the above hypotheses, the following conclusions hold.*

a. *$\mathcal{M}' \cap \mathcal{M}''$, or $\mathcal{T}' \cap \mathcal{T}''$, is a finite subset of $\mathcal{M}$, or $\mathcal{T}$, and stands in a bijective correspondence with the quotient group $\hat{\Pi}$ or, respectively, $\hat{L}$, of* (27)*,*
b. *The projection $\mathcal{T} \to \mathcal{M}$ in* (15) *maps $\mathcal{T}' \cap \mathcal{T}''$ injectively into $\mathcal{M}' \cap \mathcal{M}''$.*
c. *The cardinality $|\mathcal{M}' \cap \mathcal{M}''|$ of $\mathcal{M}' \cap \mathcal{M}''$ equals $|\mathcal{T}' \cap \mathcal{T}''|$ times $|\hat{H}|$.*
d. *The claim about $\mathcal{T}' \cap \mathcal{T}''$ in* (a) *remains true whether or not $\mathcal{E}', \mathcal{E}''$ are generic.*
e. *The two bijective correspondences in* (a) *may be chosen so that, under the resulting identifications, the injective mapping $\mathrm{pr} : \mathcal{T}' \cap \mathcal{T}'' \to \mathcal{M}' \cap \mathcal{M}''$ of* (b) *coincides with the group homomorphism $\hat{L} \to \hat{\Pi}$ induced by the inclusion $L \to \Pi$.*

Proof. We first prove (a) for $\mathcal{M}' \cap \mathcal{M}''$. The finiteness of $\mathcal{M}' \cap \mathcal{M}''$ follows as $\mathcal{M}', \mathcal{M}''$, and hence also $\mathcal{M}' \cap \mathcal{M}''$, are compact totally geodesic submanifolds of $\mathcal{M}$, while $\mathcal{M}' \cap \mathcal{M}''$, nonempty by (4), has $\dim(\mathcal{M}' \cap \mathcal{M}'') = 0$ due to (1). The mapping $\Psi : \Pi \to \mathcal{M}' \cap \mathcal{M}''$ with $\mathrm{pr}(\mathcal{E}' \cap \gamma(\mathcal{E}'')) = \{\Psi(\gamma)\}$ is well defined in view of (4) applied to $\gamma(\mathcal{E}'')$ rather than $\mathcal{E}''$, and clearly takes values in both $\mathcal{M}' = \mathrm{pr}(\mathcal{E}')$ and $\mathcal{M}'' = \mathrm{pr}(\mathcal{E}'') = \mathrm{pr}(\gamma(\mathcal{E}''))$. Surjectivity of Ψ follows: if $\mathrm{pr}(x'') \in \mathcal{M}' \cap \mathcal{M}''$, where $x'' \in \mathcal{E}''$ then, obviously, $\mathrm{pr}(x'') = \mathrm{pr}(x')$ and $x' = \gamma(x'')$ for some $x' \in \mathcal{E}'$ and $\gamma \in \Pi$, so that $x' \in \mathcal{E}' \cap \gamma(\mathcal{E}'')$ and $\mathrm{pr}(x'') = \mathrm{pr}(x')$ equals $\Psi(\gamma)$, the unique element of $\mathrm{pr}(\mathcal{E}' \cap \gamma(\mathcal{E}''))$. Furthermore, Ψ-preimages of elements of $\mathcal{M}' \cap \mathcal{M}''$ are precisely the cosets of the normal subgroup $\Pi'\Pi''$ of Π (which clearly implies

(a) for $\mathcal{M}' \cap \mathcal{M}''$). Namely, the left and right cosets coincide, and so elements γ_1, γ_2 of Π lie in the same coset of $\Pi'\Pi''$ if and only if

$$\gamma' \circ \gamma_1 = \gamma_2 \circ \gamma'' \text{ for some } \gamma' \in \Pi' \text{ and } \gamma'' \in \Pi''. \tag{28}$$

