

Null-projectability of Levi-Civita connections

Andrzej Derdzinski and Kirollos Masood

ABSTRACT. We study the natural property of projectability of a torsion-free connection along a foliation on the underlying manifold, which leads to a projected torsion-free connection on a local leaf space, focusing on projectability of Levi-Civita connections of pseudo-Riemannian metrics along foliations tangent to null parallel distributions. For the neutral metric signature and mid-dimensional distributions, Afifi showed in 1954 that projectability of the Levi-Civita connection characterizes, locally, the case of Patterson and Walker's Riemann extension metrics. We extend this correspondence to null parallel distributions of any dimension, introducing a suitable generalization of Riemann extensions.

Introduction

Projectable connections, within the more general context of transverse geometry for foliations, have been studied by many authors; see, for instance, [2, 3, 12, 17, 19]. The present paper focuses on certain special connections and foliations arising in pseudo-Riemannian geometry.

Given an integrable distribution \mathcal{V} and a torsion-free connection ∇ on a manifold, \mathcal{V} -projectability of ∇ onto a connection D on a local leaf space means that, whenever local vector fields v, w are \mathcal{V} -projectable, so must be $\nabla_v w$. This is equivalent (Section 2) to requiring the horizontal distribution of ∇ to project onto that of D . In the case of a null parallel distribution \mathcal{V} on a pseudo-Riemannian manifold with the Levi-Civita connection ∇ , Walker's theorem [18] easily leads to a local-coordinate characterization (Lemma 3.5) of \mathcal{V} -projectability of ∇ .

A less trivial aspect of the latter situation involves the resulting geometric invariants, not manifestly present in the local-coordinate description. For the neutral metric signature and mid-dimensional distributions, the underlying manifold M forms, locally, a cotangent affine bundle (Section 4) over the leaf space Σ (endowed with a torsion-free connection D) and a result due to Afifi [1, p. 313], which we reproduce as Theorem 5.2, states that the metric g is a Patterson-Walker Riemann extension for D .

In Section 8 we extend Affi's theorem to arbitrary indefinite metric signatures and null parallel distributions \mathcal{V} of all dimensions. Rather than being just a cotangent affine bundle, M is now, locally, a bundle of cotangent-principal bundles (Section 4) over Σ , meaning that there is a bundle Q over Σ , and M itself constitutes an affine bundle over the total space of Q with the pullback of $T^*\Sigma$ to Q serving as its associated vector bundle. The relevant geometric invariants include, in addition to the torsion-free connection D on Σ , also a vertical metric h on Q (in other words, a pseudo-Riemannian fibre metric in the vertical distribution of the bundle projection $Q \rightarrow \Sigma$, which makes Q a bundle of pseudo-Riemannian manifolds over Σ). The original metric g now constitutes what we call a *Riemann pullback-extension* for D, Q and h .

The mid-dimensional situation is a special case of this picture, with $Q = \Sigma$, so that the bundle Q has single-point fibres and $h = 0$.

1. Preliminaries

We always assume C^∞ -differentiability of manifolds, mappings (including bundle projections), subbundles (such as distributions), and tensor fields. Manifolds are by definition connected. Maximal connected integral manifolds of foliations (integrable distributions) are referred to as their *leaves*.

Our sign convention for the curvature tensor R of a connection ∇ on a manifold M , and any tangent vector fields v, u, u' , is

$$(1.1) \quad R(v, u)u' = \nabla_u \nabla_v u' - \nabla_v \nabla_u u' + \nabla_{[v, u]} u',$$

the coordinate form of which reads $R_{ijk}^l = \partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l + \Gamma_{jp}^l \Gamma_{ik}^p - \Gamma_{ip}^l \Gamma_{jk}^p$.

Given a mapping $\pi : M \rightarrow \Sigma$ between manifolds, a vector field w (or, a distribution \mathcal{V}) on M is said to be π -projectable if $d\pi_x w_x = u_{\pi(x)}$, or

$$(1.2) \quad d\pi_x(\mathcal{V}_x) = \mathcal{W}_{\pi(x)} \quad \text{for all } x \in M$$

and some vector field u (or, respectively, some distribution \mathcal{W}) on Σ .

Let $\pi : M \rightarrow \Sigma$ be a bundle projection with the vertical distribution $\mathcal{V} = \text{Ker } d\pi$, and let w be a vector field on M . Then

$$(1.3) \quad \begin{aligned} w \text{ is } \pi\text{-projectable if and only if, for every section } v \\ \text{of } \mathcal{V} \text{ the Lie bracket } [v, w] \text{ is also a section of } \mathcal{V}, \text{ or,} \\ \text{equivalently, the local flow of } w \text{ leaves } \mathcal{V} \text{ invariant.} \end{aligned}$$

(This is obvious if one uses local coordinates for M making π appear as a Cartesian-product projection.) Given an integrable distribution \mathcal{V} on a manifold M , every point of M has a neighborhood U such that, for some manifold Σ ,

$$(1.4) \quad \begin{aligned} \text{the leaves of } \mathcal{V} \text{ restricted to } U \text{ are the} \\ \text{fibres of a bundle projection } \pi : U \rightarrow \Sigma. \end{aligned}$$

2. Projectable connections

Suppose that \mathcal{V} is an integrable distribution on a manifold M . By \mathcal{V} -projectability of a vector field w on an open set $U' \subseteq M$ we mean its π -projectability for any π, U, Σ with (1.4) such that $U \subseteq U'$.

Let \mathcal{V} be an integrable distribution on a manifold M equipped with a torsion-free connection ∇ . We say that ∇ is \mathcal{V} -projectable (or *projectable along \mathcal{V}*) if, for any \mathcal{V} -projectable vector fields v, w on an open subset of M , the covariant derivative $\nabla_v w$ is \mathcal{V} -projectable as well.

For an integrable distribution \mathcal{V} on a manifold M and the restriction $T_L M$ of the tangent bundle TM to any given leaf L of \mathcal{V} , the normal bundle of L in M is given by $NL = T_L M / TL$, with the quotient-bundle projection $T_L M \rightarrow NL$.