Now let γ_1, γ_2 lie in the same coset of $\Pi'\Pi''$. For γ', γ'' with (28), $\gamma'(\mathcal{E}') = \mathcal{E}'$ and $\gamma''(\mathcal{E}'') = \mathcal{E}''$ by the definition (12) of Σ', Σ'' and their identification with Π', Π'' (see above). Thus, $\{\Psi(\gamma_1)\} = \mathrm{pr}(\mathcal{E}' \cap \gamma_1(\mathcal{E}'')) = \mathrm{pr}(\gamma'(\mathcal{E}' \cap \gamma_1(\mathcal{E}''))) = \mathrm{pr}(\gamma'(\mathcal{E}') \cap \gamma'(\gamma_1(\mathcal{E}''))) = \mathrm{pr}(\mathcal{E}' \cap \gamma'(\gamma_1(\mathcal{E}''))) = \mathrm{pr}(\mathcal{E}' \cap \gamma_2(\gamma''(\mathcal{E}'')))$, and so $\{\Psi(\gamma_1)\} = \mathrm{pr}(\mathcal{E}' \cap \gamma_2(\mathcal{E}'')) = \{\Psi(\gamma_2)\}$. Conversely, if $\gamma_1, \gamma_2 \in \Pi$ and $\Psi(\gamma_1) = \Psi(\gamma_2)$, the unique points x_1 of $\mathcal{E}' \cap \gamma_1(\mathcal{E}'')$ and x_2 of $\mathcal{E}' \cap \gamma_2(\mathcal{E}'')$ both lie in the same Π-orbit, and hence $x_2 = \gamma(x_1)$ with some $\gamma \in \Pi$. For $\gamma' = \gamma$ and $\gamma'' = \gamma_2^{-1} \circ \gamma \circ \gamma_1$, the image $\gamma'(\mathcal{E}')$ (or, $\gamma''(\mathcal{E}'')$) intersects $\mathcal{E}'$ (or, $\mathcal{E}''$), the common point being $x_2 = \gamma(x_1)$ or, respectively, $\gamma_2^{-1}(x_2) = \gamma_2^{-1}(\gamma(x_1))$. From (13) we thus obtain $\gamma' \in \Sigma' = \Pi'$ and $\gamma'' \in \Sigma'' = \Pi''$, which yields (28).

Now (a) for $\mathcal{T}' \cap \mathcal{T}''$, and (d), follow as special cases; see the end of Sect. 11.

Except for the word 'injective' the claim made in (e) is immediate if one uses the mapping $\Psi : \Pi \to \mathcal{M}' \cap \mathcal{M}''$ defined above and its analog $L \to \mathcal{T}' \cap \mathcal{T}''$ obtained by replacing $\Pi, \mathcal{M}', \mathcal{M}''$ and pr with $L, \mathcal{T}', \mathcal{T}''$ and the projection $\mathcal{E} \to \mathcal{T} = \mathcal{E}/L$. This yields (b), injectivity of the homomorphism $\hat{L} \to \hat{\Pi}$ being immediate: if an element of L lies in $\Pi'\Pi''$ (and hence has the form $\gamma' \circ \gamma''$, where $(\gamma', \gamma'') \in \Pi' \times \Pi''$), (23) implies that γ', γ'' are translations with $\gamma' \in L' = L \cap \mathcal{V}'$ and $\gamma'' \in L'' = L \cap \mathcal{V}''$ (see the lines preceding (26)); in other words, $\gamma' \circ \gamma''$ represents zero in $\hat{L}$.

Finally, $\hat{L}$ identified as above with a subgroup of $\hat{\Pi}$ is the kernel of the clearly-surjective homomorphism $\hat{\Pi} \to \hat{H}$, induced by $\Pi \to H$ in (8) (which, due to (e), proves (c)). Namely, $\hat{L}$ contains the kernel (the other inclusion being obvious): if the linear part of $\gamma \in \Pi$ lies in $H'H''$, and so equals the linear part of $\gamma' \circ \gamma''$ for some $(\gamma', \gamma'') \in \Pi' \times \Pi''$, then $\gamma = \lambda \circ \gamma' \circ \gamma''$, where $\lambda \in L$. □

14 Leaves and integral homology

This section once again employs the assumptions and notation of Theorem 1, with $\dim \mathcal{V} = n$ and $\dim \mathcal{V}' = k$, where $0 < k < n$. As the holonomy group $H \subseteq \mathrm{Iso}\,\mathcal{V} \cong \mathrm{O}(n)$ is finite (Remark 3), $\det(H) \subseteq \{1, -1\}$. In other words, the elements of H have the determinants ± 1. Using the covering projection $\mathcal{T} \to \mathcal{M} = \mathcal{T}/H$, cf. (15) and the line following it, we see that

$$\text{the image of } H \text{ under } \det \text{ equals } \{1\} \text{ if and only if } \mathcal{M} \text{ is orientable.} \tag{29}$$

By Theorem 3(iii), the generic leaves of $F_{\mathcal{M}}$, defined as in the line following (21), are either all orientable or all nonorientable.

Theorem 5 *Let $\mathcal{M}$ be orientable. Then all the generic leaves $\mathcal{M}'$ of $F_{\mathcal{M}}$ may be oriented so as to represent the same nonzero k-dimensional real homology class*

$[\mathcal{M}'] \in H_k(\mathcal{M}, \mathbb{R})$*, while the first two cardinalities in Theorem* 4(c) *equal the intersection numbers of the real homology classes* $[\mathcal{M}'], [\mathcal{M}'']$*, or* $[\mathcal{T}'], [\mathcal{T}'']$.

Proof. A fixed orientation of $\mathcal{V}'$, being preserved, due to (22) – (23) and (29), by the generic stabilizer group Σ', gives rise to orientations of all the leaves $\mathcal{T}'$ of $F_{\mathcal{T}}$ and all the generic leaves $\mathcal{M}'$ of $F_{\mathcal{M}}$, so as to make the covering projections $\mathcal{T}' \to \mathcal{M}'$ in the line following (16) orientation-preserving. Since the torus group $\mathcal{V}/L$ acts transitively on the oriented leaves $\mathcal{T}'$, they all represent a single real homology class $[\mathcal{T}'] \in H_k(\mathcal{T}, \mathbb{R})$, equal to the image of the fundamental class of $\mathcal{T}'$ under the inclusion $\mathcal{T}' \to \mathcal{T}$. At the same time, for generic leaves $\mathcal{M}'$, the d-fold covering projection $\mathcal{T}' \to \mathcal{M}'$ (where $d = |H'|$ does not depend on the choice of $\mathcal{M}'$, cf. Lemma 15) sends the fundamental class of $\mathcal{T}'$ to d times the fundamental class of $\mathcal{M}'$. Thus, by functoriality, $d[\mathcal{M}'] \in H_k(\mathcal{M}, \mathbb{R})$ is the image of $[\mathcal{T}'] \in H_k(\mathcal{T}, \mathbb{R})$ under the covering projection $\mathcal{T} \to \mathcal{M}$, which makes it the same for all the generic leaves $\mathcal{M}'$. Finally, $[\mathcal{M}'] \neq 0$, since a fixed constant positive differential k-form on the oriented space $\mathcal{V}'$ descends, in view of the first line of this proof, to a parallel positive volume form on each oriented generic leaf $\mathcal{M}'$ which yields a positive value when integrated over $[\mathcal{M}']$. □

Note that the final clause in Theorem 5 is, not surprisingly, consistent with the fact that – by Theorem 4(a) and Lemma 15 – the intersection numbers depend just on the two mutually complementary H-invariant L-subspaces $\mathcal{V}', \mathcal{V}''$ of $\mathcal{V}$, and not on the individual generic leaves $\mathcal{M}', \mathcal{M}'', \mathcal{T}'$ or $\mathcal{T}''$.

15 Generalized Klein bottles

This section presents some known examples [4, p. 163] to illustrate our discussion.

The symbols $\mathbb{R}^H, \mathbb{Z}^H$ used below follow the set-theoretical notational convention: Y^X is *the set of all mappings* $X \to Y$, not the fixed-point set of some group action.

Let S^1 and $r_\theta : S^1 \to S^1$ denote the unit circle in $\mathbb{C}$ and the rotation by angle θ (multiplication by $e^{i\theta}$). For a fixed integer $n \geq 2$ and the group $H = \mathbb{Z}_n \subseteq S^1$ of nth roots of unity, $\mathbb{Z}^H \cong \mathbb{Z}^n$ is a lattice in the Euclidean space $\mathcal{V} = \mathbb{R}^H \cong \mathbb{R}^n$ with the ℓ^2 inner product, and $\mathbb{Z}^H_0 = \{\psi \in \mathbb{Z}^H : \psi_{\mathrm{avg}} = 0\}$ is a subgroup of $\mathbb{Z}^H$ isomorphic to $\mathbb{Z}^{n-1}$, where $(\ )_{\mathrm{avg}}$ denotes the averaging functional $\mathcal{V} \to \mathbb{R}$. Setting $\Pi = [(1/n)\mathbb{Z}] \times \mathbb{Z}^H_0$, one easily sees that the assignment

$$((t, \psi), f) \mapsto f \circ r_{2\pi t} + t + \psi, \quad \text{where } (t, \psi) \in \Pi \text{ and } f \in \mathcal{V} = \mathbb{R}^H, \tag{30}$$

defines an affine isometric action on $\mathcal{V}$ by Π treated as a group with the group operation $(t, \psi)(t', \psi') = (t + t', \psi' \circ r_{2\pi t} + \psi)$. The term t in (30) is the constant function $t : H \to \mathbb{R}$. Note that, in the right-hand side of (30), as $(t, \psi) \in \Pi$,

$$t_{\mathrm{avg}} = t, \qquad \psi_{\mathrm{avg}} = 0, \qquad (f \circ r_{2\pi t})_{\mathrm{avg}} = f_{\mathrm{avg}}. \tag{31}$$