LEMMA 2.1. *Let a torsion-free connection ∇ on a manifold M be projectable along an integrable distribution \mathcal{V} . Then, with L denoting any given leaf of \mathcal{V} ,*

- (i) *both $\nabla_w v$, $\nabla_v w$ are local sections of \mathcal{V} , for any \mathcal{V} -projectable local vector field w in M and any local section v of \mathcal{V} with the same domain,*
- (ii) *\mathcal{V} is ∇ -parallel, and so, in particular, ∇ induces a connection in the normal bundle $NL = T_L M / TL$ of L ,*
- (iii) *the image under the quotient-bundle projection $T_L M \rightarrow NL$ of any \mathcal{V} -projectable vector field w on an open set $U \subseteq M$ intersecting L is a local section of NL , parallel relative to the connection induced by ∇ .*

PROOF. Given w, v as in (i), $\nabla_v w$ is \mathcal{V} -projectable (as v is). This remains the case after v has been multiplied by any function. Thus, $\nabla_v w$ projects onto 0 (or else, multiplied by a non-projectable function, it would cease to be \mathcal{V} -projectable); in other words, $\nabla_v w$ is a section of \mathcal{V} . Due to (1.3), $\nabla_w v$ differs from $\nabla_v w$ by the section $[w, v]$ of \mathcal{V} , and (i) follows, also proving (ii) – (iii). \square

LEMMA 2.2. *Whenever a torsion-free connection ∇ on a manifold M is projectable along an integrable distribution \mathcal{V} , it gives rise to a torsion-free connection D on each local leaf space Σ with (1.4), characterized by $D_v w = \nabla_v w$ for \mathcal{V} -projectable vector fields v, w on the open set $U \subseteq M$ appearing in (1.4), where the same symbols v, w denote their projections onto Σ .*

In fact, D is well defined: replacing v, w by other vector fields on U having the same projections onto Σ , that is, adding to them local sections of \mathcal{V} , results – by Lemma 2.1(i) – only in adding to $\nabla_v w$ a local section of \mathcal{V} , without changing the projection of $\nabla_v w$ onto Σ .

LEMMA 2.3. *In local coordinates x^1, \dots, x^n , let the s -dimensional distribution \mathcal{V} be spanned by the last s coordinate vector fields. Then a torsion-free connection ∇ is \mathcal{V} -projectable if and only if its component functions satisfy the relations $\Gamma_{a_j}^i = \Gamma_{ab}^i = \partial_a \Gamma_{jk}^i = 0$ for all $i, j, k \in \{1, \dots, n-s\}$ and $a, b \in \{n-s+1, \dots, n\}$.*

In the case of \mathcal{V} -projectability of ∇ , the projected torsion-free connection D on a local leaf space Σ with the coordinates x^i for $i = 1, \dots, n-s$, cf. Lemma 2.2, has the component functions Γ_{jk}^i .

PROOF. First, let ∇ be \mathcal{V} -projectable. By Lemma 2.1(ii), \mathcal{V} is ∇ -parallel, so that $\Gamma_{aj}^i = \Gamma_{ab}^i = 0$, while \mathcal{V} -projectability of the coordinate vector fields ∂_i implies the same for $\nabla_{\partial_j} \partial_k = \Gamma_{jk}^i \partial_i + \Gamma_{jk}^a \partial_a$, that is, for $\Gamma_{jk}^i \partial_i$ (the last term being a local section of \mathcal{V} , and hence \mathcal{V} -projectable). The functions Γ_{jk}^i are thus constant along \mathcal{V} , and so $\partial_a \Gamma_{jk}^i = 0$.

Assume now that $\Gamma_{aj}^i = \Gamma_{ab}^i = \partial_a \Gamma_{jk}^i = 0$ with index ranges as above. Due to \mathcal{V} -projectability of the coordinate vector fields, any \mathcal{V} -projectable local vector field has the form $w = w^i \partial_i + w^a \partial_a$ with $\partial_a w^i = 0$, so that, for two such vector fields w, u , since \mathcal{V} is clearly ∇ -parallel, $\nabla_u w$ equals a local section of \mathcal{V} plus $\psi^i \partial_i$, with $\psi^i = w^j \partial_j w^i + w^j w^k \Gamma_{jk}^i$. Thus, $\partial_a \psi^i = 0$, and $\nabla_u w$ is \mathcal{V} -projectable. \square

As noted in the above proof, in local coordinates chosen for an integrable distribution \mathcal{V} as in Lemma 2.3, the requirement that \mathcal{V} be ∇ -parallel amounts to

$$(2.1) \quad \Gamma_{aj}^i = \Gamma_{ab}^i = 0.$$

Projectability of a torsion-free connection ∇ on a manifold M along a ∇ -parallel distribution \mathcal{V} is equivalent to the following condition imposed on the curvature tensor R of ∇ , at every point $x \in M$:

$$(2.2) \quad R_x(v, u)u' \in \mathcal{V}_x \text{ whenever } u, u' \in T_x M \text{ and } v \in \mathcal{V}_x.$$

In fact, we may choose local coordinates for \mathcal{V} as in Lemma 2.3. Condition (2.2) reads $R_{a\bullet\bullet}{}^i = 0$ with \bullet standing for any index, that is, $R_{ajk}{}^i = R_{ajb}{}^i = R_{abj}{}^i = R_{abc}{}^i = 0$. If ∇ is \mathcal{V} -projectable, Lemma 2.3 gives (2.1) and $\partial_a \Gamma_{jk}^i = 0$, which yields $R_{a\bullet\bullet}{}^i = 0$ due to the coordinate form of (1.1). Conversely, if (2.1) holds and $R_{i\bullet\bullet}{}^a = 0$, the coordinate form of (1.1) yields $\partial_a \Gamma_{jk}^i = -R_{ajk}{}^i = 0$.