Proposition 3 *These $H, \mathcal{V}$ and Π have the following properties.*

i. *The action of Π on $\mathcal{V}$ is effective and free.*
ii. *Π is a Bieberbach group in the underlying Euclidean affine n-space of $\mathcal{V}$.*
iii. *The holonomy group and lattice subgroup of Π are our $H \cong \mathbb{Z}_n$, acting on $\mathcal{V}$ linearly by $H \times \mathcal{V} \ni (e^{i\theta}, f) \mapsto f \circ r_\theta \in \mathcal{V}$, and $L = \mathbb{Z} \times \mathbb{Z}_0^H$.*
iv. *As a transformation of $\mathcal{V}$, each $(t, \psi) \in L$ equals the translation by $t + \psi$.*
v. *L consists of all translations by vectors $\psi' \in \mathbb{Z}^H$ such that $\psi'_{\mathrm{avg}} \in \mathbb{Z}$.*

An example of two mutually complementary H-invariant L-subspaces of $\mathcal{V}$, in the sense of (1) *and Definition* 1*, is provided by the line $\mathcal{V}'$ of constant functions $H \to \mathbb{R}$ and the hyperplane $\mathcal{V}''$ consisting of all $f : H \to \mathbb{R}$ with $f_{\mathrm{avg}} = 0$. The generic stabilizer groups $\Sigma', \Sigma'' \subseteq \Pi$ associated via* (21) *with $\mathcal{V}'$ and $\mathcal{V}''$ are the translation groups $\mathbb{Z} \times \{0\}$ and $\{0\} \times \mathbb{Z}_0^H$, both contained in L. Furthermore,*

a. *under the obvious identifications of $\mathcal{V}/\mathcal{V}'$ with $\mathcal{V}''$ and $\mathcal{V}/\mathcal{V}''$ with $\mathcal{V}'$, the quotient actions of Π become $\Pi \times \mathcal{V}'' \ni ((t, \psi), f) \mapsto f \circ r_{2\pi t} + \psi \in \mathcal{V}''$ and, respectively, $\Pi \times \mathcal{V}' \ni ((t, \psi), f) \mapsto f + t \in \mathcal{V}'$,*
b. *every coset of the L-subspace $\mathcal{V}''$ is generic, as defined in Sect.* 11*,*
c. *nongeneric cosets of $\mathcal{V}'$ are precisely those cosets containing $f : H \to \mathbb{R}$ such that $f \circ r_{2\pi t} - f$ is integer-valued for some $t \in [(1/n)\mathbb{Z}] \smallsetminus \mathbb{Z}$,*
d. *the obvious homomorphism $\Pi = [(1/n)\mathbb{Z}] \times \mathbb{Z}_0^H \to (1/n)\mathbb{Z}$ maps the stabilizer group of each coset of $\mathcal{V}'$ isomorphically onto a subgroup of $(1/n)\mathbb{Z}$,*
e. *the subgroups of $(1/n)\mathbb{Z}$ resulting from* (c) *have the form $(d/n)\mathbb{Z}$, where d is a positive divisor of n or, equivalently, are the preimages, under the homomorphism $\mathbb{R} \ni t \mapsto e^{2\pi i t} \in S^1$, of subgroups of the group $H = \mathbb{Z}_n \subseteq S^1$ formed by the nth roots of unity.*

Proof. First, Π acts on $\mathcal{V}$ freely: if $f \circ r_{2\pi t} + t + \psi = f$, cf. (30), with $f : H \to \mathbb{R}$, applying $(\ )_{\mathrm{avg}}$ to both sides, we get, by (31), $t = 0$, and hence $f \circ r_{2\pi t} = f$, so that the equality $f \circ r_{2\pi t} + t + \psi = f$ reads $\psi = 0$. Secondly, H and L *defined* by (iii) arise from Π as required in (7): the claim about H is obvious, and so are (iv) – (v), showing that $L \subseteq \Pi \cap \mathcal{V}$. Conversely, $\Pi \cap \mathcal{V} \subseteq L$. To verify this, suppose that $f \circ r_{2\pi t} + t + \psi = f + \psi'$ for all $f \in \mathcal{V} = \mathbb{R}^H$, some $(t, \psi) \in \Pi$, and some $\psi' \in \mathcal{V}$. Taking the linear parts of both sides, we see that $t \in \mathbb{Z}$ and $(t, \psi) \in L$, as required.

Our Π has a compact fundamental domain in $\mathcal{V}$, since so does the lattice $L \subseteq \Pi$. Also, Π is torsion-free: $\Pi \ni (t, \psi) \mapsto t \in \mathbb{R}$ being a group homomorphism, any finite-order element (t, ψ) of Π has $t = 0$, and so, by (30), it acts via translation by ψ, which gives $\psi = 0$. Next, to establish the discreteness of the subset Π of $\mathrm{Iso}\,\mathcal{V}$ (and, consequently, (ii)), suppose that a sequence $(t_k, \psi_k) \in \Pi$ with pairwise distinct terms yields, via (30), a sequence convergent in $\mathrm{Iso}\,\mathcal{V}$. Evaluating (30) on $f = 0$, we get $(t_k, \psi_k) \to (t, \psi)$ in $\mathbb{R} \times \mathbb{R}^H$ as $k \to \infty$, for some (t, ψ) and, since $(t_k, \psi_k) \in [(1/n)\mathbb{Z}] \times \mathbb{Z}_0^H$, the sequence (t_k, ψ_k) becomes eventually constant, contrary to the fact that its terms are pairwise distinct.