Given a bundle projection $\pi : M \rightarrow \Sigma$ with the vertical distribution $\mathcal{V} = \text{Ker } d\pi$ and torsion-free connections ∇ on M and D on Σ , the following three conditions are mutually equivalent:

- (a) ∇ is \mathcal{V} -projectable onto Σ with the projected connection D ,
- (b) the horizontal distribution of ∇ is $d\pi$ -projectable onto that of the π -pull-back of D , the vector-bundle morphism $d\pi : TM \rightarrow \pi^*T\Sigma$ being treated as a mapping between the total spaces,
- (c) whenever $t \mapsto w(t) \in T_{x(t)}M$ is a ∇ -parallel vector field along a curve $t \mapsto x(t) \in M$, its $d\pi$ -image $t \mapsto d\pi_{x(t)}w(t) \in T_{y(t)}\Sigma$ is D -parallel along the image curve $t \mapsto y(t) = \pi(x(t))$.

In fact, (b) and (c) imply each other: the left-to-right inclusion in (1.2) for $d\pi$ rather than π trivially follows from (c). For the opposite inclusion, in suitable local coordinates, $\pi : M \rightarrow \Sigma$ (and, consequently, $d\pi : TM \rightarrow \pi^*T\Sigma$) appears

as a linear projection $(y, \xi) \mapsto y$ (or, respectively, $(y, \xi, \dot{y}, \dot{\xi}) \mapsto (y, \xi, \dot{y})$), which realizes any D -horizontal vector as the image of a vector tangent to TM , while a TM -vertical correction allows us to replace the latter with a ∇ -horizontal one.

To establish equivalence of (a) and (c), use local coordinates x^1, \dots, x^n in M in which \mathcal{V} is spanned by the last s coordinate vector fields, and so, with the index ranges $i, j, k \in \{1, \dots, n-s\}$ and $a, b \in \{n-s+1, \dots, n\}$, we may treat x^i as coordinates in Σ , denoting by $\bar{\Gamma}_{jk}^i$ the component functions of D and reserving the usual Γ notation for those of ∇ . Condition (c) now amounts to requiring that $\dot{w}^i + \bar{\Gamma}_{jk}^i \dot{x}^j w^k = 0$ whenever $\dot{w}^i + \Gamma_{jk}^i \dot{x}^j w^k + \Gamma_{ja}^i \dot{x}^j w^a + \Gamma_{ak}^i \dot{x}^a w^k + \Gamma_{ab}^i \dot{x}^a w^b = 0$. Choosing appropriate initial data we see that this amounts to $\Gamma_{aj}^i = \Gamma_{ab}^i = 0$ and $\Gamma_{jk}^i = \bar{\Gamma}_{jk}^i$. As the last relation yields $\partial_a \Gamma_{jk}^i = 0$, our claim follows from Lemma 2.3.

3. Null-projectability of Levi-Civita connections

For the Levi-Civita connection ∇ of a pseudo-Riemannian metric on a manifold M and an integrable distribution \mathcal{P} on M ,

(3.1) \mathcal{P} -projectability of ∇ is equivalent to its \mathcal{P}^\perp -projectability.

(Note that projectability implies, by Lemma 2.1(ii), that the distribution in question is parallel; thus, \mathcal{P}^\perp must be integrable here if \mathcal{P} is, and vice versa.) In fact, in terms of the $(0, 4)$ curvature tensor, also denoted by R , (2.2) reads $R(\mathcal{P}, \cdot, \cdot, \mathcal{P}^\perp) = \{0\}$ which, due to symmetries of R , amounts to $R(\mathcal{P}^\perp, \cdot, \cdot, \mathcal{P}) = \{0\}$, that is, to (2.2) with \mathcal{P} replaced by \mathcal{P}^\perp .

This situation is of rather little interest when \mathcal{P} is nondegenerate (meaning nondegeneracy of the metric restricted to \mathcal{P}) since, the local version of the de Rham decomposition theorem, originally due to Thomas [16], then implies that \mathcal{P} is, locally, a factor distribution in a product decomposition of the metric, and so ∇ projects via (1.2) onto the Levi-Civita connection of the other factor metric. The extreme opposite case, in which \mathcal{P} is null, leads to a much more diverse family of examples, such as those listed below.

EXAMPLE 3.1. Any nonzero null parallel vector field w on a pseudo-Riemannian manifold has $R(w, \cdot, \cdot, \cdot) = 0$, which implies (2.2) for the distribution \mathcal{P} spanned by w , and hence \mathcal{P} -projectability of the Levi-Civita connection.

EXAMPLE 3.2. *ECS manifolds* [7] are pseudo-Riemannian manifolds of dimensions $n \geq 4$ which have parallel Weyl tensor without being conformally flat or locally symmetric. They exist for every $n \geq 4$ as shown by Roter [15, Corollary 3], their metrics are all indefinite [6, Theorem 2], and compact examples are known in all dimensions $n \geq 5$ [10, 11]. Every ECS manifold carries a distinguished null parallel distribution \mathcal{P} of dimension $d \in \{1, 2\}$, discovered by Olszak [13], and its Levi-Civita connection is \mathcal{P} -projectable: for $d = 2$, [8, Lemma 17.3(ii)] yields (2.2) while, if $d = 1$, the metric has, locally, according to [9, Theorem 4.1], the

coordinate form of [11, formula (3.2)] with \mathcal{P} spanned by a null parallel vector field w [11, lines following formula (3.6)], and we can invoke Example 3.1.

EXAMPLE 3.3. In a cotangent affine bundle over a manifold carrying a torsion-free connection, equipped with any Riemann extension metric, the vertical distribution \mathcal{V} is null and parallel, and the Levi-Civita connection is \mathcal{V} -projectable. See Section 5 below, especially Theorem 5.2.

Let us call the Levi-Civita connection of an n -dimensional pseudo-Riemannian manifold (M, g) *null-projectable* if it is projectable along some null parallel distribution \mathcal{P} of dimension r , with $0 < r < n$. We will use the index ranges

$$(3.2) \quad i, j, k \in \{1, \dots, r\}, \quad p, q \in \{r+1, \dots, n-r\} \quad a, b \in \{n-r+1, \dots, n\},$$

where $n = \dim M$, the range of p, q being thus empty when $n = 2r$. As shown by Walker [18], a null parallel distribution \mathcal{P} of dimension r on an n -dimensional pseudo-Riemannian manifold is, in some local coordinates, spanned by the last r coordinate vector fields, while, for the components of the metric g ,

$$(3.3) \quad \begin{aligned} g_{ab} = g_{ap} = 0, \quad \det[g_{ia}] \neq 0 \neq \det[g_{pq}], \\ \partial_b g_{ia} = \partial_p g_{ia} = \partial_j g_{ia} = \partial_a g_{pq} = \partial_a g_{pi} = 0, \end{aligned}$$

$\det[g_{ia}] \neq 0 \neq \det[g_{pq}]$ reflecting nondegeneracy of g . Conversely, (3.3) with (3.2) always defines a metric g for which the span \mathcal{P} of $\partial_{n-r+1}, \dots, \partial_n$ is null and parallel. Note that $[g_{ia}]$ is here a nonsingular $r \times r$ matrix of constants (and may always be assumed equal to the identity matrix).

LEMMA 3.4. *For the Levi-Civita connection of a metric g satisfying (3.3) with (3.2), or even the weaker assumption that $g_{ab} = g_{ap} = \partial_j g_{ia} = 0$, one has $2\Gamma_{jk}^i = -g^{ai} \partial_a g_{jk}$, where the matrix $[g^{ai}]$ is the inverse of $[g_{ia}]$.*

In fact, $0 = \delta_a^j = g^{ji} g_{ia}$ and $0 = \delta_a^p = g^{pi} g_{ia}$, so that $g^{ji} = g^{pi} = 0$ since $\det[g_{ia}] \neq 0$, and our claim follows.

LEMMA 3.5. *The Levi-Civita connection of a metric g as in (3.3) with (3.2) is projectable along the distribution \mathcal{P} spanned by the last r coordinate vector fields if and only if $\partial_a \partial_b g_{jk} = \partial_a \partial_p g_{jk} = 0$ or, equivalently,*

$$(3.4) \quad g_{jk} = x^a B_{ajk} + \lambda_{jk} \quad \text{for some } B_{ajk}, \lambda_{jk} \quad \text{with } \partial_b B_{ajk} = \partial_p B_{ajk} = \partial_a \lambda_{jk} = 0.$$

PROOF. According to Lemma 2.3, the requirement that ∇ be \mathcal{V} -projectable, where $\mathcal{V} = \mathcal{P}^\perp$, equivalent – by (3.1) – to its \mathcal{P} -projectability, reads, $\Gamma_{aj}^i = \Gamma_{ab}^i = \Gamma_{pj}^i = \Gamma_{ap}^i = \Gamma_{pq}^i = \partial_a \Gamma_{jk}^i = \partial_p \Gamma_{jk}^i = 0$. However, $\Gamma_{aj}^i = \Gamma_{ab}^i = \Gamma_{pj}^i = \Gamma_{ap}^i = \Gamma_{pq}^i = 0$ amounts, in view of (2.1), to \mathcal{V} being ∇ -parallel, which is the case here, so that ∇ is \mathcal{V} -projectable (or \mathcal{P} -projectable) if and only if $\partial_a \Gamma_{jk}^i = \partial_p \Gamma_{jk}^i = 0$, and our claim follows from Lemma 3.4. \square

Lemma 3.5 completely describes the local picture of null-projectability for Levi-Civita connections, illustrating the relative strength of the projectability requirement versus just assuming the distribution to be null and parallel.

In addition to being essentially trivial, Lemma 3.5 also fails to identify various geometric aspects of this situation which have a coordinate-free description. We address such aspects in Section 6.

4. Cotangent affine bundles

An *affine bundle* over a manifold Σ is defined in the usual way, so that there is a total space M with a bundle projection $\pi : M \rightarrow \Sigma$, each fibre $M_y = \pi^{-1}(y)$ of which carries a structure of an affine space depending smoothly on $y \in \Sigma$. It has its associated vector bundle N , the fibre of which over each $y \in \Sigma$ forms the translation vector space of the affine space M_y . Any fixed section S of M , treated as a submanifold of the total space M , allows us to identify M with N by providing in each fibre M_y the origin o_y given by $S \cap M_y = \{o_y\}$. Note that, the fibres being contractible, a global section S always exists.

We call such M a *cotangent affine bundle* over Σ if $N = T^*\Sigma$.

Cotangent affine bundles are encountered in a variety of interesting situations. One example arises in the case of a real line bundle or a complex Hermitian line bundle Λ over Σ . The linear connections (or, respectively, Hermitian linear connections) in Λ then constitute precisely all the sections of an affine bundle over Σ associated with $T^*\Sigma$.

In the complex holomorphic category, with a fixed holomorphic vector bundle N over a complex manifold Σ , the set of equivalence classes of holomorphic affine bundles over Σ associated with N stands in a natural one-to-one correspondence with the first sheaf cohomology group $H^1(\Sigma, \mathcal{F})$, for the sheaf \mathcal{F} of local holomorphic sections of N . This applies, in particular, to $N = T^*\Sigma$, the holomorphic cotangent bundle of Σ .

We will introduce a generalization of cotangent affine bundles in Section 7.

5. Riemann extensions

Given a cotangent affine bundle M over a manifold Σ (Section 4), every 1-form ω on Σ may be viewed as a fibre-preserving diffeomorphism $M \rightarrow M$,

$$(5.1) \quad \text{acting in each fibre } M_y \text{ via the translation by } \omega_y \in T_y^*\Sigma.$$

Let $\pi : M \rightarrow \Sigma$ and $\mathcal{V} = \text{Ker } d\pi$ denote the bundle projection and the vertical distribution. By a *standard-type metric* on M we mean any pseudo-Riemannian metric g on M having $g_x(\xi, w) = \xi(d\pi_x w)$ for any $x \in M$, any $w \in T_x M$, and any vertical vector $\xi \in \mathcal{V}_x = T_y^*\Sigma$, with $y = \pi(x)$. To define such g , it suffices to fix a horizontal distribution \mathcal{H} on M , with $TM = \mathcal{V} \oplus \mathcal{H}$, and prescribe the restriction of g to \mathcal{H} (which may be any section of $[\mathcal{H}^*]^{\odot 2}$, since nondegeneracy of g then

follows as g_x has a nonsingular matrix in a vertical-horizontal basis of $T_x M$). The vertical distribution \mathcal{V} being g -null, every standard-type metric g has the neutral metric signature.