As for $\mathcal{V}'$ and $\mathcal{V}''$, note that, by (iv), a $\mathbb{Z}$-basis of $L \cap \mathcal{V}'$ (or, $L \cap \mathcal{V}''$) may be defined to consist just of the constant function 1 (or, respectively, of the $n-1$ functions $\psi_q : H \to \mathbb{Z}$, labeled by $q \in H \smallsetminus \{1\}$, where $\psi_q(q) = 1 = -\psi_q(1)$ and

$\psi_q = 0$ on $H \smallsetminus \{1, q\}$). Specifically, $\psi' = \sum_q \psi'(q)\psi_q$ whenever $\psi' \in \mathbb{Z}^H$ and $\psi'_{\mathrm{avg}} = 0$. Our descriptions of Σ' and Σ'' are in turn immediate from (a), which itself is a trivial consequence of (30) – (31), and easily implies (b) – (c). Now (d) follows as the relation $f \circ r_{2\pi t} + \psi = f$, with fixed $f \in \mathcal{V}''$, uniquely determines ψ, once t is given. Finally, we have (e) since the first assignment in (a) only depends on t through $e^{2\pi i t}$, which completes the proof. □

The compact flat Riemannian manifold $\mathcal{V}/\Pi$ arising here from our Bieberbach group Π as in Sect. 6 is called the n-dimensional *generalized Klein bottle* [4, p. 163]. The linear functional $\mathcal{V} \ni f \mapsto f_{\mathrm{avg}} \in \mathbb{R}$ is equivariant, due to (31), with respect to the actions of Π and $\mathbb{Z}$ (the latter, on $\mathbb{R}$, via translations by multiples of $1/n$), relative to the homomorphism $\Pi \ni (t, \psi) \mapsto t \in (1/n)\mathbb{Z}$. Thus, it descends, in view of Remark 1(c), to a bundle projection $\mathcal{V}/\Pi \to \mathbb{R}/[(1/n)\mathbb{Z}]$, making $\mathcal{V}/\Pi$ a bundle of tori over the circle. The fibres of this bundle are, obviously, the images, under the projection $\mathrm{pr} : \mathcal{V} \to \mathcal{V}/\Pi$, of cosets of the L-subspace $\mathcal{V}'' \subseteq \mathcal{V}$ mentioned in Proposition 3(b), all of them generic. On the other hand, $\mathcal{V}'$ has some nongeneric cosets – by Proposition 3(c), an example is $\mathcal{V}'$ itself, with the stabilizer group easily seen to be $[(1/n)\mathbb{Z}] \times \{0\}$. The pr-images of the cosets of $\mathcal{V}'$ are embedded circles, forming the leaves of the foliation $F_{\mathcal{M}}$ of $\mathcal{M} = \mathcal{V}/\Pi$ arising as in Theorem 1.

For the foliation $F_{\mathcal{M}}$ of $\mathcal{M} = \mathcal{V}/\Pi$ obtained in the general case of Theorem 1, the stabilizer group of a leaf $\mathcal{M}'$ of $F_{\mathcal{M}}$ is only defined as a conjugacy class of subgroups of Π (the subgroups being the stabilizer groups Σ' of leaves $\mathcal{E}'$ of $F_{\mathcal{E}}$ with $\mathrm{pr}(\mathcal{E}') = \mathcal{M}'$). Genericity of $\mathcal{M}'$, mentioned in the line following (21), amounts to genericity of all such $\mathcal{E}'$, and Σ' is then uniquely associated with $\mathcal{M}'$ (due to its being the generic stabilizer group, normal in Π). In the subsequent discussion Σ' is also treated as uniquely defined, for a different reason: each Σ' is replaced by its image under a homomorphism from Π into the Abelian group $(1/n)\mathbb{Z}$.