Patterson and Walker's *Riemann extensions* [14, p. 26] form a class of neutral pseudo-Riemannian metrics on cotangent affine bundles M over any manifold Σ equipped with a fixed torsion-free connection D . We define them here to be those standard-type metrics on M which, for every 1-form ω on Σ treated as a diffeomorphism $\omega : M \rightarrow M$ with (5.1), satisfy [14, §8] the transformation rule

$$(5.2) \quad \omega^*g = g + \pi^*\mathcal{L}\omega,$$

\mathcal{L} being the *Killing operator* associated with D , sending any 1-form ω on Σ to the symmetric twice-covariant tensor field

$$(5.3) \quad \mathcal{L}\omega = D\omega + [D\omega]^* \text{ or, in coordinates, } [\mathcal{L}\omega]_{ij} = \omega_{i,j} + \omega_{j,i}.$$

Thus, ω is a g -isometry if $\mathcal{L}\omega = 0$.

We use the term ‘Riemann extension’ narrowly. Wider classes of Riemann extensions have been discussed in the literature. See, for instance, [14], [1] and, more recently, [4]. As the next lemma shows, the above definition of Riemann extensions is equivalent to the standard one, appearing in [14, formula (28)].

LEMMA 5.1. *Let M be the total space of a cotangent affine bundle over an r -dimensional manifold Σ carrying a torsion-free connection D . Given any submanifold S of M forming a global section of the affine bundle, and any symmetric twice-covariant tensor field λ on S , there exists a unique Riemann extension metric g for D the restriction of which to S equals λ .*

If S is used to identify M with T^Σ, so as to turn S into the zero section $\Sigma \subseteq T^*\Sigma$, then, in local coordinates x^i, ξ_i for $T^*\Sigma$ arising from a coordinate system x^i for Σ in which D has the components Γ_{jk}^i ,*

$$(5.4) \quad g = 2d\xi_i \odot dx^i + (\lambda_{jk} - 2\xi_i \Gamma_{jk}^i) dx^j \odot dx^k.$$

With the index ranges $i, j, k \in \{1, \dots, r\}$ and $a, b \in \{r+1, \dots, 2r\}$, using any fixed nonsingular $r \times r$ matrix $[g_{ia}]$ of constants and its inverse $[g^{ai}]$, we may set $x^a = g^{ai}\xi_i$. In the resulting coordinates x^1, \dots, x^{2r} , (5.4) reads $g = 2g_{ia}dx^i \odot dx^a + (\lambda_{jk} - 2g_{ia}x^a \Gamma_{jk}^i) dx^j \odot dx^k$, that is, g has the components

$$(5.5) \quad g_{jk} = \lambda_{jk} - 2g_{ia}x^a \Gamma_{jk}^i \text{ and } g_{ab} = 0, \text{ along with our fixed constants } g_{ia}.$$

PROOF. The existence claim is immediate since (5.4) defines a metric g with the required properties: the zero section $\Sigma \subseteq T^*\Sigma$ being given by $\xi_i = 0$, the restriction of g to Σ is nothing else than λ , while (5.2) follows as the ω -pullback operation applied to differential forms on M commutes with d , leaves invariant functions on Σ treated as defined on M , and $\omega^*\xi_i = \xi_i \circ \omega = \xi_i + \omega_i$, so that ω^*g equals the right-hand side of (5.4) plus $(\partial_j \omega_i + \partial_i \omega_j - 2\omega_k \Gamma_{ij}^k) dx^i \odot dx^j = \mathcal{L}\omega$.

Uniqueness of g follows easily as well: given $x \in M$, let $y = \pi(x) \in \Sigma$, so that x lies in the affine space $M_y = \pi^{-1}(y)$. If we fix a 1-form ω on Σ for which $o = x + \omega_y$ is the unique intersection point of S and M_y (the origin in M_y provided by the global section S) then, by (5.2), $g_x = \omega_x^* g_o - [\pi^* \mathcal{L}\omega]_x$, which proves our claim as x and S uniquely determine o . \square

The following intrinsic local characterization of Riemann extension metrics is a special case of a result of Afifi [1, p. 313]. See also [5, p. 369, Theorem 4.5]. We provide a proof here for the reader's convenience.

THEOREM 5.2. *Given a torsion-free connection D on a manifold Σ and a Riemann extension metric g for D on the total space M of a cotangent affine bundle over Σ , the Levi-Civita connection ∇ of g is projectable along the vertical distribution \mathcal{V} of the bundle projection $M \rightarrow \Sigma$, while \mathcal{V} itself is g -null as well as ∇ -parallel, and the projected torsion-free connection described in Lemma 2.2 coincides with D .*

Conversely, let the Levi-Civita connection ∇ of a pseudo-Riemannian manifold (M, g) with $\dim M = 2r$ be projectable along an r -dimensional null parallel distribution \mathcal{V} . Then for every point $x \in M$, there exist a manifold Σ of dimension r , a torsion-free connection D on Σ , and a diffeomorphic identification of a neighborhood of x in M with an open subset of the total space of a cotangent affine bundle over Σ , under which g, \mathcal{V} and the projected torsion-free connection on a local leaf space correspond to a Riemann extension metric for D , the vertical distribution of the affine-bundle projection, and D .

PROOF. The first part is immediate from Lemmas 3.5 and 2.3: (5.5) amounts to (3.3), where the indices p, q have an empty range, with (3.4) for $B_{ajk} = -2g_{ia}\Gamma_{jk}^i$ and our λ_{jk} . For the second part, Walker's theorem [18] applied to $\mathcal{P} = \mathcal{V}$ gives (3.3) with an empty range for the indices p, q , while (3.4) now becomes (5.5) if one defines Γ_{jk}^i by $B_{ajk} = -2g_{ia}\Gamma_{jk}^i$ using any fixed nonsingular $r \times r$ matrix $[g_{ia}]$ of constants. Lemma 3.5 and the final clause of Lemma 2.3 then yield our claim. \square

The transformation rule (5.2) involves two actions by the infinite-dimensional Abelian group $\Omega^1\Sigma$ of all 1-forms ω on Σ , one via the ω -pullback, the other – via the addition of $\pi^*\mathcal{L}\omega$. The two actions of $\Omega^1\Sigma$ commute with each other, which allows us to think of *Riemann extension metrics on M as the fixed points of a specific action of $\Omega^1\Sigma$ on the set of standard-type metrics on M (which both actions leave invariant).*

6. Geometric consequences of null-projectability

Let \mathcal{P} be an r -dimensional null parallel distribution on a pseudo-Riemannian manifold (M, g) with $\dim M = n$. Every point of M has a neighborhood U such

that, for some manifolds Σ and Q of dimensions r and $n - r$, there are three bundle projections, and two vertical distributions of interest to us:

$$(6.1) \quad p : U \rightarrow Q, \quad q : Q \rightarrow \Sigma, \quad \pi = q \circ p : U \rightarrow \Sigma, \quad \mathcal{P}^\perp = \text{Ker } d\pi, \quad \mathcal{P} = \text{Ker } dp,$$

as one sees applying (1.4) twice, first to realize \mathcal{P} , locally, as the vertical distribution of a bundle projection and, noting that $\mathcal{V} = \mathcal{P}^\perp$ then projects onto an integrable distribution on the base manifold, to similarly realize this projected distribution.

For the remainder of this section, assume that the Levi-Civita connection ∇ of (M, g) is projectable along \mathcal{P} or, equivalently due to (3.1), along $\mathcal{V} = \mathcal{P}^\perp$.

With π and p appearing in (6.1), and any $y \in \Sigma$, the assignment

$$(6.2) \quad T_y^*\Sigma \ni \xi \mapsto v \text{ such that } \pi^*\xi = g(v, \cdot),$$

is a linear isomorphism between $T_y^*\Sigma$ and the space of all vector fields v tangent to \mathcal{P} , defined just on the leaf $\pi^{-1}(y)$ of $\mathcal{V} = \mathcal{P}^\perp$ and parallel along this leaf. That v must be tangent to \mathcal{P} follows as $\pi^*\xi$ vanishes on $\mathcal{V} = \mathcal{P}^\perp$. To show that $\nabla_w v = 0$ for any vector field w on U tangent to \mathcal{V} , we are free to assume that the 1-form ξ is defined on Σ rather than just at y . Projectability of ∇ along \mathcal{V} , with the projected connection D on Σ , easily gives $g(\nabla_w v, \cdot) = \nabla_w[\pi^*\xi] = \pi^*[D_w\xi]$ for all π -projectable vector fields, where w also denotes the projected image of w . In our case, w projects onto 0, and the claim follows.

On the other hand, for each fibre $Q_y = q^{-1}(y)$ of the bundle Q , where $y \in \Sigma$,

$$(6.3) \quad \begin{array}{l} \text{there exists a unique pseudo-Riemannian metric } h_y \text{ on } Q_y \text{ such that} \\ p^*h_y \text{ equals the restriction of } g \text{ to the leaf } U_y = \pi^{-1}(y) \text{ of } \mathcal{V} = \mathcal{P}^\perp. \end{array}$$

In fact, the restriction of g to U_y is projectable under $p : U_y \rightarrow Q_y$. Namely, if p -projectable vector fields v, w on U are tangent to \mathcal{V} , and a vector field u tangent to \mathcal{P} , we have $d_u[g(v, w)] = g(\nabla_u v, w) + g(v, \nabla_u w) = g(\nabla_v u, w) + g(v, \nabla_w u) = 0$ since $[u, v]$ and $[u, w]$ are tangent to \mathcal{P} by (1.3), and so are $\nabla_v u, \nabla_w u$ as \mathcal{P} is parallel, while $\mathcal{V} = \mathcal{P}^\perp$ (which also proves nondegeneracy of the projected metric).

7. Bundles of cotangent-principal bundles

We call a manifold M a *bundle of cotangent-principal bundles* over a base manifold Σ if we are given a bundle projection $q : Q \rightarrow \Sigma$ with some total space Q , while M itself is the total space of an affine bundle over Q having the associated vector bundle $q^*T^*\Sigma$ (the q -pullback of $T^*\Sigma$). This leads to three bundle projections and two relevant vertical distributions:

$$(7.1) \quad p : M \rightarrow Q, \quad q : Q \rightarrow \Sigma, \quad \pi = q \circ p : M \rightarrow \Sigma, \quad \mathcal{V} = \text{Ker } d\pi, \quad \mathcal{P} = \text{Ker } dp.$$

The projection $\pi : M \rightarrow \Sigma$ turns M into a bundle over Σ with the fibre M_y over each $y \in \Sigma$ arising as the restriction of the affine bundle M over Q to the fibre $Q_y = q^{-1}(y) \subseteq Q$, and hence forming a $T_y^*\Sigma$ -principal bundle over Q_y (in the usual sense, for the additive group $T_y^*\Sigma$).

Every section ω of the vector bundle $q^*T^*\Sigma$ over Q is now a fibre-preserving

(7.2) diffeomorphism $\omega : M \rightarrow M$, acting as the translation by $\omega_z \in T_y^*\Sigma$

in each fibre $p^{-1}(z)$ over $z \in Q$, the fibre being an affine space with the translation vector space $T_y^*\Sigma$, where $y = q(z)$.

We are particularly interested in the case where, for a given bundle of cotangent-principal bundles, with M, Q, Σ as above and (7.1),

(a) Σ carries a torsion-free connection D .