We denote by $|$ the divisibility relation in the set Δ_n of all positive divisors of n and by $\mathcal{M}$ the n-dimensional generalized Klein bottle $\mathcal{V}/\Pi$. Let us also use Proposition 3(d)-(e) and the final sentence of the last paragraph to treat

$$\begin{array}{l}\text{the stabilizer groups of leaves of } F_{\mathcal{M}} \text{ as subgroups}\\ \text{of } (1/n)\mathbb{Z} \text{ having the form } (d/n)\mathbb{Z}, \text{ where } d \in \Delta_n.\end{array} \tag{32}$$

Proposition 4 *There exists a family* $\{\mathcal{M}[d] : d \in \Delta_n\}$ *of compact connected immersed submanifolds of* $\mathcal{M}$ *with the following properties, for all* $d, d' \in \Delta_n$.

i. *Each* $\mathcal{M}[d]$ *has the dimension* d *and is foliated by circle leaves of* $F_{\mathcal{M}}$.
ii. $\mathcal{M}[d'] \subseteq \mathcal{M}[d]$ *whenever* $d'|d$.
iii. $\mathcal{M}[d] \smallsetminus \bigcup_k \mathcal{M}[k]$, *with* k *ranging over* $\Delta_d \smallsetminus \{d\}$, *equals the union of all leaves of* $F_{\mathcal{M}}$ *having the stabilizer group* $(d/n)\mathbb{Z}$.
iv. $\mathcal{M}[n] = \mathcal{M}$ *and* $\mathcal{M}[1] = \mathrm{pr}(\mathcal{V}')$.
v. *If* n *is prime,* $\mathcal{M}[1] = \mathrm{pr}(\mathcal{V}')$ *is the only nongeneric leaf of* $F_{\mathcal{M}}$.

Proof. The zero-average functions $h : H \to \mathbb{R}/\mathbb{Z}$, from the group $H = \mathbb{Z}_n \subseteq S^1$ of nth roots of unity into the circle $\mathbb{R}/\mathbb{Z}$, form a manifold $[\mathbb{R}/\mathbb{Z}]^H$ diffeomorphic to the torus T^{n-1}, and the covering projection $\mathcal{V}'' \to [\mathbb{R}/\mathbb{Z}]^H$, sending a zero-average function $f : H \to \mathbb{R}$ to its composition h with the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$, is obviously equivariant relative to the first action in Proposition 3(a),

$$\text{the action of } H \text{ on } [\mathbb{R}/\mathbb{Z}]^H \text{ given by } (q,h) \mapsto h \circ r_{2\pi t}, \tag{33}$$

for $t \in (1/n)\mathbb{Z}$ with $q = e^{2\pi i t}$, and the homomorphism $\Pi \ni (t,\psi) \mapsto e^{2\pi i t} \in H$. Thus, (33) is the translational action of $H = \mathbb{Z}_n$ on mappings $H \to \mathbb{R}/\mathbb{Z}$, and the preimages of its isotropy groups under the homomorphism $(1/n)\mathbb{Z} \ni t \mapsto e^{2\pi i t} \in H$ are precisely the stabilizer groups in (32). We now define $\mathcal{M}[d]$, for any $d \in \Delta_n$, to be the union of circle leaves of $F_{\mathcal{M}}$ having stabilizer groups contained in $(d/n)\mathbb{Z}$. Thus, $\mathcal{M}[d]$ is the image under the quotient projection $\mathrm{pr} : \mathcal{V} \to \mathcal{V}/\Pi$ of the union of all cosets $v + \mathcal{V}'$ for vectors $v \in \mathcal{V}''$ which the covering projection $\mathcal{V}'' \to [\mathbb{R}/\mathbb{Z}]^H$ sends to (zero-average) functions $h : H \to \mathbb{R}/\mathbb{Z}$ having $h \circ r_{2\pi d/n} = h$. Since such h form a manifold diffeomorphic to the torus T^{d-1}, via the assignment to h of the zero-sum d-tuple consisting of $h(e^{2\pi i k/n}) \in \mathbb{R}/\mathbb{Z}$, with $k = 0, 1, \ldots, d-1$, all our claims about $\mathcal{M}[d]$ easily follow from the fact that both projections just mentioned are locally diffeomorphic. ◻

The n-dimensional generalized Klein bottle, for any $n \geq 2$, is an example illustrating the fact that the last inclusion of Theorem 1(ii-c) may be proper. In fact, the stabilizer group $[(1/n)\mathbb{Z}] \times \{0\}$ of $\mathcal{V}'$, mentioned six lines before Proposition 4, although not contained in the lattice L, acts on $\mathcal{V}'$ by translations. Also, unless n is prime, Proposition 4(iii) shows that the dependence on u in the last line of Lemma 13 is actually possible, as it occurs for $\mathcal{M}' = \mathcal{M}[k]$, with any $k \in \Delta_n \smallsetminus \{n\}$.