Then the Killing operator \mathcal{L} given by (5.3) can be extended to act on sections ω of $q^*T^*\Sigma$. Namely, in $q^*T^*\Sigma$ one has the q -pullback of the connection in $T^*\Sigma$ dual to D . For simplicity, the dual connection and its q -pullback are still denoted here by D . Thus, $[D\omega]_z$, at every $z \in Q$, is a linear operator $T_zQ \rightarrow T_y^*\Sigma$, with $y = q(z)$, assigning to $v \in T_zQ$ the 1-form $\beta = D_v\omega \in T_y^*\Sigma$, and hence giving rise to the bilinear form $T_zQ \times T_zQ \ni (v, u) \mapsto [D\omega]_z(v, u) = \beta(dq_z u)$. In this way $D\omega$ is interpreted as a twice-covariant tensor field on Q , and the Killing operator \mathcal{L} sends ω to twice the symmetrization of $D\omega$, that is, again,

$$(7.3) \quad \mathcal{L}\omega = D\omega + [D\omega]^*.$$

The other assumption to be made about a given bundle of cotangent-principal bundles, for M, Q, Σ as above, reads

(b) Q is a bundle of pseudo-Riemannian manifolds over Σ .

Equivalently, (b) states that we have an assignment

$$(7.4) \quad \Sigma \ni y \mapsto h_y$$

of a pseudo-Riemannian metric h_y on $Q_y = q^{-1}(y)$ to each $y \in \Sigma$, depending smoothly on y . In other words, Q is endowed with a *vertical metric* h , meaning a pseudo-Riemannian fibre metric in the vertical distribution $\text{Ker } dq$.

Bundles of cotangent-principal bundles include cotangent affine bundles (Section 4) as a special case, with $Q = \Sigma$ and $q = \text{Id}$, so that Q has the single-point fibres $Q_y = \{y\}$, and the $T_y^*\Sigma$ -principal bundle over each $\{y\}$ is a single affine space associated with $T_y^*\Sigma$.

8. Riemann pullback-extensions

Let M be a bundle of cotangent-principal bundles over Σ (Section 7). In analogy with Section 5, we define a *standard-type metric* on M to be any pseudo-Riemannian metric g on M such that, with the notation of (7.1), $g_x(\xi, w) = \xi(d\pi_x w)$ for any $x \in M$, any $w \in T_x M$, and any vertical vector $\xi \in \mathcal{P}_x = T_y^*\Sigma$, where $y = \pi(x) \in \Sigma$ (and $\mathcal{P}_x = T_y^*\Sigma$ since \mathcal{P}_x is the tangent space at x of the affine space $p^{-1}(p(x))$ with the translation vector space $T_y^*\Sigma$).

To define such g , it suffices, again, to fix a horizontal distribution \mathcal{H} in the affine bundle M over Q , with $TM = \mathcal{P} \oplus \mathcal{H}$, and prescribe the restriction of g to

\mathcal{H} . The restriction may, this time, be any section of $[\mathcal{H}^*]^{\odot 2}$ nondegenerate on the subbundle \mathcal{H}° of \mathcal{H} with the fibre \mathcal{H}_x° at each $x \in M$ equal to the preimage under the isomorphism $dp_x : \mathcal{H}_x \rightarrow T_z Q$ of the subspace $T_z Q_y \subseteq T_z Q$, for $z = p(x) \in Q$ and $Q_y = q^{-1}(y)$, where $y = \pi(x) \in \Sigma$. Nondegeneracy of g then follows since, in a basis of $T_x M$ containing bases of \mathcal{P}_x and \mathcal{H}_x° , the matrix of g_x has the form

$$\begin{bmatrix} 0 & 0 & G \\ 0 & H & * \\ G' & * & * \end{bmatrix}$$

for some nonsingular square matrices G, H, G' .

Suppose that a bundle M of cotangent-principal bundles over Σ , with the three bundle projections in (7.1), also satisfies (a) – (b) in Section 7: one has a fixed torsion-free connection D on Σ and a vertical metric h on Q .

By a *Riemann pullback-extension* for D, Q and h we then mean any standard-type metric g on M having the following two properties: first, the transformation rule (5.2) holds for g and for all sections ω of the bundle $q^* T^* \Sigma$ over Q , and, secondly, the restriction of g to each fibre $M_y = \pi^{-1}(y)$ of the bundle M over Σ equals $p^* h_y$. Both requirements make sense: the former due to (7.2) – (7.3), the latter since p maps M_y into Q_y , and h_y is a metric on Q_y .

For a global section S of the bundle M over Q , treated as a submanifold of M , and a symmetric twice-covariant tensor field λ on S , we call λ *consistent with our vertical metric h* if the push-forward of λ under the diffeomorphism $p : S \rightarrow Q$, when restricted to the q -vertical subbundle of TQ , equals h .

The next result generalizes Lemma 5.1. Note that the affine bundle M over Q admits global sections.

LEMMA 8.1. *Let there be given a bundle of cotangent-principal bundles, with M, Q, Σ, D, h as in Section 7, (7.1) – (7.4) and (a) – (b), Then, for any submanifold S of M forming a global section of the bundle M over Q , and any symmetric twice-covariant tensor field λ on S consistent with h , there exists a unique Riemann pullback-extension metric g for D, Q and h with the restriction to S equal to λ .*

Using the index ranges (3.2) and any fixed nonsingular $r \times r$ matrix $[g_{ia}]$ of constants, in suitable local coordinates x^1, \dots, x^n , we can express this unique g as

$$(8.1) \quad g = 2g_{ia} dx^i \odot dx^a + (\lambda_{jk} - 2g_{ia} x^a \Gamma_{jk}^i) dx^j \odot dx^k + (2\lambda_{iq} dx^i + h_{pq} dx^p) \odot dx^q,$$

with functions $\lambda_{jk}, \lambda_{iq}, h_{pq}, \Gamma_{jk}^i$ satisfying the conditions

$$(8.2) \quad \partial_a \lambda_{jk} = \partial_a \lambda_{ip} = \partial_a h_{pq} = \partial_p \Gamma_{jk}^i = \partial_a \Gamma_{jk}^i = 0, \quad |\det[g_{pq}]| > 0.$$

Equivalently, in addition to our fixed constants g_{ia} , the components of g are

$$(8.3) \quad g_{jk} = \lambda_{jk} - 2g_{ia} x^a \Gamma_{jk}^i, \quad g_{ip} = \lambda_{ip}, \quad g_{pq} = h_{pq}, \quad g_{pa} = g_{ab} = 0.$$