16 Remarks on holonomy

The correspondence between Bieberbach groups and compact flat manifolds mentioned in Remark 4 has an extension to almost-Bieberbach groups and infra-nilmanifolds [6] obtained by using – instead of the translation vector space of a Euclidean affine space – a connected, simply connected nilpotent Lie group $\mathcal{G}$ acting simply transitively on a manifold $\mathcal{E}$, and replacing the Bieberbach group with a torsion-free uniform discrete subgroup Π of $\mathrm{Diff}\,\mathcal{E}$ contained in a semidirect product (canonically transplanted so as to act on $\mathcal{E}$) of $\mathcal{G}$ and a maximal compact subgroup of $\mathrm{Aut}\,\mathcal{G}$. Here 'uniform' means admitting a compact fundamental domain, cf. Remark 2. The analogs of (8) and (15) remain valid, reflecting the fact that any infra-nilmanifold is the quotient of a nilmanifold under the action of a finite group H.

A somewhat similar picture may arise in some cases where $\mathcal{G}$ is not assumed nilpotent. As an example, let $\mathcal{G} \cong \mathrm{Spin}(m,1)$ be the universal covering group of the identity component $\mathcal{G}/\mathbb{Z}_2 \cong \mathrm{SO}^+(m,1)$ of the pseudo-orthogonal group of an $(m+1)$-dimensional Lorentzian vector space $\mathcal{L}$, $m \geq 3$. Here $\mathcal{E}$ is the (two-fold) universal covering manifold of the orthonormal-frame bundle of the future unit pseudosphere $\mathcal{S} \subseteq \mathcal{L}$, isometric to the hyperbolic m-space, and $\mathcal{G}/\mathbb{Z}_2$ acts on $\mathcal{S}$ via hyperbolic isometries, leading to an action of $\mathcal{G}$ on $\mathcal{E}$. The orthonormal-frame bundles of compact hyperbolic manifolds obtained as quotients of $\mathcal{S}$ give rise to the required torsion-free uniform discrete subgroups Π.

The resulting compact quotient manifolds $\mathcal{M} = \mathcal{E}/\Pi$ can be endowed with various interesting Riemannian metrics coming from Π-invariant metrics on $\mathcal{E}$. For Π and $\mathcal{E}$ of the preceding paragraph, a particularly natural choice of an invariant *indefinite* metric is provided by the Killing form of $\mathcal{G}$, turning $\mathcal{M}$ into a compact locally symmetric pseudo-Riemannian Einstein manifold.

Outside of the Bieberbach-group case, however, these metrics are not flat, and finite groups H such as mentioned above cannot serve as their holonomy groups. The holonomy interpretation of H still makes sense, though, if – instead of metrics – one uses either of the two Π-invariant flat connections, with (parallel) torsion, naturally distinguished on $\mathcal{E}$. Here $\mathcal{E}$ is, again, a manifold on which a connected, simply connected Lie group $\mathcal{G}$ acts simply transitively. Two natural bi-invariant connections with the stated properties exist on $\mathcal{G}$, rather than $\mathcal{E}$, and are characterized by the requirement that they make all the left-invariant (or, right-invariant) vector fields parallel. Due to their naturality, these two connections on $\mathcal{G}$ are also invariant under all the Lie-group automorphisms of $\mathcal{G}$. It is the two connections on $\mathcal{G}$ that induce, in an obvious way, the ones on $\mathcal{E}$, mentioned six lines earlier.

Appendix: Hiss and Szczepański's reducibility theorem

Consider an *abstract Bieberbach group*, that is, any torsion-free group Π containing a finitely generated free Abelian normal subgroup L of finite index, which is at the same time a maximal Abelian subgroup of Π. As shown by Zassenhaus [14], up to isomorphisms these groups coincide with the Bieberbach groups of Sect. 6, we again summarize their structure using the short exact sequence

$$L \to \Pi \to H, \quad \text{where } H = \Pi/L. \tag{34}$$

For Abelian groups G_1, G_2 and G' one has canonical isomorphisms

$$\mathbb{Z}\otimes G \cong G, \quad (G_1\oplus G_2)\otimes G' \cong (G_1\otimes G')\oplus(G_2\otimes G'), \quad L\otimes \mathbb{Q} \cong \mathrm{Hom}(L^*, \mathbb{Q}), \tag{35}$$

where $L^* = \mathrm{Hom}(L, \mathbb{Z})$ and, for simplicity, L is assumed to be finitely generated and free. In the last case, with a suitable integer $n \geq 0$, there are further noncanonical isomorphisms

$$\text{a)}\ \ L \cong \mathbb{Z}^n, \qquad \text{b)}\ \ L \otimes \mathbb{Q} \cong \mathbb{Q}^n, \tag{36}$$

while, using the injective homomorphism $L \ni \lambda \mapsto \lambda \otimes 1 \in L \otimes \mathbb{Q}$ to treat L as a subgroup of $L \otimes \mathbb{Q}$, we see that, under suitably chosen identifications (36),

$$\text{the inclusion } L \subseteq L \otimes \mathbb{Q} \text{ corresponds to the standard inclusion } \mathbb{Z}^n \subseteq \mathbb{Q}^n. \tag{37}$$

Finally, if L as above is a (full) lattice in an finite-dimensional real vector space $\mathcal{V}$, a further canonical isomorphic identification arises:

$$L \otimes \mathbb{Q} \cong \mathrm{Span}_{\mathbb{Q}} L, \tag{38}$$

that is, we may view $L \otimes \mathbb{Q}$ as the rational vector subspace of $\mathcal{V}$ spanned by L.