The coordinates x^i, x^p, x^a to may be chosen so that x^i and $\xi_i = g_{ia}x^a$ are local coordinates for $T^*\Sigma$ arising from the coordinate system x^i for Σ in which D has the components Γ_{jk}^i , while x^i, x^p form a coordinate system in Q in which λ and h have the components λ_{ij} , $\lambda_{iq} = \lambda_{qi}$ and $\lambda_{pq} = h_{pq}$, and the first two projections in (7.1) send x^i, x^p, x^a to x^i, x^p and, respectively, x^i, x^p to x^i . Also,

$$(8.4) \quad g = 2d\xi_i \odot dx^i + (\lambda_{jk} - 2\xi_i \Gamma_{jk}^i) dx^j \odot dx^k + (2\lambda_{iq} dx^i + h_{pq} dx^p) \odot dx^q,$$

PROOF. Existence: with $\xi_i = g_{ia}x^a$, (8.1) rewritten as (8.4) defines a metric g with the required properties. Namely, S , identified with the zero section Q of $q^*T^*\Sigma$ is given by $\xi_i = 0$, and so g restricted to Q equals λ (as $\lambda_{pq} = h_{pq}$ due to the assumption about consistency). Finally, (5.2) follows since the ω -pullback operation applied to differential forms on M commutes with d , preserves functions on Q viewed as defined on M , and $\omega^*\xi_i = \xi_i \circ \omega = \xi_i + \omega_i$, so that ω^*g equals the right-hand side of (8.4) plus $(\partial_j \omega_i + \partial_i \omega_j - 2\omega_k \Gamma_{ij}^k) dx^i \odot dx^j + 2(\partial_p \omega_i) dx^i \odot dx^p = \mathcal{L}\omega$.

Uniqueness of g follows easily as well: given $x \in M$, let $z = p(x) \in Q$ and $y = q(z) = \pi(x) \in \Sigma$, so that x lies in the affine space $p^{-1}(z)$. If we fix a 1-form ω on Σ for which $o = x + \omega_y$ is the unique intersection point of S and $p^{-1}(z)$ (the origin in $p^{-1}(z)$ provided by the global section S) and treat it as a section of the pullback bundle $q^*T^*\Sigma$, still denoted by ω , then, by (5.2), $g_x = \omega_x^*g_o - [\pi^*\mathcal{L}\omega]_x$, and our claim follows since x and S uniquely determine o . \square

We can now prove the following generalization of Theorem 5.2.

THEOREM 8.2. *Let there be given a bundle M of cotangent-principal bundles with the bundle projections $M \rightarrow Q \rightarrow \Sigma$ as in (7.1), a torsion-free connection D on Σ , and a vertical metric h on Q . If g is a Riemann pullback-extension metric on M , for D, Q and h , then the vertical distributions \mathcal{P} and \mathcal{V} of the bundle projections $M \rightarrow Q$ and $M \rightarrow \Sigma$ are each other's g -orthogonal complements, \mathcal{P} is g -null and g -parallel, the Levi-Civita connection ∇ of g is projectable along both \mathcal{P} and \mathcal{V} , while the \mathcal{V} -projected torsion-free connection on Σ , cf. Lemma 2.2, coincides with D , and h arises from g via (6.3).*

Conversely, let the Levi-Civita connection ∇ of a pseudo-Riemannian metric g on an n -dimensional manifold be projectable along a null parallel distribution \mathcal{P} of dimension r . Then, for every point x of the manifold, there exist data M, Q, Σ, D, h as above and a diffeomorphic identification of a neighborhood of x with an open set in M under which g, \mathcal{P} , the connection projected along $\mathcal{V} = \mathcal{P}^\perp$, cf. Lemma 2.2, and the vertical metric on a local leaf space of \mathcal{P} arising from g via (6.3) correspond to a Riemann pullback-extension metric for D, Q and h , the vertical distribution of the affine bundle M over Q , along with D and h .

PROOF. The first part follows from Lemmas 8.1, 3.5 and the final clause of Lemma 2.3: (8.2) – (8.3) are just (3.3) – (3.4) for $B_{ajk} = -2g_{ia}\Gamma_{jk}^i$ and our λ_{jk} .

Under the hypotheses of the second part, Walker's theorem [18] applied to \mathcal{P} gives (3.3) with (3.2) on a coordinate neighborhood U of x which may also be assumed to carry the three bundle projections as in (6.1), the first two of which send x^i, x^p, x^a to x^i, x^p and, respectively, x^i, x^p to x^i . Now (3.3) – (3.4) become (8.3) with (8.2) if one fixes a nonsingular $r \times r$ matrix $[g_{ia}]$ of constants and defines λ_{ip}, h_{pq} and Γ_{jk}^i by $\lambda_{ip} = g_{ip}$, $h_{pq} = g_{pq}$ and $B_{ajk} = -2g_{ia}\Gamma_{jk}^i$. Setting $\xi_i = g_{ia}x^a$, we may treat x^i, ξ_i as the local coordinates for $T^*\Sigma$, associated with the coordinate system x^i in Σ , which leads to the new coordinates x^i, x^p, ξ_i on U and, for the corresponding coordinate vector fields, (8.4) gives $g(\partial/\partial x^i, \partial/\partial \xi_j) = \delta_j^i$. In other words, each $v = \partial/\partial \xi_i$, restricted to any each leaf of \mathcal{P} , is the image under the isomorphism (6.2) of the 1-form $\xi = dx^i$ on Σ , which realizes U (made smaller, if necessary) as an open subset of the total space M of a bundle of cotangent-principal bundles, so as to identify (6.1) with a restriction version of (7.1).

In view of Lemma 8.1, our g , given by (8.3), is thus a Riemann pullback-extension metric for D, Q and h , since the components of D , the \mathcal{V} -projected torsion-free connection, are Γ_{jk}^i (see the final clause of Lemma 2.3), and h , arising from g via (6.3), has the components $h_{pq} = g_{pq}$. This completes the proof. \square

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DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210
Email address: `andrzej@math.ohio-state.edu`

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210
Email address: `masood.24@buckeyemail.osu.edu`