Let Π now be an abstract Bieberbach group. Hiss and Szczepański [10, the corollary in Sect. 1] proved that, if L in (34) satisfies (36.a) with $n \geq 2$, then *the* (obviously $\mathbb{Q}$-linear) *action of* H *on* $L \otimes \mathbb{Q}$ *must be reducible*, in the sense of admitting a nonzero proper invariant rational vector subspace $\mathcal{W}$.

Next, using (37), we may write $L = \mathbb{Z}^n \subseteq \mathbb{Q}^n = L \otimes \mathbb{Q}$, so that $\mathcal{W} \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n$, and the closure $\mathcal{V}'$ of $\mathcal{W}$ in $\mathbb{R}^n$ has the real dimension $\dim_{\mathbb{Q}} \mathcal{W}$ (any $\mathbb{Q}$-basis of $\mathcal{W}$ being, obviously, an $\mathbb{R}$-basis of $\mathcal{V}'$). By clearing denominators, one can replace such a $\mathbb{Q}$-basis with one consisting of vectors in $L = \mathbb{Z}^n$, and so, by Lemma 10(a), the intersection $L' = L \cap \mathcal{W} = L \cap \mathcal{V}'$ is a lattice in $\mathcal{V}'$. We thus obtain (11).

A stronger version of Hiss and Szczepański's reducibility theorem, established more recently by Lutowski [12], states that the rational holonomy representations of any compact flat manifold other than a torus has at least two nonequivalent irreducible subrepresentations.

References

1. D. M. Arnold, *Finite Rank Torsion-Free Abelian Groups and Rings*, Lecture Notes in Mathematics **931**, Springer-Verlag, New York, 1982.
2. R. G. Bettiol, A. Derdzinski, R. Mossa, P. Piccione, *On the collapse of compact flat manifolds*, Math. Nachrichten **295** (2022), 2338–2356.
3. N. Bourbaki, *Topologie générale*, 3ème éd., Chap. V – VIII, Actualités Sci. Ind., No. 1235, Hermann, Paris, 1963.
4. L. S. Charlap, *Bieberbach Groups and Flat Manifolds*, Universitext, Springer-Verlag, New York, 1986.
5. M. W. Davis, *Lectures on orbifolds and reflection groups*, in: Advanced Lectures in Mathematics **16**: Transformation Groups and Moduli Spaces of Curves (L. Ji and S.-T. Yau, eds.), International Press, Somerville, MA, USA/Higher Education Press, Beijing, China, 2011, 63–93.
6. K. Dekimpe, *Almost-Bieberbach Groups: Affine and Polynomial Structures*, Lecture Notes in Mathematics **1639**. Springer-Verlag, Berlin, 1996.
7. A. Derdzinski, P. Piccione, *Flat manifolds and reducibility*, arXiv:1903.10479v2.
8. B. I. Dundas, *A Short Course in Differential Topology*, Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 2018.
9. R. E. Edwards, *Fourier Series: A Modern Introduction, Volume 1*, 2nd ed., Springer-Verlag, New York, 1979.
10. G. Hiss, A. Szczepański, *On torsion free crystallographic groups.* J. Pure Appl. Algebra **74** (1991), 39–56.
11. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry, Vol. I.* Interscience, New York, 1963.
12. R. Lutowski, *Flat manifolds with homogeneous holonomy representation*, Publ. Math. Debrecen, **99** (2021), no. 1–2, art. 6.
13. A. Szczepański, *Geometry of Crystallographic Groups*. Algebra and Discrete Mathematics, **4**. World Scientific, Hackensack, NJ, 2012.

14. H. Zassenhaus, *Beweis eines Satzes über diskrete Gruppen.* Abh. Math. Sem. Univ. Hamburg **12** (1938), 289–312.

Acknowledgements Both authors' research was supported in part by a FAPESP-OSU 2015 Regular Research Award (FAPESP grant: 2015/50265-6). The authors wish to thank Andrzej Szczepański for helpful comments. We also greatly appreciate suggestions made by anonymous referees of earlier versions of the manuscript, which allowed us to make the exposition much easier to follow.

Competing Interests The authors have no conflicts of interest to declare that are relevant to the content of this chapter.